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Introduction générale

Cette thèse s'inscrit dans le domaine des mathématiques appelé **optimisation de formes**. Plus précisément, nous étudions ici un **problème inverse de détection** à l'aide du **calcul de forme** et de l'**analyse asymptotique**. L'objectif est de localiser un objet immergé dans un fluide visqueux, incompressible et stationnaire. Les questions principales qui ont motivé ce travail sont les suivantes :

- *peut-on détecter un objet immergé dans un fluide à partir d'une mesure effectuée à la surface ?*
- *peut-on reconstruire numériquement cet objet, c'est-à-dire approcher sa position et sa forme, à partir de cette mesure ?*
- *peut-on connaître le nombre d'objets présents dans le fluide en utilisant cette mesure ?*

Dans cette introduction, nous commençons par détailler les motivations de la détection d'objets immergés dans un fluide en faisant un point sur les différentes études menées sur ce sujet jusqu'ici. Nous positionnons ensuite les résultats que nous avons obtenus par rapport à ces précédents travaux. Enfin, nous décrivons les cinq chapitres de cette thèse dont nous donnons brièvement le contenu principal ici :

1. le premier est une étude détaillée de l'**existence des dérivées de forme** d'ordre un et deux concernant les équations de Stokes et Navier-Stokes stationnaires incompressibles. Nous étudions ici des conditions de bord de type Dirichlet et Neumann. De plus, nous montrons ces résultats en minimisant les contraintes de régularité de domaine. Nous établissons ainsi un cadre mathématique rigoureux pour démontrer une idée intuitive : les dérivées de forme associées à des problèmes de détection d'obstacles existent même si le bord du domaine dans lequel vit l'objet est peu régulier (à savoir Lipschitz) ;
2. le chapitre II analyse le problème de détection d'obstacles immergés dans un fluide à partir d'une mesure effectuée sur ce fluide à l'aide de méthode d'**optimisation géométrique de forme**. Nous commençons par démontrer rigoureusement, pour chacun des problèmes étudiés, un résultat d'identifiabilité : celui-ci garantit que deux mesures identiques effectuées à la surface du fluide correspondent à deux objets identiques. Nous étudions ensuite la reconstruction des inclusions par la minimisation d'une fonctionnelle de forme. Nous calculons alors son gradient de forme en soulignant les difficultés engendrées lorsque des conditions de Neumann sont imposées sur le bord de l'obstacle. Nous caractérisons enfin la Hessienne de forme : cette étude d'ordre deux nous permet de démontrer par des résultats de régularité locale des solutions que ce problème inverse est sévèrement mal posé. Finalement, nous illustrons la dégénérescence de la fonctionnelle de forme par des calculs explicites de la Hessienne de forme pour des situations géométriques particulières ;

3. le troisième chapitre traite de la reconstruction numérique d'objets dans le cas du système de Stokes. Pour cela, nous minimisons ici une fonctionnelle de forme à l'aide d'un **algorithme d'optimisation** de descente en utilisant le **gradient de forme** calculé dans le chapitre précédent. L'étude théorique menée précédemment montre que cette fonctionnelle de forme est dégénérée pour les *hautes fréquences*. Ainsi, une méthode de régularisation par paramétrisation est utilisée ici pour compenser le caractère mal posé du problème. Nous proposons alors une *méthode adaptative* permettant de travailler avec ces hautes fréquences afin de pouvoir reconstruire des obstacles non réguliers. Nous considérons également la reconstruction numérique de plusieurs objets lorsque nous connaissons le nombre d'inclusions ;
4. le chapitre IV étudie la détection de petites inclusions dans le cas du système de Stokes à l'aide d'une analyse asymptotique. Cette hypothèse de petitesse des objets permet de détecter le nombre d'obstacles et leur localisation approximative en utilisant des méthodes d'**optimisation topologique de forme**. La localisation des inclusions se fait ici à nouveau par minimisation d'une fonctionnelle de forme. Ainsi, nous démontrons un développement asymptotique de la solution des équations de Stokes avec des conditions de Dirichlet ou mixtes sur le bord extérieur et de Dirichlet sur le bord des objets. Nous en déduisons le gradient topologique de la fonctionnelle considérée afin de pouvoir réaliser un algorithme d'optimisation de descente ;
5. le dernier chapitre s'attache à la localisation numérique de petits obstacles en utilisant la dérivée topologique calculée dans le chapitre précédent. Pour cela, nous minimisons, comme dans le chapitre III, une fonctionnelle de forme à l'aide d'un **algorithme d'optimisation** de descente en utilisant cette fois le **gradient topologique**. L'efficacité et les limites de cette méthode concernant notre cadre d'étude (c'est-à-dire les équations de Stokes incompressibles) sont étudiées en effectuant plusieurs simulations. En particulier, nous soulignons le fait que la dérivée topologique dans ce cas statique (ici, stationnaire) semble peu sensible par rapport aux objets éloignés du domaine où la mesure est effectuée.

Le problème inverse de détection d'objets immergés dans un fluide

En sciences, un **problème inverse** consiste à déterminer des *causes* connaissant des *effets*. Il s'oppose ainsi au problème direct visant à décrire les effets connaissant les causes. De nombreuses études portant sur des problèmes inverses sont menées tant pour leurs intérêts mathématiques que pour leurs utilités industrielles. En effet, la résolution de ce type de problème intervient dans des domaines variés comme en sismologie (localisation de l'origine d'un tremblement de terre à partir de mesures effectuées par des stations sismiques à la surface du globe), en imagerie médicale (échographie utilisant des ultrasons, radiographie ou scanner X utilisant des rayons X), en ingénierie pétrolière (prospection par méthodes sismiques ou magnétiques), en chimie (détermination de constantes de réaction), en traitement d'images (restauration d'images floues), etc. Une méthode pour résoudre ces problèmes inverses, en particulier dans le cas de la mécanique des fluides, est l'optimisation de forme. L'**optimisation de forme** consiste à rechercher la meilleure forme possible pour un certain problème (ou critère) : la meilleure aile d'avion, le meilleur mur anti-bruit, le meilleur pare-brise, etc. L'objectif est alors de minimiser une certaine fonction coût dépendant de la forme du domaine : une *fonctionnelle de forme*. Par exemple, il est bien connu (et naturel) que la forme ayant une surface minimale pour un volume fixe est une boule. Dans cette thèse, nous traitons la détection et la reconstruction d'objets immergés dans un fluide visqueux, incompressible et stationnaire par des méthodes d'optimisation de forme.

Récemment, en 2005, Alvarez *et al.* ont étudié dans [6] le problème inverse suivant : un corps rigide

inaccessible ω est immergé dans un fluide visqueux s'écoulant dans un plus grand domaine borné Ω . Ils souhaitent déterminer ω (*i.e.* sa forme et sa position) par des mesures effectuées sur le bord $\partial\Omega$. Sous des hypothèses de régularité raisonnables sur Ω et ω , ils montrent que l'on peut identifier ω si l'on connaît la vitesse du fluide et les forces de Cauchy sur une partie du bord extérieur $\partial\Omega$. Ils donnent également un résultat de stabilité directionnelle pour le problème inverse (voir [6, Théorème 1.3]). Après ce premier résultat, plusieurs études ont été menées. En 2007, Heck *et al.* donnent dans [71] une estimation de la distance entre un point choisi et l'obstacle lorsque le mouvement du fluide est régi par les équations de Stokes. La même année, dans [7], Alves *et al.* se servent d'une méthode basée sur l'analyse d'un système non linéaire d'équations intégrales pour déterminer la forme et la position d'un corps rigide immergé dans un fluide visqueux et incompressible. En 2008, Conca *et al.* étudient dans [41] le problème de détection d'un objet se déplaçant dans un fluide parfait à l'aide d'une mesure effectuée sur le bord. Ils montrent que, lorsque l'obstacle est une boule, on peut identifier la position et la vitesse de son centre de masse à partir d'une simple mesure sur le bord. En 2010, en utilisant l'analyse complexe, Conca *et al.* montrent dans [42] que ce résultat ne peut pas être généralisé à tout solide. Cependant, ils étendent celui-ci à des ellipses mobiles et ils montrent que, lorsque le solide a des propriétés de symétrie, il peut être *partiellement détecté*. La même année, Ballerini étudie dans [19] la détection d'un corps immergé dans un domaine plus grand rempli d'un fluide visqueux à partir d'une simple mesure des forces et des vitesses sur une partie de la surface (voir également [20] du même auteur). Enfin, des expériences numériques ont été menées et montrent que la reconstruction numérique est difficile comme on peut s'y attendre pour un problème inverse.

Dans cette thèse, nous allons suivre principalement deux approches afin de détecter un (ou plusieurs) objet(s) immergé(s) dans un fluide visqueux, incompressible et stationnaire : une basée sur l'optimisation géométrique et la dérivation par rapport au domaine (à l'aide du *gradient de forme*) et une autre basée sur l'optimisation topologique et l'analyse asymptotique (à l'aide du *gradient topologique*). La dérivation par rapport au domaine consiste à dériver la solution d'une EDP, une EDP, une fonctionnelle de forme, etc. par rapport au domaine. L'optimisation géométrique représente alors la minimisation d'une certaine fonctionnelle de forme coût à l'aide d'un algorithme de gradient de forme par exemple. L'analyse asymptotique topologique est l'étude de l'influence de l'inclusion d'un petit trou (ou d'un petit objet) à l'intérieur d'un domaine sur la solution d'une EDP ou sur une fonctionnelle de forme. On étudie ainsi la variation (d'une fonctionnelle de forme) par rapport à la modification de la topologie du domaine. L'optimisation topologique utilisant le gradient topologique peut alors également se présenter comme la minimisation d'une certaine fonctionnelle de forme coût à l'aide d'un algorithme de gradient.

Concernant l'approche par gradient de forme, nous nous servons des outils de dérivation par rapport au domaine présentés dans le livre de Henrot *et al.* [72, Chapitre 5]. On renvoie également au livre de Sokołowski *et al.* [94] et aux papiers de Simon [90, 91]. Ainsi, la première étape est de démontrer l'existence des dérivées de forme. En 1991, Simon l'a fait rigoureusement dans le cas de Stokes dans [92] et le cas de Navier-Stokes a été traité par Bello *et al.* en 1992 dans [21, 22]. Concernant la dérivation d'ordre un des problème de Stokes, c'est-à-dire la caractérisation du problème satisfait par les dérivées de forme, Simon l'étudie dans [92] afin de minimiser la traînée. Il y étudie également les dérivées d'ordre deux. Les équations stationnaires de Navier-Stokes sont dérivées par rapport à la forme par Bello *et al.* dans [21, 22]. Le cas des équations instationnaires de Navier-Stokes incompressibles est étudié en 2004 par Dziri *et al.* dans [52]. Nous pouvons également mentionner les études menées en 2008 par Gao *et al.* en introduisant la transformation de Piola sur le problème de Stokes avec des conditions de Dirichlet dans [57] et sur le problème de Navier-Stokes stationnaire avec des conditions de Dirichlet dans [58]. Ils y caractérisent également le gradient de forme de différentes fonctionnelles. La dérivation par rapport à la forme du problème de Navier-Stokes stationnaire est également traité pour un problème de formes optimales en 2008 par Henrot *et al.* dans [73, 74]. Plus récemment, Ballerini a dérivé les problèmes de

Stokes et Navier-Stokes stationnaires par rapport à la forme dans [19] et [20] afin d'étudier la stabilité du problème de détection d'obstacles.

La deuxième approche par gradient topologique utilise des formules asymptotiques permettant de changer de point de vue (voir les travaux de Ammari *et al.* [8–12]) : pour cela, nous faisons l'hypothèse de petitesse des obstacles. Une méthode d'optimisation basée sur le gradient topologique permet alors de minimiser une certaine fonctionnelle de forme coût afin de déterminer le nombre d'inclusions et leur emplacement. L'analyse de la sensibilité topologique a été introduite en 1995 par Schumacher dans [88] suivi en 1999 par Sokolowski *et al.* dans [93] pour la minimisation de la compliance en élasticité linéaire. Ensuite, Masmoudi a étudié l'équation de Laplace en introduisant une généralisation de la méthode de l'adjoint (voir [39]) et l'utilisation d'une technique de troncature pour donner un cadre d'étude de la sensibilité topologique sur un espace fonctionnel fixé. En utilisant cette approche, le développement asymptotique topologique d'une large classe de fonctionnelles de forme a été donné pour l'élasticité linéaire par Garreau *et al.* en 2001 dans [59] et pour les équations de Poisson et de Stokes par Guillaume *et al.* en 2002 et 2004 dans [65, 66]. En 2003 et 2004, les équations de Helmholtz ont également été étudiées par Amstutz *et al.* dans [17] et par Pommier *et al.* dans [87] et le problème de quasi-Stokes par Hassine *et al.* dans [69]. La sensibilité topologique a également été utilisée pour l'imagerie d'ondes élastiques de corps solides comportant des cavités par Bonnet *et al.* en 2004 dans [28], ou dans l'étude des équations de Maxwell par Masmoudi *et al.* en 2005 dans [79], ou encore dans l'étude de diffusion inverse électrodynamique et acoustique en temps par Bonnet dans [27] en 2006. Nous pouvons pour finir citer les travaux de Carpio *et al.* dans [34] concernant la reconstruction d'objets à l'aide de la dérivée topologique dans le cas des équations de Helmholtz en 2008.

Le reconstruction d'obstacles suscite ainsi beaucoup d'intérêts dans la communauté mathématique. Dans cette thèse, nous allons étudier la détection d'objets immergés dans un fluide visqueux, incompressible et stationnaire en utilisant les deux approches précédentes : l'approche d'Hadamard du début du XX^{ème} siècle dans [67] utilisant l'optimisation géométrique et le gradient de forme et l'approche plus récente, née à la fin de ce même siècle, utilisant l'optimisation topologique et la dérivée topologique.

Dans cette thèse, nous commençons par établir un cadre mathématique rigoureux et général pour la démonstration des dérivées de forme d'ordre un et deux par rapport à un objet immergé dans un domaine plus grand. Ce cadre permet de démontrer l'idée intuitive que la régularité du bord du domaine extérieur n'a alors pas d'importance (il peut être supposé seulement Lipschitz). Nous utilisons pour cela des espaces à poids permettant de tenir compte de la régularité locale des solutions au voisinage de l'inclusion. De plus, nous démontrons cette existence dans le cas de conditions de Neumann sur l'objet en soulignant les difficultés entraînées par celles-ci : nous ne pouvons plus utiliser directement le théorème des fonctions implicites classique. Nous traitons ensuite le problème de reconstruction d'obstacles par minimisation d'une fonctionnelle de forme coût en utilisant le gradient de forme (comme par exemple dans [3, 18, 54, 75, 76]). Nous étudions deux types de fonctionnelles : une fonctionnelle de type moindres carrés et une fonctionnelle de type Kohn-Vogelius. Nous soulignons alors les difficultés engendrées lorsque des conditions de Neumann sont imposées sur le bord de l'objet pour la caractérisation du problème dérivé par rapport à la forme et la caractérisation du gradient de forme que nous calculons explicitement. De plus, nous présentons le cas non-linéaire des équations de Navier-Stokes qui implique des modifications non triviales des démonstrations. Nous menons également une étude d'ordre deux en calculant la Hessienne de forme des fonctionnelles étudiées. Nous démontrons alors, à l'aide d'outils élémentaires pouvant s'adapter à de nombreux problèmes de ce type, que l'opérateur de Riesz associé à cette Hessienne est compact. Nous démontrons ainsi que ce problème de détection est sévèrement mal posé au sens où les fonctionnelles sont dégénérées pour certaines directions de perturbation du domaine : le gradient n'a pas une sensibilité uniforme par rapport aux directions de perturbation. Nous illustrons alors cette dégénérescence par des calculs

explicites de la Hessienne de forme dans le cas de deux cercles concentriques. Ce résultat de compacité motive ainsi nos simulations numériques. En effet, compte tenu de cette compacité, nous utilisons une régularisation (par paramétrisation) afin de pouvoir réaliser des simulations numériques en écartant les *hautes fréquences* pour lesquelles la fonctionnelle est dégénérée. Nous proposons alors une *méthode adaptative* permettant de travailler avec ces hautes fréquences : cette méthode semble ainsi efficace pour reconstruire des objets non réguliers nécessitant l'utilisation de ces fréquences. La reconstruction numérique de plusieurs objets est également considérée lorsque le nombre d'inclusion est connu.

Cependant, cette approche géométrique ne permet pas de modifier la topologie du domaine. En particulier, nous devons connaître le nombre d'inclusions pour reconstruire plusieurs objets. À l'aide d'une hypothèse de petitesse sur les obstacles, nous changeons alors d'approche et utilisons l'optimisation topologique à l'aide du gradient topologique pour dépasser cette limite. En effet, nous détectons alors le nombre d'objets et leur emplacement approximatif. Nous nous plaçons dans le cadre des équations de Stokes incompressibles stationnaires. Afin de localiser les inclusions, nous minimisons ici une fonctionnelle de forme de type Kohn-Vogelius. Nous donnons alors explicitement l'expression du gradient topologique de la fonctionnelle considérée après avoir établi le développement asymptotique des solutions du problème de Stokes. Ce dernier est notamment démontré avec des conditions mixtes sur le bord du domaine extérieur (et de Dirichlet sur le bord de l'objet) du fait de notre approche de type Kohn-Vogelius. Cette expression explicite de la dérivée topologique nous permet alors de réaliser des simulations numériques à l'aide d'une méthode d'optimisation de type gradient. Ces tests sont concluants dans le cas où les inclusions sont proches du domaine de mesure. Cependant, il ressort que la dérivée topologique dans ce cas statique (ici stationnaire) semble peu sensible par rapport aux objets éloignés du domaine de mesure et qu'il est donc très difficile de les détecter.

Chapitre I :

Existence des dérivées de forme

Dans ce chapitre, nous montrons l'existence des dérivées de forme pour les équations de Stokes et Navier-Stokes incompressibles stationnaires. Démontrer rigoureusement cette existence est évidemment la première étape avant de pouvoir dériver un problème (une EDP) par rapport à la forme, c'est-à-dire expliciter, en le justifiant, le problème satisfait par les dérivées de forme.

Pour ce faire, nous suivons ici la recette classique exposée dans le livre de Henrot *et al.* [72, Chapitre 5]. Nous considérons une géométrie de référence et une déformation de ce domaine dite géométrie déformée (ou perturbée). Le problème (par exemple une EDP) étudié sur ce domaine déformé est dit problème perturbé et sa solution est nommée solution perturbée. Nous caractérisons alors cette dernière comme solution d'un nouveau problème sur la géométrie de référence par changement de variables. Nous pouvons ainsi y appliquer un théorème des fonctions implicites pour obtenir la différentiabilité de la solution perturbée transportée sur le domaine de référence. Par composition, nous obtenons alors la différentiabilité par rapport à la forme de la solution de l'EDP sur le domaine de référence.

Ce chapitre se décompose en deux parties. La première présente le cas des équations de Stokes, avec des conditions de Dirichlet puis des conditions de Neumann sur le bord intérieur $\partial\omega$. Nous verrons que les conditions de Neumann compliquent la preuve car nous ne pourrions pas utiliser directement le théorème des fonctions implicites classique. Nous utilisons alors une extension de ce théorème due à Simon (voir [92, Théorème 6]). Dans la deuxième partie, le cas du problème non linéaire de Navier-Stokes stationnaire avec des conditions de Dirichlet est traité (ainsi que le cas de Navier-Stokes linéarisé).

Des résultats de ce type ont été montrés par Simon dans [92] pour le cas de Stokes (avec des

conditions de Dirichlet) et par Bello *et al.* dans [21, 22] pour le cas de Navier-Stokes stationnaire. Cependant, nous affinons ici ces résultats, en particulier concernant la régularité des domaines. Nous établissons ainsi un cadre mathématique général pour la démonstration des dérivées de forme par rapport à un objet immergé dans un domaine plus grand. Ce cadre permet de démontrer rigoureusement l'idée intuitive que la régularité du bord du domaine extérieur n'a alors pas d'importance. Nous utilisons pour cela des espaces à poids permettant de tenir compte de la régularité locale des solutions au voisinage de l'inclusion. Ce cadre est également appliqué pour démontrer l'existence des dérivées de forme d'ordre deux et peut se généraliser aux dérivées d'ordre quelconque. De plus, nous démontrons cette existence dans le cas de conditions de Neumann sur l'objet en soulignant les difficultés entraînées par celles-ci : nous ne pouvons pas utiliser directement le théorème des fonctions implicites classique dans ce cas. Cependant, nous montrons que dans le cas des conditions de Dirichlet, le théorème des fonctions implicites classique peut être appliqué contrairement à ce que fait Simon dans [92] où il utilise une extension de ce théorème. Enfin, nous simplifions la preuve de l'existence des dérivées de forme pour les équations de Navier-Stokes. En effet, dans [21, 22], l'argument principal pour démontrer ce résultat est que tout espace

$$Y_{\boldsymbol{\theta}} := \left\{ \boldsymbol{\phi} \in \mathbf{H}^1(\Omega \setminus \bar{\omega}), \int_{\Omega \setminus \bar{\omega}} |\det \frac{\partial}{\partial x_j} (\mathbf{I} + \boldsymbol{\theta})_i| \boldsymbol{\phi} = 0 \right\}$$

est isomorphe à $Y_{\mathbf{0}}$ (où $\boldsymbol{\theta}$ est une déformation admissible). Ils travaillent alors avec un isomorphisme $\Lambda_{\boldsymbol{\theta}} : Y_{\boldsymbol{\theta}} \rightarrow Y_{\mathbf{0}}$ afin de garantir la condition de compatibilité concernant la divergence dans le problème de Navier-Stokes linéarisé (lorsqu'ils montrent que $D_{(\mathbf{w}, p)}(\mathbf{0}, (\mathbf{w}, p))(\mathbf{v}, q)$ est un isomorphisme afin d'appliquer le théorème des fonctions implicites). Ici, nous n'utilisons pas cet artifice. Nous soulignons simplement que la condition de compatibilité est automatiquement satisfaite si on l'impose dans les espaces avec lesquels on travaille (l'espace E_3 dans la preuve du Lemme I.2.3).

Comme mentionné plus haut, nous nous intéressons ici à la dérivabilité d'ordre un et deux des solutions des problèmes de Stokes et Navier-Stokes stationnaires dans un ouvert borné lipschitzien connexe $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) par rapport à un domaine $\omega \subset\subset \Omega$. Afin de pouvoir montrer cette dérivabilité (d'ordre deux), nous introduisons un certain type de domaines perturbés appelés *domaines admissibles*. Pour $\delta > 0$ fixé (petit), on définit

$$\mathcal{O}_{\delta} := \left\{ \omega \subset\subset \Omega \text{ de classe } C^{2,1} \text{ tel que } d(x, \partial\Omega) > \delta \ \forall x \in \omega \text{ et tel que } \Omega \setminus \bar{\omega} \text{ est connexe} \right\}.$$

Afin d'utiliser la régularité locale des solutions, on définit également Ω_{δ} un ouvert de classe C^{∞} tel que

$$\{x \in \Omega; d(x, \partial\Omega) > \delta/2\} \subset \Omega_{\delta} \subset \{x \in \Omega; d(x, \partial\Omega) > \delta/3\}.$$

Ainsi, on note

$$\mathcal{U} := \{\boldsymbol{\theta} \in \mathbf{W}^{3,\infty}(\mathbb{R}^d); \text{supp } \boldsymbol{\theta} \subset \Omega_{\delta}\} \quad \text{et} \quad \mathcal{U} := \left\{ \boldsymbol{\theta} \in \mathcal{U}; \|\boldsymbol{\theta}\|_{3,\infty} < \min\left(\frac{\delta}{3}, 1\right) \right\}$$

l'espace des *déformations admissibles*. Pour tout $\boldsymbol{\theta} \in \mathcal{U}$ et $\omega \in \mathcal{O}_{\delta}$, on vérifie que $\Omega = (\mathbf{I} + \boldsymbol{\theta})(\Omega)$ et on définit le domaine perturbé $\omega_{\boldsymbol{\theta}} := (\mathbf{I} + \boldsymbol{\theta})(\omega)$ qui est tel que $\Omega \setminus \bar{\omega}_{\boldsymbol{\theta}} \in \mathcal{O}_{\delta}$. Les hypothèses de régularité présentées ici correspondent aux hypothèses utilisées pour démontrer la dérivabilité par rapport à la forme d'ordre deux. Cependant, si l'on s'intéresse seulement à la dérivabilité d'ordre un, nous pouvons travailler avec ω de classe $C^{1,1}$ et $\boldsymbol{\theta} \in \mathbf{W}^{2,\infty}(\mathbb{R}^d)$.

Nous étudions dans ce chapitre trois types de problème. Le premier est le problème de Stokes avec

des conditions de Dirichlet homogènes sur le bord intérieur et non homogène sur le bord extérieur :

$$(SD) \begin{cases} -\nu \Delta \mathbf{u} + \nabla p = \mathbf{0} & \text{dans } \Omega \setminus \overline{\omega} \\ \operatorname{div} \mathbf{u} = 0 & \text{dans } \Omega \setminus \overline{\omega} \\ \mathbf{u} = \mathbf{g} & \text{sur } \partial\Omega \\ \mathbf{u} = \mathbf{0} & \text{sur } \partial\omega, \end{cases}$$

où $\mathbf{g} \in \mathbf{H}^{1/2}(\partial\Omega)$ est tel que $\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n} = 0$. Le second est le problème de Stokes avec second membre et des conditions de type Neumann homogènes sur les bords intérieur et extérieur :

$$(SN) \begin{cases} -\nu \Delta \mathbf{u} + \nabla p = \mathbf{f} & \text{dans } \Omega \setminus \overline{\omega} \\ \operatorname{div} \mathbf{u} = 0 & \text{dans } \Omega \setminus \overline{\omega} \\ -\nu \partial_{\mathbf{n}} \mathbf{u} + p \mathbf{n} = \mathbf{0} & \text{sur } \partial\Omega \\ -\nu \partial_{\mathbf{n}} \mathbf{u} + p \mathbf{n} = \mathbf{0} & \text{sur } \partial\omega, \end{cases}$$

où $\mathbf{f} \in [\mathbf{H}^1(\Omega \setminus \overline{\omega})]'$ est tel que $\int_{\Omega \setminus \overline{\omega}} \mathbf{f} = \mathbf{0}$. Le fait de travailler avec ou sans second membre $\mathbf{f}$ et avec des conditions de bord extérieur homogènes ou non homogènes permet juste de souligner le fait que les résultats d'existence ne changent pas. Il suffit alors que $\mathbf{f}$ ait une régularité suffisante au moins au voisinage du bord intérieur. Notons également que la régularité sur la condition de bord extérieur $\mathbf{g}$ peut d'ailleurs être *minimale* (au regard des espaces d'énergie classiques $\mathbf{H}^1(\Omega \setminus \overline{\omega}) \times L^2(\Omega \setminus \overline{\omega})$), c'est-à-dire $\mathbf{H}^{1/2}(\partial\Omega)$ pour des conditions de Dirichlet et $\mathbf{H}^{-1/2}(\partial\Omega)$ pour des conditions de Neumann. Enfin, nous étudions le problème de Navier-Stokes stationnaire avec des conditions de Dirichlet sur les bords intérieur et extérieur :

$$(NS) \begin{cases} -\nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = \mathbf{0} & \text{dans } \Omega \setminus \overline{\omega} \\ \operatorname{div} \mathbf{u} = 0 & \text{dans } \Omega \setminus \overline{\omega} \\ \mathbf{u} = \mathbf{g} & \text{sur } \partial\Omega \\ \mathbf{u} = \mathbf{0} & \text{sur } \partial\omega. \end{cases}$$

Pour un ouvert borné connexe Lipschitzien Ω de $\mathbb{R}^d$, si $\mathbf{f} \in \mathbf{H}^3(\Omega_\delta)$ (ou dans $\mathbf{H}^1(\Omega_\delta)$ si l'on étudie seulement la dérivabilité d'ordre un par rapport à la forme) et $\mathbf{g} \in \mathbf{H}^{1/2}(\partial\Omega)$, alors le résultat principal démontré dans ce chapitre est le suivant :

Théorème 1. *Les solutions $(\mathbf{u}, p) \in \mathbf{H}^1(\Omega \setminus \overline{\omega}) \times L^2(\Omega \setminus \overline{\omega})$ des trois problèmes ci-dessus sont deux fois différentiables par rapport au domaine $\omega \in \mathcal{O}_\delta$, i.e., pour tout $\mathcal{D} \subset \subset \Omega \setminus \overline{\omega}$, l'application*

$$\boldsymbol{\theta} \in \mathcal{U} \mapsto (\mathbf{u}_{\boldsymbol{\theta}|_{\mathcal{D}}}, p_{\boldsymbol{\theta}|_{\mathcal{D}}}) \in \mathbf{H}^1(\mathcal{D}) \times L^2(\mathcal{D}),$$

où $(\mathbf{u}_{\boldsymbol{\theta}}, p_{\boldsymbol{\theta}})$ est la solution du problème dans $\Omega \setminus \overline{\omega_\theta}$, est Fréchet différentiable en $\mathbf{0}$.

Remarque 2. *Nous verrons que dans l'étude de notre problème inverse du chapitre suivant, la condition $\mathbf{f} \in \mathbf{H}^3(\Omega_\delta)$ sera automatiquement vérifiée du fait que, pour garantir un résultat d'identifiabilité, nous allons supposer que $\mathbf{f} = \mathbf{0}$ dans Ω_δ .*

À l'aide de ces résultats, nous allons alors dériver les problèmes (SD), (SN) et (NS) par rapport à la forme, c'est-à-dire expliciter, en les justifiant, les problèmes satisfaits par les dérivées de formes. Ainsi nous étudierons un problème de détection en utilisant des outils d'optimisation de forme géométrique : nous minimiserons une fonctionnelle de forme à l'aide d'une méthode de descente utilisant le gradient de forme de cette fonctionnelle. Dans le chapitre suivant, nous allons donc analyser un problème inverse à l'aide de ces méthodes et en particulier étudier la stabilité du problème de reconstruction d'un objet immergé dans un fluide visqueux, incompressible et stationnaire.

Chapitre II :

Analyse du problème de détection : identifiabilité, dérivation par rapport à la forme et instabilité

Dans ce chapitre, nous analysons plus concrètement le problème de détection d'obstacles immergés dans un fluide visqueux dans le cas où le mouvement du fluide est régi par les équations de Stokes ou Navier-Stokes incompressibles stationnaires.

Nous voulons ici reconstruire l'objet (ou les objets) supposé(s) immobile(s), c'est-à-dire détecter sa (leur) position et sa (leur) forme. L'obstacle ω est supposé situé dans un domaine plus grand Ω dans lequel s'écoule un fluide visqueux, incompressible et stationnaire. Nous effectuons alors une mesure sur une partie O du bord extérieur (c'est-à-dire à la surface du fluide) et nous voulons reconstruire l'objet à partir de cette mesure. Par exemple, si des conditions de Dirichlet sont imposées sur le bord extérieur, nous mesurons les forces de Cauchy sur une partie de ce bord comme nous le résumons dans la figure 1. Pour reconstruire l'obstacle, nous définissons des fonctionnelles de forme coûts permettant d'évaluer

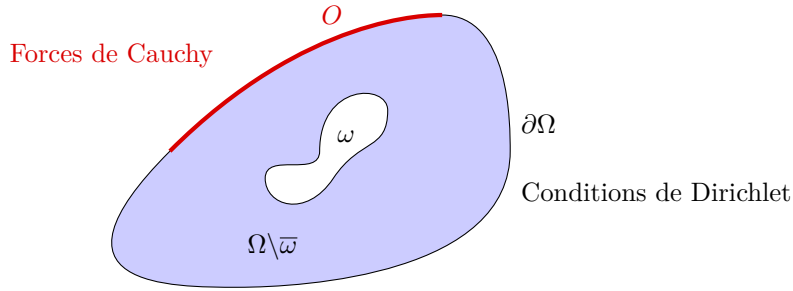


Figure 1 – Configuration type de notre étude

l'erreur entre la solution réelle (au travers de la mesure effectuée) et la solution approchée. L'objectif est alors de minimiser ces fonctionnelles à l'aide d'un algorithme d'optimisation de descente utilisant leur gradient de forme afin de se rapprocher de la solution réelle (comme par exemple dans [3, 18, 54, 75, 76]). Nous travaillons ici avec deux types de fonctionnelles : une fonctionnelle de type moindres carrés et une de type Kohn-Vogelius.

Le premier résultat à étudier dans ce type de problème de reconstruction d'un obstacle à partir d'une mesure est un résultat d'identifiabilité : il faut garantir l'unicité de la solution en montrant que deux mesures identiques correspondent à deux objets identiques. Autrement dit, ce résultat nous assure que ce problème est bien posé au sens où il admet un unique minimum : on peut alors espérer reconstruire l'objet à partir de cette mesure du fait que cette dernière est liée de façon unique à l'obstacle inconnu. Ce résultat d'identifiabilité dans le cadre de Stokes et Navier-Stokes (stationnaire et instationnaire) est dû à Alvarez *et al.* dans [6] (nous renvoyons également au papier [51] de Doubova *et al.*). Nous rappelons ici ce résultat pour le cas stationnaire :

Théorème 3 (Alvarez *et al.* [6]). *Soit $\Omega \subset \mathbb{R}^d$, $d = 2$ ou $d = 3$, un domaine borné de classe $C^{1,1}$ et O un sous-ensemble ouvert non vide de $\partial\Omega$. Soit*

$$\omega_0, \omega_1 \in \mathcal{D}_{\text{ad}} := \{\omega \subset \subset \Omega; \omega \text{ est ouvert Lipschitzien et } \Omega \setminus \bar{\omega} \text{ est connexe}\}$$

et $\mathbf{g} \in \mathbf{H}^{3/2}(\partial\Omega)$ avec $\mathbf{g} \neq \mathbf{0}$, satisfaisant la condition de flux $\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n} = 0$. Pour $\epsilon_ = 0$ ou $\epsilon_* = 1$,*

soit $(\mathbf{u}_j, p_j)$ pour $j = 0, 1$, une solution de

$$\begin{cases} -\operatorname{div}(\sigma(\mathbf{u}_j, p_j)) + \epsilon_* \operatorname{div}(\mathbf{u}_j \otimes \mathbf{u}_j) &= \mathbf{0} & \text{dans } \Omega \setminus \overline{\omega_j} \\ \operatorname{div} \mathbf{u}_j &= 0 & \text{dans } \Omega \setminus \overline{\omega_j} \\ \mathbf{u}_j &= \mathbf{g} & \text{sur } \partial\Omega \\ \mathbf{u}_j &= \mathbf{0} & \text{sur } \partial\omega_j. \end{cases}$$

Supposons que $(\mathbf{u}_j, p_j)$ sont tels que

$$\sigma(\mathbf{u}_j, p_j) \mathbf{n} = \sigma(\mathbf{u}_j, p_j) \mathbf{n} \quad \text{sur } O.$$

Alors $\omega_0 \equiv \omega_1$.

Ici σ est le tenseur des contraintes défini par $\sigma(\mathbf{u}, p) := \nu(\nabla \mathbf{u} + {}^t \nabla \mathbf{u}) - p\mathbf{I}$. Ce théorème repose sur un résultat de continuation unique démontré par Fabre et Lebeau dans [55]. Nous re-démontrons ce résultat en détaillant la preuve. En effet, les preuves faites dans [6] et [51], ne traitent pas le cas où $\omega_0 \setminus \overline{\omega_1}$ n'est pas Lipschitz et connexe (avec ω_0 et ω_1 sont deux objets inclus dans Ω). Nous proposons donc ici une preuve complète en utilisant entre autres un résultat de densité pour le cas où $\omega_0 \setminus \overline{\omega_1}$ n'est pas Lipschitz. Nous mentionnons également le fait que l'on peut mesurer $-\nu \partial_{\mathbf{n}} \mathbf{u} + p \mathbf{n}$ à la place de $\sigma(\mathbf{u}, p) \mathbf{n}$. Finalement, nous soulignons le fait que les mêmes techniques (à savoir le résultat de densité) ne peuvent pas être utilisées si l'on impose des conditions de Neumann sur l'objet. Dans, ce cas, la preuve complète (prenant en compte le cas où $\omega_0 \setminus \overline{\omega_1}$ n'est pas Lipschitz) que nous proposons ici nécessite de supposer que les objets ω sont plus réguliers (à savoir de classe $C^{1,1}$).

Ensuite, nous nous intéressons à la résolution de ce problème inverse par minimisation d'une fonctionnelle de forme à l'aide du gradient de forme (comme par exemple dans [3, 18, 54, 75, 76]). Nous avons ici deux approches : une approche utilisant une fonctionnelle de type moindres carrés et une utilisant une fonctionnelle de type Kohn-Vogelius. Afin d'illustrer plus concrètement ces deux approches, considérons le cas des équations de Stokes avec des conditions de Dirichlet (SD) exposé ci-dessus. La mesure effectuée sur une partie $O \subset \partial\Omega$ est alors $-\nu \partial_{\mathbf{n}} \mathbf{u} + p \mathbf{n} = \mathbf{f}_b$. On définit alors, pour $\omega \in \mathcal{O}_\delta$ (où $\mathcal{O}_\delta$ est défini comme précédemment), la fonctionnelle des moindres carrés suivante :

$$\mathcal{J}_{LS}(\omega) := \int_O m^2 |-\nu \partial_{\mathbf{n}} \mathbf{u} + p \mathbf{n} - \mathbf{f}_b|^2,$$

où la fonction $m \in C_c^\infty(\partial\Omega)$ est telle que $\operatorname{supp}(m) = \overline{O}$ et où $(\mathbf{u}(\omega), p(\omega)) \in \mathbf{H}^1(\Omega \setminus \overline{\omega}) \times L^2(\Omega \setminus \overline{\omega})$ est solution du problème de Stokes

$$\begin{cases} -\nu \Delta \mathbf{u} + \nabla p &= \mathbf{0} & \text{dans } \Omega \setminus \overline{\omega} \\ \operatorname{div} \mathbf{u} &= 0 & \text{dans } \Omega \setminus \overline{\omega} \\ \mathbf{u} &= \mathbf{g} & \text{sur } \partial\Omega \\ \mathbf{u} &= \mathbf{0} & \text{sur } \partial\omega. \end{cases}$$

On définit également la fonctionnelle de Kohn-Vogelius suivante :

$$\mathcal{J}_{KV}(\omega) := \frac{1}{2} \int_{\Omega \setminus \overline{\omega}} \nu |\nabla(\mathbf{u}_D - \mathbf{u}_N)|^2, \quad (1)$$

où $(\mathbf{u}_D, p_D) \in \mathbf{H}^1(\Omega \setminus \overline{\omega}) \times L^2(\Omega \setminus \overline{\omega})$ et $(\mathbf{u}_N, p_N) \in \mathbf{H}^1(\Omega \setminus \overline{\omega}) \times L^2(\Omega \setminus \overline{\omega})$ sont solutions respectives des problèmes de Stokes suivants

$$\begin{cases} -\nu \Delta \mathbf{u}_D + \nabla p_D &= \mathbf{0} & \text{dans } \Omega \setminus \overline{\omega} \\ \operatorname{div} \mathbf{u}_D &= 0 & \text{dans } \Omega \setminus \overline{\omega} \\ \mathbf{u}_D &= \mathbf{g} & \text{sur } \partial\Omega \\ \mathbf{u}_D &= \mathbf{0} & \text{sur } \partial\omega \end{cases} \quad \text{et} \quad \begin{cases} -\nu \Delta \mathbf{u}_N + \nabla p_N &= \mathbf{0} & \text{dans } \Omega \setminus \overline{\omega} \\ \operatorname{div} \mathbf{u}_N &= 0 & \text{dans } \Omega \setminus \overline{\omega} \\ -\nu \partial_{\mathbf{n}} \mathbf{u}_N + p_N \mathbf{n} &= \mathbf{f}_b & \text{sur } O \\ \mathbf{u}_N &= \mathbf{g} & \text{sur } \partial\Omega \setminus \overline{O} \\ \mathbf{u}_N &= \mathbf{0} & \text{sur } \partial\omega. \end{cases} \quad (2)$$

Introduction générale

Afin de minimiser ces fonctionnelles et de réaliser des simulations numériques, nous caractérisons leur gradient de forme. Pour les définir, nous utilisons ici la *méthode des vitesses* introduite par Murat et Simon en 1976 dans [84]. Alors, pour une direction de perturbation admissible $\mathbf{V} \in \mathbf{U}$ (où $\mathbf{U}$ est défini comme précédemment) et $t \in [0, T]$ (où $T > 0$ est un réel fixé suffisamment petit), on définit la fonction suivante

$$\phi : t \in [0, T] \mapsto \mathbf{I} + t \mathbf{V} \in \mathbf{W}^{3,\infty}(\mathbb{R}^d).$$

Pour $t \in [0, T]$, on définit alors le domaine perturbé $\omega_t := \phi(t)(\omega)$. Afin de caractériser au mieux les gradients de forme et en vue de réaliser des simulations numériques, un objectif est de faire apparaître explicitement la dépendance en la direction de perturbation $\mathbf{V}$ dans leur expression. Pour cela, les deux approches diffèrent légèrement. En effet, l'approche des moindres carrés nécessite alors l'introduction d'un problème adjoint. La dérivation par rapport au domaine des équations de Stokes (et Navier-Stokes) dans le cas des conditions de Dirichlet est classique (voir par exemple [21, 22, 57, 58, 92]). Cependant, nous verrons que les conditions de Neumann compliquent très largement celle-ci ainsi que la caractérisation du gradient. Afin de fixer les idées, nous donnons ici les caractérisations dans le cas de Stokes avec des conditions de Dirichlet pour les deux fonctionnelles considérées :

Proposition 4. *Pour $\mathbf{V}$ dans $\mathbf{U}$, la fonctionnelle des moindres carrés $\mathcal{J}_{LS}$ est différentiable en ω dans la direction $\mathbf{V}$ avec*

$$\mathrm{D} \mathcal{J}_{LS}(\omega) \cdot \mathbf{V} = - \int_{\partial\omega} [(\nu \partial_{\mathbf{n}} \mathbf{w} - q \mathbf{n}) \cdot \partial_{\mathbf{n}} \mathbf{u}] (\mathbf{V} \cdot \mathbf{n}),$$

où $(\mathbf{w}, q) \in \mathbf{H}^1(\Omega \setminus \bar{\omega}) \times L^2(\Omega \setminus \bar{\omega})$ est solution du problème de Stokes suivant :

$$\left\{ \begin{array}{ll} -\operatorname{div}(\sigma(\mathbf{w}, q)) &= \mathbf{0} & \text{dans } \Omega \setminus \bar{\omega} \\ \operatorname{div} \mathbf{w} &= 0 & \text{dans } \Omega \setminus \bar{\omega} \\ \mathbf{w} &= 2m^2(\nu \partial_{\mathbf{n}} \mathbf{u} - p \mathbf{n} - \mathbf{f}_b) & \text{sur } \partial\Omega \\ \mathbf{w} &= \mathbf{0} & \text{sur } \partial\omega. \end{array} \right.$$

Proposition 5. *Pour $\mathbf{V}$ dans $\mathbf{U}$, la fonctionnelle de Kohn-Vogelius $\mathcal{J}_{KV}$ est différentiable en ω dans la direction $\mathbf{V}$ avec*

$$\mathrm{D} \mathcal{J}_{KV}(\omega) \cdot \mathbf{V} = - \int_{\partial\omega} (\nu \partial_{\mathbf{n}} \mathbf{w} - q \mathbf{n}) \cdot \partial_{\mathbf{n}} \mathbf{u}_D (\mathbf{V} \cdot \mathbf{n}) + \frac{1}{2} \nu \int_{\partial\omega} |\nabla \mathbf{w}|^2 (\mathbf{V} \cdot \mathbf{n}),$$

où $\mathbf{w} := \mathbf{u}_D - \mathbf{u}_N$ et $q := p_D - p_N$.

Avant de réaliser des simulations numériques, nous étudions la stabilité du problème. Ainsi, nous analysons la Hessienne de forme des fonctionnelles considérées en un point critique. Nous commençons alors par caractériser la Hessienne de forme en procédant de manière similaire à la caractérisation du gradient. À l'aide de cette analyse d'ordre deux, nous montrons le résultat suivant qui est démontré pour tous les problèmes présentés dans ce chapitre :

Théorème 6. *Soit $\omega^* \in \mathcal{O}_\delta$ un point critique de $\mathcal{J}_{LS}$ (respectivement $\mathcal{J}_{KV}$). L'opérateur de Riesz associé à la Hessienne de forme $\mathrm{D}^2 \mathcal{J}_{LS}(\omega^*)$ (respectivement $\mathrm{D}^2 \mathcal{J}_{KV}(\omega^*)$) défini de $\mathbf{H}^{1/2}(\partial\omega^*)$ dans $\mathbf{H}^{-1/2}(\partial\omega^*)$ est compact.*

Ce résultat est démontré ici en utilisant des arguments de régularité locale des solutions. En effet, étant donné la régularité supposée des objets (à savoir qu'ils sont de classe $C^{2,1}$), nous savons que les solutions sont plus régulières au voisinage de ces objets (la régularité globale étant limitée en particulier par le fait que Ω est supposé seulement lipschitzien). En écrivant la Hessienne de forme comme composée d'opérateurs linéaires continus dont un compact (par injection compacte d'un Sobolev dans un autre

moins régulier), nous montrons alors ce théorème. Cette démonstration met en avant le fait que plus l'objet à trouver est régulier, *plus l'injection est compacte* et plus il sera difficile à détecter. Cet énoncé explique ainsi les difficultés rencontrées pour résoudre numériquement ce problème de détection. En effet, le gradient n'a pas une sensibilité uniforme par rapport aux directions de perturbation. Puisque $\mathcal{J}_{LS}$ (respectivement $\mathcal{J}_{KV}$) est deux fois différentiable, elle se comporte au voisinage de ω^* comme son approximation du second ordre. Ce résultat de compacité signifie grosso modo que, dans un voisinage de ω^* (*i.e.* pour t petit), nous ne pouvons pas espérer une estimation du type $C t \leq \sqrt{\mathcal{J}_{LS}(\omega_t)}$ (respectivement $\mathcal{J}_{KV}(\omega_t)$) avec une constante C uniforme en $\mathbf{V}$. Cependant nous montrons que, pour un espace de déformations de dimension fini, la Hessienne de forme discrète est coercive : le résultat d'identifiabilité montre que le domaine à reconstruire est un minimum local strict de la fonctionnelle de forme considérée. Donnons un exemple pour un domaine étoilé ω en dimension deux. Supposons que $\partial\omega$ est paramétré par

$$\begin{aligned} \partial\omega &= \left\{ \begin{pmatrix} g_0 \\ g_1 \end{pmatrix} + \left(g_2 + \sum_{k=1}^{\infty} (g_{2k+1} \cos(kt) + g_{2k+2} \sin(kt)) \right) \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} \right. \\ &\quad \left. = \sum_{k=0}^{\infty} g_k \mathbf{V}_k(t); t \in (0, 2\pi) \right\}, \end{aligned}$$

où $g_k \in \mathbb{R}$. Alors pour tout $n \in \mathbb{N}$, on a une estimation du type

$$\forall \mathbf{V} \in \text{Vect}(\mathbf{V}_k)_{0 \leq k \leq 2n+2}, \quad D^2 J(\omega^*) \cdot (\mathbf{V}, \mathbf{V}) \geq C_n |\mathbf{V}|^2,$$

où C_n est une constante positive. Mais cette constante C_n tend vers 0 quand n tend vers $+\infty$. Ainsi notre fonctionnelle est dégénérée pour les déformations fortement oscillantes, *i.e.* pour les directions de déformations $\mathbf{V}_k$ avec $k \gg 1$. Par conséquent, pour une résolution numérique, nous veillerons à écarter ces déformations fortement oscillantes. Ce manque de réelle stabilité est aussi remarqué dans un récent papier de Ballerini [19] : il obtient une estimation de stabilité de type log-log. Ainsi, notre résultat complète celui-ci au sens où nous montrons que ce problème est sévèrement mal posé.

Dans un dernier temps, nous étudions un peu plus concrètement la dégénérescence de la fonctionnelle. Pour cela, nous effectuons des calculs explicites de la Hessienne de forme dans le cas des équations de Stokes avec une approche de Kohn-Vogelius. Nous nous plaçons dans le cadre de géométries particulières à savoir dans le cas deux cercles concentriques. Nous commençons par résoudre explicitement les équations de Stokes dans cet anneau à l'aide d'un développement en séries de Laurent. Nous nous servons ensuite des conditions de bord pour obtenir une expression des solutions en résolvant des systèmes 4×4 . Enfin, nous calculons la Hessienne de forme à l'aide de l'expression que nous avons obtenue précédemment. Nous montrons alors que la décroissance de la constante C_n mentionnée précédemment est exponentielle. Nous précisons que ces calculs ont été effectués avec l'aide précieuse de Djalil Kateb¹.

Ce chapitre explique donc en particulier les difficultés rencontrées dans la résolution numérique de ce problème de détection. Cependant, cette instabilité peut être évitée par régularisation, en particulier en écartant les *hautes fréquences*. Le chapitre suivant vise ainsi à réaliser des simulations numériques de reconstruction d'objets en utilisant le gradient de forme de la fonctionnelle de Kohn-Vogelius caractérisé ici et en tenant compte de notre résultat de compacité (voir Théorème 6).

1. Université de Technologie de Compiègne, Laboratoire de Mathématiques Appliquées

Chapitre III :

Reconstruction numérique avec la méthode de variation frontières en deux dimensions

Dans ce chapitre, nous reconstruisons numériquement un (ou des) objet(s) immergé(s) dans un fluide lorsque le mouvement du fluide est supposé régi par les équations de Stokes incompressibles. Nous utilisons pour cela l'approche de Kohn-Vogelius présentée précédemment.

Afin de minimiser cette fonctionnelle de forme, nous adoptons un algorithme d'optimisation de descente utilisant le gradient de forme caractérisé dans la proposition 5 du chapitre précédent. Cependant, puisque le problème est sévèrement mal posé d'après le théorème 6 du chapitre précédent, nous avons besoin de méthodes de régularisation pour le résoudre numériquement comme mentionné ci-dessus. Une solution est par exemple l'ajout d'un terme de pénalisation sur le périmètre à la fonctionnelle de forme considérée. En effet, ce terme entraîne alors des problèmes bien posés (voir [33] ou [45]) : la fonctionnelle n'est alors plus dégénérée. Ici, nous choisissons d'utiliser une méthode de régularisation par paramétrisation en utilisant un modèle paramétrique de variations de forme afin de mettre en évidence dans un premier temps le mauvais conditionnement de la matrice Hessienne de forme. Une *méthode adaptative* est ensuite proposée afin de pouvoir réaliser des simulations plus complexes. En effet, cette méthode nous permet d'éviter l'apparition d'oscillations du bord et d'obtenir des résultats satisfaisants de reconstruction d'objets non réguliers, contenant par exemple des angles, voire même de plusieurs inclusions.

Les simulations numériques présentées ici sont faites en dimension deux en utilisant la librairie éléments finis MÉLINA [78]. On se restreint ici à des domaines étoilés et on utilise les coordonnées polaires pour paramétrer le domaine : le bord $\partial\omega$ de l'objet est alors paramétré par

$$\partial\omega = \left\{ \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + r(\theta) \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \theta \in [0, 2\pi) \right\},$$

où $x_0, y_0 \in \mathbb{R}$ et où r est une fonction $C^{2,1}$ et 2π -périodique. En tenant compte du résultat de compacité du chapitre précédent (voir Théorème 6), on approche le rayon polaire r par sa série de Fourier tronquée

$$r_N(\theta) := a_0^N + \sum_{k=1}^N a_k^N \cos(k\theta) + b_k^N \sin(k\theta),$$

où $N \in \mathbb{N}^*$, pour les simulations numériques. En effet, cette régularisation par projection permet de supprimer les *hautes fréquences* générées par $\cos(k\theta)$ et $\sin(k\theta)$ avec $k \gg 1$ pour lesquelles la fonctionnelle est dégénérée. Les formes avec lesquelles nous travaillons sont alors entièrement définies par les coefficients (a_i, b_i) . Ainsi, pour $k = 1, \dots, N$, les directions de déformation correspondantes sont respectivement

$$\mathbf{V}_1 := \mathbf{V}_{x_0} := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{V}_2 := \mathbf{V}_{y_0} := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathbf{V}_3(\theta) := \mathbf{V}_{a_0}(\theta) := \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix},$$

$$\mathbf{V}_{2k+2}(\theta) := \mathbf{V}_{a_k}(\theta) := \cos(k\theta) \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \quad \mathbf{V}_{2k+3}(\theta) := \mathbf{V}_{b_k}(\theta) := \sin(k\theta) \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix},$$

$\theta \in [0, 2\pi)$. Le gradient de forme est alors calculé composante par composante en utilisant sa caractérisation démontrée dans le chapitre précédent (voir Proposition 5). Nous précisons également que ces simulations numériques ont été réalisées avec la précieuse contribution de Grégory Vial².

2. École Centrale de Lyon

Les premiers tests réalisés montrent la dégénérescence de la fonctionnelle pour les *hautes fréquences*. En effet, si on travaille avec des perturbations $\cos(k\theta)$, $\sin(k\theta)$ avec $k \gg 1$, on voit alors apparaître des oscillations sur le bord $\partial\omega$ entraînant une mauvaise approximation de l'obstacle. Il faut donc travailler avec *peu* de modes de Fourier. Cependant, il est nécessaire d'utiliser ces *hautes fréquences* pour reconstruire des objets non réguliers, contenant par exemple des angles. Nous proposons alors une méthode permettant de tenir compte de celles-ci : une *méthode adaptative*. Celle-ci consiste à augmenter progressivement le nombre de modes de Fourier avec lesquels on travaille. Cela semble efficace et permet d'éviter l'apparition d'oscillations du bord lors de la reconstruction. Cette méthode permet ainsi de travailler avec les *hautes fréquences* et donc de reconstruire des objets non réguliers comme présenté dans la figure 2. L'influence du domaine de mesure O est également testé et les

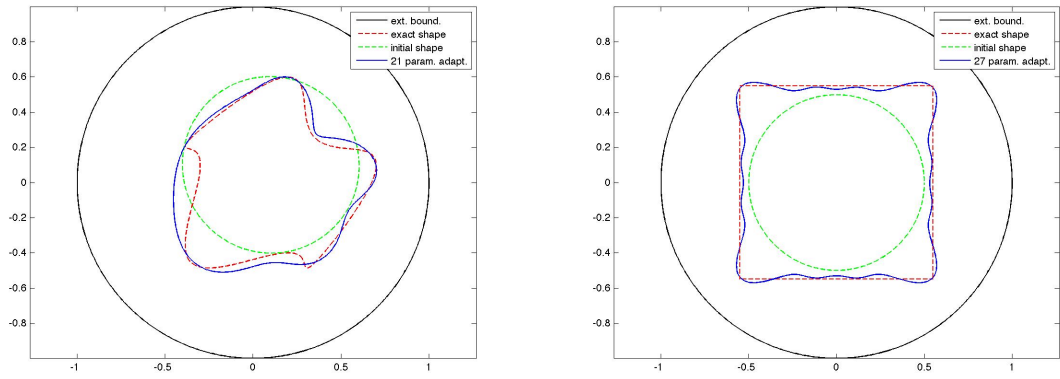


Figure 2 – Reconstruction d'objets à l'aide de la paramétrisation couplée à la *méthode adaptative*

simulations montrent que, naturellement, il est difficile de reconstruire des objets (ou des parties d'un objet) éloignés du domaine de mesure.

Pour finir, nos simulations montrent qu'il est possible de reconstruire plusieurs objets. Cependant, cette méthode d'optimisation géométrique nécessite de connaître le nombre d'obstacles à reconstruire avant de les détecter. Cette constatation a motivé l'étude menée dans le chapitre suivant qui vise à répondre à la question suivante : *peut-on connaître le nombre d'objets et leur position approximative à partir de la mesure effectuée sur une partie du bord extérieur ?*

Chapitre IV :

Cas de petites inclusions : analyse asymptotique

Pour répondre à la question précédente, nous faisons une hypothèse supplémentaire concernant la taille des objets à détecter. Dans ce chapitre, nous traitons ainsi le cas de petites inclusions immergées dans un fluide visqueux, incompressible et stationnaire lorsque le mouvement du fluide est régi par les équations de Stokes. Nous utilisons à nouveau une approche de type Kohn-Vogelius.

L'hypothèse de petitesse faite sur les obstacles nous permet d'utiliser des formules asymptotiques et donc de changer de point de vue (voir les travaux d'Ammari *et al.* [8–12]). Nous déterminons ainsi un développement asymptotique de la solution des problèmes de Stokes considérés lorsque l'on ajoute un petit obstacle à l'intérieur. Nous en déduisons alors une caractérisation de la *dérivée topologique* de la fonctionnelle de Kohn-Vogelius. Nous pouvons alors la minimiser pour résoudre notre problème inverse initial à l'aide d'un algorithme d'optimisation de type gradient. L'utilisation de cette notion de

gradient topologique permet de déterminer le nombre d'objets présents et leur position approximative et peut, par exemple, fournir une initialisation pour un algorithme de gradient de forme (c'est-à-dire une méthode de variations frontière comme celle utilisée dans le chapitre précédent).

L'idée de l'utilisation de la dérivée topologique est la suivante. On considère une fonctionnelle de forme $\mathcal{J}(\Omega) = J_\Omega(u_\Omega)$, où u_Ω est solution d'une EDP définie sur un domaine $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. Nous nous intéressons ici à la fonctionnelle de Kohn-Vogelius $\mathcal{J}_{KV}$ dans le cas des équations de Stokes présentée plus haut (voir (1)). Pour un petit paramètre $\varepsilon > 0$ et un point $z \in \Omega$, on considère un domaine perturbé $\Omega_{z,\varepsilon}$ créé en insérant un petit trou (ou un petit obstacle dans notre cas) à l'intérieur du domaine initial Ω au point z , *i.e.*

$$\Omega_{z,\varepsilon} := \Omega \setminus \overline{\omega_{z,\varepsilon}},$$

avec $\omega_{z,\varepsilon} := z + \varepsilon\omega$, où ω est un ouvert borné fixé de $\mathbb{R}^d$ contenant l'origine. Nous voulons alors savoir l'influence de la modification de la topologie sur les variations de $\mathcal{J}$. Nous pouvons généralement obtenir un développement asymptotique de la fonctionnelle $\mathcal{J}$ de la forme suivante :

$$\mathcal{J}(\Omega_{z,\varepsilon}) = \mathcal{J}(\Omega) + \xi(\varepsilon) G(z) + o(\xi(\varepsilon)) \quad \forall z \in \Omega,$$

où ξ est une fonction scalaire positive tendant vers 0 avec ε et où G est appelé le *gradient topologique* (ou *dérivée topologique*). Ce développement est appelé *développement asymptotique topologique* et le gradient topologique G est noté $\delta\mathcal{J}$. Cette dérivée topologique fournit des informations sur l'insertion d'un petit trou au point z . En effet, si $\delta\mathcal{J}(z) < 0$, alors $\mathcal{J}(\Omega_{z,\varepsilon}) < \mathcal{J}(\Omega)$ pour ε petit. Ainsi, le meilleur endroit pour créer ce trou est le point où $\delta\mathcal{J}$ atteint son minimum. Une difficulté est alors de connaître la taille du trou que l'on doit rajouter. Nous illustrons l'idée dans la figure 3 ci-dessous.

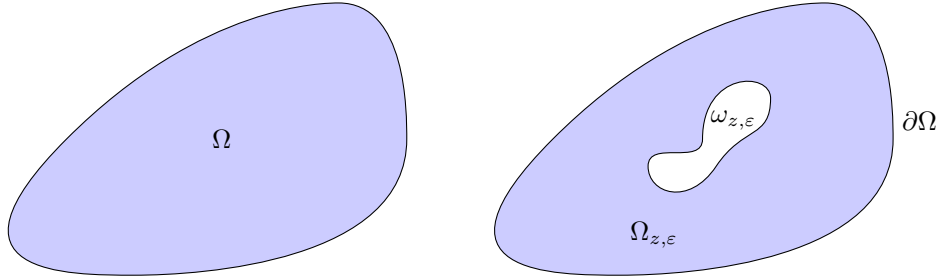


Figure 3 – Le domaine initial et le domaine après l'inclusion d'un objet

Nous voulons à l'aide de cette technique détecter de petits obstacles immergés dans un fluide vivant dans un domaine plus grand Ω . Nous nous concentrons ici sur la dimension trois. Nous effectuons une mesure sur une partie O du bord extérieur $\partial\Omega$ et étudions alors la fonctionnelle de type Kohn-Vogelius $\mathcal{J}_{KV}$ définie précédemment par (1). Afin de pouvoir obtenir une expression pour le gradient topologique de $\mathcal{J}_{KV}$, nous démontrons dans un premier temps le développement asymptotique de la solution des équations de Stokes considérées, *i.e.* (2), lorsque l'on ajoute un obstacle à l'intérieur. Nous nous inspirons principalement des travaux [1, 23, 66, 68]. Le fait de considérer ici des conditions de Dirichlet sur le bord intérieur et des conditions mixtes sur le bord extérieur (du fait de notre approche par une fonctionnelle de Kohn-Vogelius) ajoute des difficultés pour montrer ce développement asymptotique et nous reprenons donc l'ensemble des démonstrations des références précédentes en détails. Nous obtenons le résultat suivant à l'aide d'estimations uniformes *a priori* :

Théorème 7. Les solutions respectives $\mathbf{u}_D^\varepsilon \in \mathbf{H}^1(\Omega_{z,\varepsilon})$ et $\mathbf{u}_N^\varepsilon \in \mathbf{H}^1(\Omega_{z,\varepsilon})$ des problèmes

$$\left\{ \begin{array}{ll} -\nu \Delta \mathbf{u}_D^\varepsilon + \nabla p_D^\varepsilon = \mathbf{0} & \text{dans } \Omega \setminus \overline{\omega_\varepsilon} \\ \operatorname{div} \mathbf{u}_D^\varepsilon = 0 & \text{dans } \Omega \setminus \overline{\omega_\varepsilon} \\ \mathbf{u}_D^\varepsilon = \mathbf{g} & \text{sur } \partial\Omega \\ \mathbf{u}_D^\varepsilon = \mathbf{0} & \text{sur } \partial\omega_\varepsilon \end{array} \right. \quad \text{et} \quad \left\{ \begin{array}{ll} -\nu \Delta \mathbf{u}_N^\varepsilon + \nabla p_N^\varepsilon = \mathbf{0} & \text{dans } \Omega \setminus \overline{\omega_\varepsilon} \\ \operatorname{div} \mathbf{u}_N^\varepsilon = 0 & \text{dans } \Omega \setminus \overline{\omega_\varepsilon} \\ -\nu \partial_{\mathbf{n}} \mathbf{u}_N^\varepsilon + p_N^\varepsilon \mathbf{n} = \mathbf{f}_b & \text{sur } O \\ \mathbf{u}_N^\varepsilon = \mathbf{g} & \text{sur } \partial\Omega \setminus \overline{O} \\ \mathbf{u}_N^\varepsilon = \mathbf{0} & \text{sur } \partial\omega_\varepsilon, \end{array} \right.$$

admettent le développement asymptotique suivant (avec l'indice $\mathfrak{h} = D$ et $\mathfrak{h} = N$ respectivement) :

$$\mathbf{u}_\mathfrak{h}^\varepsilon(x) = \mathbf{u}_\mathfrak{h}^0(x) + \mathbf{U}_\mathfrak{h} \left(\frac{x-z}{\varepsilon} \right) + O_{\mathbf{H}^1(\Omega_{z,\varepsilon})}(\varepsilon)$$

où $(\mathbf{U}_\mathfrak{h}, P_\mathfrak{h}) \in \mathbf{W}_0^{1,2}(\mathbb{R}^3 \setminus \overline{\omega}) \times L^2(\mathbb{R}^3 \setminus \overline{\omega})$ est solution du problème de Stokes extérieur suivant :

$$\left\{ \begin{array}{ll} -\nu \Delta \mathbf{U}_\mathfrak{h} + \nabla P_\mathfrak{h} = \mathbf{0} & \text{dans } \mathbb{R}^3 \setminus \overline{\omega} \\ \operatorname{div} \mathbf{U}_\mathfrak{h} = 0 & \text{dans } \mathbb{R}^3 \setminus \overline{\omega} \\ \mathbf{U}_\mathfrak{h} = -\mathbf{u}_\mathfrak{h}^0(z) & \text{sur } \partial\omega. \end{array} \right.$$

Nous étudions ensuite le développement asymptotique de la fonctionnelle de Kohn-Vogelius considérée. À l'aide du résultat précédent et de manipulations judicieuses de la fonctionnelle, nous montrons le résultat suivant :

Théorème 8. Pour $z \in \Omega$, la fonctionnelle $\mathcal{J}_{KV}$ admet le développement asymptotique topologique suivant :

$$\mathcal{J}_{KV}(\Omega_{z,\varepsilon}) - \mathcal{J}_{KV}(\Omega) = \varepsilon \left[- \left(\int_{\partial\omega} \boldsymbol{\eta}_D \right) \cdot \mathbf{u}_D^0(z) + \left(\int_{\partial\omega} \boldsymbol{\eta}_N \right) \cdot \mathbf{u}_N^0(z) \right] + o(\varepsilon),$$

où $\boldsymbol{\eta}_D \in \mathbf{H}^{-1/2}(\partial\omega)/\mathbb{R}\mathbf{n}$ and $\boldsymbol{\eta}_N \in \mathbf{H}^{-1/2}(\partial\omega)/\mathbb{R}\mathbf{n}$ sont respectivement l'unique solution de

$$\int_{\partial\omega} E(y-x) \boldsymbol{\eta}_\mathfrak{h}(x) \, ds(x) = -\mathbf{u}_\mathfrak{h}^0(y) \quad \forall y \in \partial\omega$$

avec $\mathfrak{h} = D$ et $\mathfrak{h} = N$ respectivement.

Dans le cas particulier où l'on ajoute des objets sphériques, on en déduit une formule plus explicite pour le gradient topologique. Concrètement, nous utiliserons la formule suivante pour les simulations numériques :

Corollaire 9. Si $\omega = B(0,1)$, alors, pour $z \in \Omega$, la fonctionnelle $\mathcal{J}_{KV}$ admet le développement asymptotique topologique suivant :

$$\mathcal{J}_{KV}(\Omega_{z,\varepsilon}) - \mathcal{J}_{KV}(\Omega) = 6\pi\nu\varepsilon \left(|\mathbf{u}_D^0(z)|^2 - |\mathbf{u}_N^0(z)|^2 \right) + o(\varepsilon).$$

Nous disposons ainsi d'un critère nous permettant d'ajouter des objets à l'intérieur d'un domaine afin de diminuer le coût de la fonctionnelle de Kohn-Vogelius $\mathcal{J}_{KV}$. Le chapitre suivant utilise alors cette notion de gradient topologique pour réaliser des simulations numériques et connaître le nombre d'objets présents dans le fluide et leur position approximative.

Chapitre V :

Cas de petites inclusions : détection numérique à l'aide de la dérivée topologique

Dans ce chapitre, nous reconstruisons numériquement un (ou des) petit(s) objet(s) immergé(s) dans un fluide stationnaire lorsque le mouvement du fluide est supposé régi par les équations de

Stokes incompressibles. Nous utilisons la dérivée topologique de la fonctionnelle de Kohn-Vogelius présentée dans le chapitre précédent (voir Corollaire (9)) pour minimiser celle-ci.

L'utilisation de la dérivée topologique a pour objectif de fournir le nombre d'inclusions ainsi que leur emplacement approximatif. En utilisant un algorithme de type gradient topologique, nous détectons ici de petits objets immergés dans un fluide à partir d'une mesure effectuée sur une partie du bord extérieur. Des résultats satisfaisants sont obtenus mais nous soulignons également les inconvénients de cette méthode dans le cadre de notre étude. En particulier, il s'avère très difficile de détecter des inclusions éloignées du domaine sur lequel nous effectuons la mesure.

Les simulations numériques présentées ici sont faites en dimension trois en utilisant la librairie éléments finis FREEFEM++ [70]. L'algorithme utilisé est l'algorithme de gradient topologique classique. Ainsi, l'idée est de calculer la dérivée topologique $\delta\mathcal{J}_{KV}$ pour une situation donnée et d'ajouter un objet à l'endroit où il atteint son minimum. La taille de l'objet est ici déterminée par une méthode de seuillage.

Nous explorons l'efficacité et les limites de cette méthode. Nous commençons ainsi par détecter de petits objets situés près du bord, qu'ils soient éloignés ou proches les uns des autres (voir la figure 4). Ces simulations donnent alors de bons résultats. Nous étudions ensuite l'influence de la distance de

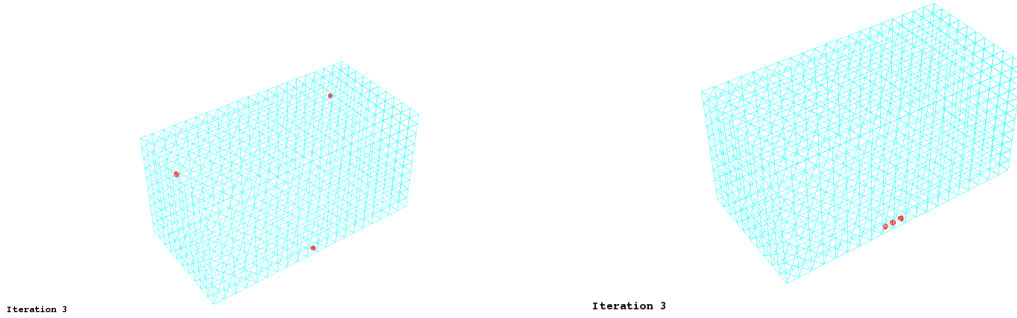


Figure 4 – Détection de petits obstacles à l'aide du gradient topologique

l'objet avec le domaine de mesure. Il apparaît alors qu'il est très difficile de détecter les objets éloignés du domaine de mesure. Pour être plus précis, on étudie la détection d'un seul objet et on définit d la distance entre l'objet (de taille r^*) et le bord extérieur du domaine sur lequel on effectue la mesure. On définit alors la distance non dimensionnelle $\eta := \frac{d}{2r^*}$. On veut étudier l'erreur entre $P^* := (x^*, y^*, z^*)$ et $P(\eta) := (x(\eta), y(\eta), z(\eta))$ qui sont les coordonnées respectives des centres de l'objet réel et de son approximation. On définit alors les deux erreurs suivantes

$$\text{Err}_1(\eta) := \frac{\|P^* - P(\eta)\|}{d} \quad \text{et} \quad \text{Err}_2(\eta) := \frac{\|P^* - P(\eta)\|}{2r^*},$$

où $\|\cdot\|$ représente la norme euclidienne. La figure 5, représente Err_1 et Err_2 par rapport à la distance non dimensionnelle η . Nous nous apercevons qu'il est effectivement difficile de reconstruire l'objet lorsque la distance avec le domaine de mesure devient grande. Ce résultat avait également été souligné par Ben Abda *et al.* dans [23, Fig. 4.2.(b)] dans le cas de conditions de Neumann sur le bord de l'objet. Notre intuition pour expliquer ces résultats numériques est que la profondeur de pénétration de notre approche est faible dans notre cas stationnaire. Dans le cas élastique, cette profondeur augmente pour un problème dynamique ou ondulatoire comme dans l'article [34] de Carpio *et al.*

Nous analysons ensuite l'influence de la taille et la forme de l'objet sur la détection de celui-ci. Les simulations montrent que si la taille caractéristique de l'objet devient trop importante, la méthode

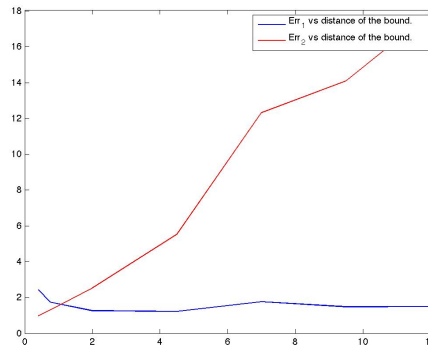


Figure 5 – La norme euclidienne de l’erreur entre les coordonnées exactes et approchées du centre de l’objet par rapport à la distance non dimensionnelle η

devient inefficace. Ces résultats ne sont pas surprenants étant donné que l’hypothèse de petitesse est un argument essentiel pour obtenir les résultats théoriques du chapitre précédent. De plus, les différents tests réalisés montrent que les objets possédant des angles (comme des cubes) sont plus facilement détectés par la dérivée topologique.

Ces simulations numériques permettent ainsi de détecter le nombre d’objets et leur emplacement approximatif dans le cas de petits obstacles proches du domaine de mesure. Une fois ces restrictions satisfaites, la détection est efficace, d’autant plus si les objets possèdent des angles. Cependant, notre cadre statique de l’étude du problème de Stokes stationnaire ne permet pas de détecter des objets éloignés du bord du fait que la sensibilité de la fonctionnelle et de son gradient topologique décroît lorsque la distance augmente.

Ces travaux ont donné lieu à quatre publications dont une publiée et trois soumises :

- *Detecting an obstacle immersed in a fluid by shape optimization methods*, co-écrit avec Mehdi Badra³ et Marc Dambrine³ (voir [18]) ;
- *Instability of an inverse problem for the stationary Navier-Stokes equations* (voir [36]) ;
- *A Kohn-Vogelius formulation to detect an obstacle immersed in a fluid*, co-écrit avec Marc Dambrine³, Djalil Kateb⁴ et Chahnaz Zakia Timimoun⁵ (voir [38]) ;
- *Localisation of small obstacles in Stokes flow*, co-écrit avec Marc Dambrine³ (voir [37]), ainsi qu’à un acte de conférence :
- *Detecting an obstacle immersed in a fluid: the Stokes case* (voir [35])

Nous les avons ici réorganisés afin de rendre la lecture de ce manuscrit plus ordonnée et cohérente.

3. Université de Pau et des Pays de l’Adour

4. Université de Technologie de Compiègne

5. Université d’Oran

Notations

Notations générales

d	:	entier naturel non nul, dimension de l'espace de travail ($d = 2, 3$)
ν	:	constante strictement positive, vitesse cinématique du fluide
$\mathbf{n}$	:	normale extérieure unitaire
Ω	:	ouvert borné connexe lipschitzien de $\mathbb{R}^d$
$\partial\Omega$	:	bord de Ω
$ \Omega $	:	volume de Ω
ω	:	ouvert borné strictement inclus dans Ω
ω^*	:	solution d'un problème d'optimisation de forme
O	:	sous-ensemble ouvert non vide de $\partial\Omega$, domaine de mesure
$\mathbb{1}_O$	:	fonction indicatrice de O
$\mathcal{M}_{d,d}$	:	espace des matrices de taille $d \times d$
$\mathbf{I}$	:	matrice identité de taille $d \times d$
$\mathbf{I}$	$:=$	${}^t(1, \dots, 1)$, vecteur unité de taille d
supp	:	support d'une fonction
$(\mathbf{u}, p)$	:	solution de l'EDP considérée (Stokes ou Navier-Stokes) dans $\Omega \setminus \bar{\omega}$
${}^t\nabla \mathbf{u}$	:	matrice transposée de $\nabla \mathbf{u}$
$\partial_{\mathbf{n}} \mathbf{u}$	:	dérivée normale de $\mathbf{u}$
$\partial_{\mathbf{nn}} \mathbf{u}$	:	dérivée normale seconde de $\mathbf{u}$
$\sigma(\mathbf{u}, p)$	$:=$	$\nu(\nabla \mathbf{u} + {}^t\nabla \mathbf{u}) - p\mathbf{I}$, tenseur des contraintes
$\mathcal{D}(\mathbf{u})$	$:=$	$(\nabla \mathbf{u} + {}^t\nabla \mathbf{u})$, gradient symétrisé de $\mathbf{u}$
C^k	:	espace des fonctions k fois continûment différentiables
$\mathcal{D}$	:	espace des fonctions C^∞ à support compact
$\mathcal{D}'$	:	espace des distributions, dual de $\mathcal{D}$
L^p	:	espace de Lebesgue classique
$W^{m,p}$	:	espace de Sobolev classique
H^m	$:=$	$W^{m,2}$
$C^k, \mathbf{W}^{m,p}$, etc.	:	espaces des fonctions vectorielles
$L_0^2(\Omega)$	:	espace des fonctions $p \in L^2(\Omega)$ telles que $\int_\Omega p = 0$
$\ \cdot\ _{m,\Omega}$	:	norme $\ \cdot\ _{H^m(\Omega)}$
$\ \cdot\ _{s,\partial\Omega}$	:	norme $\ \cdot\ _{H^s(\partial\Omega)}$
$\langle \cdot, \cdot \rangle_\Omega$	:	produit de dualité entre $H^{-1}(\Omega)$ et $H_0^1(\Omega)$
$\langle \cdot, \cdot \rangle_{\partial\Omega}$	:	produit de dualité entre $H^{-1/2}(\partial\Omega)$ et $H^{1/2}(\partial\Omega)$

Chapitre I

m	:	fonction de $\mathcal{D}(\partial\Omega)$ telle que $\text{supp}(m) = \overline{O}$
δ	:	réel strictement positif
$\mathcal{O}_\delta$	$:=$	$\left\{ \omega \subset\subset \Omega \text{ de classe } C^{2,1} \text{ tel que } d(x, \partial\Omega) > \delta \ \forall x \in \omega \text{ et tel que } \Omega \setminus \overline{\omega} \text{ est connexe} \right\}$
Ω_δ	:	ouvert régulier tel que $\{x \in \Omega; d(x, \partial\Omega) > \delta/2\} \subset \Omega_\delta \subset \{x \in \Omega; d(x, \partial\Omega) > \delta/3\}$
$\mathbf{U}$	$:=$	$\{\boldsymbol{\theta} \in \mathbf{W}^{3,\infty}(\mathbb{R}^d); \text{supp } \boldsymbol{\theta} \subset \Omega_\delta\}$
$\mathcal{U}$	$:=$	$\{\boldsymbol{\theta} \in \mathbf{U}; \ \boldsymbol{\theta}\ _{3,\infty} < \min(\frac{\delta}{3}, 1)\}$
ω	:	domaine de référence appartenant aux ensembles admissibles Ω_δ
$\boldsymbol{\theta}$	:	direction de perturbation appartenant à $\mathcal{U}$
$\mathbf{V}$	:	direction de perturbation appartenant à $\mathbf{U}$
ϕ	:	$t \in [0, T) \mapsto \mathbf{I} + t \mathbf{V} \in \mathbf{W}^{3,\infty}(\mathbb{R}^d)$
ω_θ	$:=$	$(\mathbf{I} + \boldsymbol{\theta})(\omega)$, domaine perturbé
ω_t	$:=$	$(\mathbf{I} + t\mathbf{V})(\omega)$, domaine perturbé
$\mathbf{n}_\theta$	:	normale unitaire extérieure de $\Omega \setminus \overline{\omega_\theta}$
J_θ	$:=$	$\det(\mathbf{I} + \nabla \boldsymbol{\theta})$, jacobien associé à $x = (\mathbf{I} + \boldsymbol{\theta})y$
$(\mathbf{u}_\theta, p_\theta)$	:	solution de l'EDP considérée (Stokes ou Navier-Stokes) dans $\Omega \setminus \overline{\omega_\theta}$
$(\mathbf{u}_t, p_t)$	:	solution de l'EDP considérée (Stokes ou Navier-Stokes) dans $\Omega \setminus \overline{\omega_t}$
$b_\Omega(\mathbf{u}, \mathbf{v}, \mathbf{w})$	$:=$	$\sum_{i,j=1}^N \int_{\mathcal{D}} \mathbf{u}_i \frac{\partial v_j}{\partial x_i} \mathbf{w}_j$, forme trilinéaire définie sur $\mathbf{H}^1(\mathcal{D}) \times \mathbf{H}^1(\mathcal{D}) \times \mathbf{H}^1(\mathcal{D})$
$X^{k,m}(\Omega_1, \Omega_2)$	:	espace des fonctions u telles que $u \in H^k(\Omega_1)$ et $u _{\Omega_2} \in H^m(\Omega_2)$ (avec $\Omega_2 \subset \Omega_1$)

Chapitre II

J_D, J	:	fonctionnelles de forme de type moindres carrés
J_{KV}	:	fonctionnelle de forme de Kohn-Vogelius
m	:	fonction de $\mathcal{D}(\partial\Omega)$ telle que $\text{supp}(m) = \overline{O}$
δ	:	réel strictement positif
$\mathcal{O}_\delta$	$:=$	$\left\{ \omega \subset\subset \Omega \text{ de classe } C^{2,1} \text{ tel que } d(x, \partial\Omega) > \delta \ \forall x \in \omega \text{ et tel que } \Omega \setminus \overline{\omega} \text{ est connexe} \right\}$
Ω_δ	:	ouvert régulier tel que $\{x \in \Omega; d(x, \partial\Omega) > \delta/2\} \subset \Omega_\delta \subset \{x \in \Omega; d(x, \partial\Omega) > \delta/3\}$
$\mathbf{U}$	$:=$	$\{\boldsymbol{\theta} \in \mathbf{W}^{3,\infty}(\mathbb{R}^d); \text{supp } \boldsymbol{\theta} \subset \Omega_\delta\}$
ω	:	domaine de référence appartenant aux ensembles admissibles Ω_δ
$\mathbf{V}$	:	direction de perturbation appartenant à $\mathbf{U}$
V_n	$:=$	$\mathbf{V} \cdot \mathbf{n}$, composante normale de $\mathbf{V}$
ϕ	:	$t \in [0, T) \mapsto \mathbf{I} + t \mathbf{V} \in \mathbf{W}^{3,\infty}(\mathbb{R}^d)$
ω_t	$:=$	$(\mathbf{I} + t\mathbf{V})(\omega)$, domaine perturbé
$\mathbf{n}_t$	:	normale unitaire extérieure de $\Omega \setminus \overline{\omega_t}$
$(\mathbf{u}_t, p_t)$	:	solution de l'EDP considérée (Stokes ou Navier-Stokes) dans $\Omega \setminus \overline{\omega_t}$
$(\mathbf{u}', p')$	:	dérivée de forme de $(\mathbf{u}, p)$
$b_\Omega(\mathbf{u}, \mathbf{v}, \mathbf{w})$	$:=$	$\sum_{i,j=1}^N \int_{\mathcal{D}} \mathbf{u}_i \frac{\partial v_j}{\partial x_i} \mathbf{w}_j$, forme trilinéaire définie sur $\mathbf{H}^1(\mathcal{D}) \times \mathbf{H}^1(\mathcal{D}) \times \mathbf{H}^1(\mathcal{D})$
$\nabla_\Gamma, \text{div}_\Gamma, \Delta_\Gamma$	:	opérateurs différentiels tangentiels classiques
H	:	courbure moyenne de $\partial\omega$

Chapitre III

$\mathcal{J}_{KV}$	: fonctionnelle de forme de Kohn-Vogelius
$\omega_i(1 : m)$	: m premiers coefficients paramétrisant la forme ω_i
$(\mathbf{u}_D, p_D)$	: solution du système Stokes dans $\Omega \setminus \overline{\omega}$ avec des conditions de Dirichlet
$(\mathbf{u}_N, p_N)$	: solution du système Stokes dans $\Omega \setminus \overline{\omega}$ avec des conditions mixtes

Chapitre IV

z	: un point de Ω
ω	: ouvert borné connexe lipschitzien contenant l'origine
ε	: réel strictement positif (petit)
$\omega_{z,\varepsilon}$	$:= z + \varepsilon\omega$, objet situé en z de taille ε
$\Omega_{z,\varepsilon}$	$:= \Omega \setminus \overline{\omega_{z,\varepsilon}}$, domaine perforé
$\mathcal{J}_{KV}$	: fonctionnelle de forme de Kohn-Vogelius
$(E, \mathbf{P})$	: solution fondamentale de Stokes dans $\mathbb{R}^d$
$(\mathbf{u}_D^0, p_D^0)$	: solution du système Stokes dans Ω avec des conditions de Dirichlet
$(\mathbf{u}_N^0, p_N^0)$	: solution du système Stokes dans Ω avec des conditions mixtes
$(\mathbf{u}_D^\varepsilon, p_D^\varepsilon)$	: solution du système Stokes dans $\Omega_{z,\varepsilon}$ avec des conditions de Dirichlet
$(\mathbf{u}_N^\varepsilon, p_N^\varepsilon)$	: solution du système Stokes dans $\Omega_{z,\varepsilon}$ avec des conditions mixtes
Ω_R^z	$:= \Omega \setminus \overline{B(z, R)}$
D_ε^z	$:= B(z, R) \setminus \overline{\omega_{z,\varepsilon}}$

Chapitre V

$\mathcal{J}_{KV}$	: fonctionnelle de forme de Kohn-Vogelius
ω_ε	: obstacle(s) inclus dans Ω
$\delta \mathcal{J}_{KV}$	: gradient topologique de $\mathcal{J}_{KV}$
$(\mathbf{u}_D^0, p_D^0)$	: solution du système Stokes dans Ω avec des conditions de Dirichlet
$(\mathbf{u}_N^0, p_N^0)$	: solution du système Stokes dans Ω avec des conditions mixtes

Notations

Chapitre I

Existence des dérivées de forme

Dans ce chapitre, nous montrons l'existence des dérivées de forme pour les équations de Stokes et Navier-Stokes incompressibles stationnaires. Démontrer rigoureusement cette existence est évidemment la première étape avant de pouvoir dériver un problème (une EDP) par rapport à la forme, c'est-à-dire expliciter, en le justifiant, le problème satisfait par les dérivées de formes.

Pour ce faire, nous suivons ici la recette classique exposée dans le livre de Henrot et al. [72, Chapitre 5]. Nous considérons une géométrie de référence et une déformation de ce domaine dite géométrie déformée (ou perturbée). Le problème (par exemple une EDP) étudié sur ce domaine déformé est dit problème perturbé et sa solution est nommée solution perturbée. Nous caractérisons alors cette dernière comme solution d'un nouveau problème sur la géométrie de référence par changement de variables. Nous pouvons ainsi y appliquer un théorème des fonctions implicites pour obtenir la différentiabilité de la solution perturbée transportée sur le domaine de référence. Par composition, nous obtenons alors la Fréchet différentiabilité par rapport à la forme de la solution de l'EDP considérée sur le domaine de référence. Nous pouvons finalement obtenir un résultat de Gâteaux différentiabilité dans une certaine direction de perturbation du domaine.

Ce chapitre se décompose en deux parties. La première présente le cas des équations de Stokes, avec des conditions de Dirichlet puis des conditions de Neumann sur le bord intérieur $\partial\omega$. Les conditions de Neumann compliquent la preuve car nous ne pouvons pas utiliser directement le théorème des fonctions implicites classique. Nous utilisons alors une extension de ce théorème due à Simon (voir [92, Théorème 6]). Dans la deuxième partie, le cas du problème non linéaire de Navier-Stokes stationnaire avec des conditions de Dirichlet est traité (ainsi que le cas de Navier-Stokes linéarisé).

Des résultats de ce type ont été montrés par Simon dans [92] pour le cas de Stokes (avec des conditions de Dirichlet) et par Bello et al. dans [21, 22] pour le cas de Navier-Stokes stationnaire. Cependant, nous affinons ici ces résultats, en particulier concernant la régularité des domaines. Nous établissons ainsi un cadre mathématique général pour la démonstration de l'existence des dérivées de forme par rapport à un objet immergé dans un domaine plus grand. Ce cadre permet de démontrer rigoureusement l'idée intuitive que la régularité du bord du domaine extérieur n'a alors pas d'importance. Nous utilisons pour cela des espaces à poids permettant de tenir compte de la régularité locale des solutions au voisinage de l'inclusion. Ce cadre est également appliqué pour démontrer l'existence des dérivées de forme d'ordre deux et peut se généraliser aux dérivées d'ordre quelconque. De plus, nous démontrons cette existence dans le cas de conditions de Neumann sur l'objet en soulignant les difficultés entraînées par celles-ci : nous ne pouvons pas utiliser directement le théorème des fonctions implicites classique dans ce cas. Cependant, nous montrons que dans le cas des conditions de Dirichlet, le théorème des fonctions implicites classique peut être appliqué contrairement à ce que fait Simon dans [92]. Enfin, nous simplifions la preuve de l'existence des dérivées de forme pour les équations de Navier-Stokes. En effet, dans [21, 22], l'argument principal pour démontrer ce résultat est

que tout espace

$$Y_{\boldsymbol{\theta}} := \left\{ \boldsymbol{\phi} \in \mathbf{H}^1(\Omega \setminus \overline{\omega}), \int_{\Omega \setminus \overline{\omega}} |\det \frac{\partial}{\partial x_j} (\mathbf{I} + \boldsymbol{\theta})_i| \boldsymbol{\phi} = 0 \right\}$$

est isomorphe à Y_0 (où $\boldsymbol{\theta}$ est une déformation admissible). Ils travaillent alors avec un isomorphisme $\Lambda_{\boldsymbol{\theta}} : Y_{\boldsymbol{\theta}} \rightarrow Y_0$ afin de garantir la condition de compatibilité concernant la condition sur la divergence dans le problème de Navier-Stokes linéarisé (lorsqu'ils montrent que $D_{(\mathbf{w}, p)}(\mathbf{0}, (\mathbf{w}, p))(\mathbf{v}, q)$ est un isomorphisme afin d'appliquer le théorème des fonctions implicites). Ici, nous n'utilisons pas cet artifice. Nous soulignons simplement que la condition de compatibilité est automatiquement satisfaite si on l'impose dans les espaces avec lesquels on travaille (l'espace E_3 dans la preuve du Lemme I.2.3).

Nous nous intéressons ici à la dérivabilité d'ordre un et deux des solutions des problèmes de Stokes et Navier-Stokes stationnaires dans un ouvert borné lipschitzien connexe $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) par rapport à un domaine $\omega \subset \subset \Omega$. Afin de pouvoir montrer cette dérivabilité (d'ordre deux), nous introduisons un certain type de domaines perturbés appelés domaines admissibles. Pour $\delta > 0$ fixé (petit), on définit

$$\mathcal{O}_{\delta} := \left\{ \omega \subset \subset \Omega \text{ de classe } C^{2,1} \text{ tel que } d(x, \partial\Omega) > \delta \ \forall x \in \omega \text{ et tel que } \Omega \setminus \overline{\omega} \text{ est connexe} \right\}. \quad (\text{I.0.1})$$

Afin d'utiliser la régularité locale des solutions (des problèmes de Stokes et Navier-Stokes), nous définissons également un ouvert Ω_{δ} de classe C^{∞} tel que

$$\{x \in \Omega; d(x, \partial\Omega) > \delta/2\} \subset \Omega_{\delta} \subset \{x \in \Omega; d(x, \partial\Omega) > \delta/3\}$$

(voir Figure I.1). Ainsi, on note

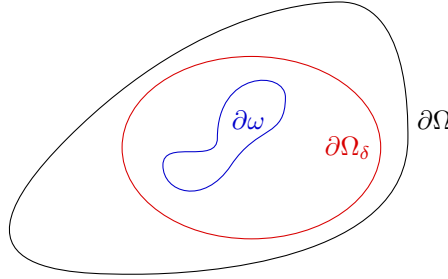


Figure I.1 – Le domaine Ω_{δ}

$$\mathcal{U} := \{\boldsymbol{\theta} \in \mathbf{W}^{3,\infty}(\mathbb{R}^d); \text{supp } \boldsymbol{\theta} \subset \Omega_{\delta}\} \quad \text{et} \quad \mathcal{U} := \left\{ \boldsymbol{\theta} \in \mathcal{U}; \|\boldsymbol{\theta}\|_{3,\infty} < \min\left(\frac{\delta}{3}, 1\right) \right\} \quad (\text{I.0.2})$$

l'espace des déformations admissibles. L'espace $\mathcal{U}$ permet de perturber seulement l'objet ω et non le domaine fixe Ω . En effet, pour tout $\boldsymbol{\theta} \in \mathcal{U}$, on a $\Omega = (\mathbf{I} + \boldsymbol{\theta})(\Omega)$. De plus, pour tout $\boldsymbol{\theta} \in \mathcal{U}$, on vérifie que $(\mathbf{I} + \boldsymbol{\theta})$ est un difféomorphisme d'après le Théorème de point fixe de Banach et, pour $\omega \in \mathcal{O}_{\delta}$, on définit le domaine perturbé

$$\omega_{\boldsymbol{\theta}} := (\mathbf{I} + \boldsymbol{\theta})(\omega)$$

qui est tel que $\Omega \setminus \overline{\omega_{\boldsymbol{\theta}}} \in \mathcal{O}_{\delta}$.

Remarque I.0.1. Les hypothèses de régularité présentées ici correspondent aux hypothèses utilisées pour démontrer la dérivabilité par rapport à la forme d'ordre deux. Cependant, si l'on s'intéresse seulement à la dérivabilité d'ordre un, nous pouvons travailler avec ω de classe $C^{1,1}$ et $\boldsymbol{\theta} \in \mathbf{W}^{2,\infty}(\mathbb{R}^d)$.

Afin d'être plus précis, nous rappelons ici l'idée de la méthode des vitesses introduite par Murat et Simon en 1976 dans [84] afin de définir les dérivées de forme. Pour $\mathbf{V} \in \mathcal{U}$ et $t \in [0, T)$ (où $T > 0$ est un réel fixé suffisamment petit), on définit la fonction

$$\phi : t \in [0, T) \mapsto \mathbf{I} + t\mathbf{V} \in \mathbf{W}^{3,\infty}(\mathbb{R}^d). \quad (\text{I.0.3})$$

Remarquons que, pour t petit, $\phi(t)$ est un difféomorphisme de $\mathbb{R}^d$. Pour $t \in [0, T)$, on définit alors

$$\omega_t := \phi(t)(\omega).$$

Dans notre situation (i.e. pour $(\mathbf{u}, p)$ solution des équations de Stokes ou Navier-Stokes), on définit alors les dérivées de forme par rapport à l'obstacle $\omega \subset \subset \Omega$ comme suit (voir [72] pour plus de détails) :

- si l'application $\boldsymbol{\theta} \in \mathcal{U} \mapsto (\mathbf{u}_{\boldsymbol{\theta}} \circ (\mathbf{I} + \boldsymbol{\theta}), p_{\boldsymbol{\theta}} \circ (\mathbf{I} + \boldsymbol{\theta})) \in \mathbf{H}^1(\Omega \setminus \bar{\omega}) \times L^2(\Omega \setminus \bar{\omega})$ est Fréchet différentiable en $\mathbf{0}$, on dit que $\boldsymbol{\theta} \mapsto (\mathbf{u}_{\boldsymbol{\theta}}, p_{\boldsymbol{\theta}})$ possède une variation première totale (ou dérivée première totale) en $\mathbf{0}$. Dans ce cas, cette dérivée première totale en $\mathbf{0}$ dans la direction $\boldsymbol{\theta}$ est notée $(\mathbf{u}'_{\boldsymbol{\theta}}, p'_{\boldsymbol{\theta}})$;
- si, pour tout $\mathcal{D} \subset \subset \Omega \setminus \bar{\omega}$, l'application $\boldsymbol{\theta} \in \mathcal{U} \mapsto (\mathbf{u}_{\boldsymbol{\theta}|_{\mathcal{D}}}, p_{\boldsymbol{\theta}|_{\mathcal{D}}}) \in \mathbf{H}^1(\mathcal{D}) \times L^2(\mathcal{D})$ est Fréchet différentiable en $\mathbf{0}$, on dit que $\boldsymbol{\theta} \mapsto (\mathbf{u}_{\boldsymbol{\theta}}, p_{\boldsymbol{\theta}})$ possède une variation première locale (ou dérivée première locale) en $\mathbf{0}$. Dans ce cas, cette dérivée première locale en $\mathbf{0}$ dans la direction $\mathbf{V} \in \mathcal{U}$ est notée $(\mathbf{u}'_{\mathbf{V}}, p'_{\mathbf{V}})$ et est bien définie dans le domaine entier $\Omega \setminus \bar{\omega}$:

$$(\mathbf{u}'_{\mathbf{V}}, p'_{\mathbf{V}}) = \frac{d}{dt}(\mathbf{u}_{t\mathbf{V}|_{\mathcal{D}}}, p_{t\mathbf{V}|_{\mathcal{D}}})|_{t=0} \quad \text{dans tout } \mathcal{D} \subset \subset \Omega \setminus \bar{\omega}.$$

On définit de manière similaire les dérivées de forme d'ordres supérieurs.

Pour finir cette introduction, nous définissons, pour $(\mathbf{u}, p) \in \mathbf{H}^1 \times L^2$ et $\nu > 0$ représentant la vitesse cinématique du fluide considéré, le tenseur des contraintes suivant :

$$\sigma(\mathbf{u}, p) := \nu(\nabla \mathbf{u} + {}^t\nabla \mathbf{u}) - p\mathbf{I},$$

où ${}^t\nabla \mathbf{u}$ représente la matrice transposée de $\nabla \mathbf{u}$. Enfin, nous définissons de nouveaux espaces fonctionnels. Pour $k \in \mathbb{Z}$, $k \geq -1$ et $m \in \mathbb{N}$ avec $k < m$, on note $X^{k,m}(\Omega \setminus \bar{\omega}, \Omega_{\delta} \setminus \bar{\omega})$ l'espace des fonctions de $\mathbf{H}^k(\Omega \setminus \bar{\omega})$ telles que leur restriction à $\Omega_{\delta} \setminus \bar{\omega}$ appartienne à $\mathbf{H}^m(\Omega_{\delta} \setminus \bar{\omega})$. Cet espace muni de la norme

$$\|u\|_{X^{k,m}(\Omega \setminus \bar{\omega}, \Omega_{\delta} \setminus \bar{\omega})} := \left(\|u\|_{\mathbf{H}^k(\Omega \setminus \bar{\omega})}^2 + \|u\|_{\mathbf{H}^m(\Omega_{\delta} \setminus \bar{\omega})}^2 \right)^{1/2}$$

est hilbertien. On définit de manière similaire l'espace $X^{*,m}(\Omega \setminus \bar{\omega}, \Omega_{\delta} \setminus \bar{\omega})$ correspondant aux fonctions de $[\mathbf{H}^1(\Omega \setminus \bar{\omega})]'$ telles que leur restriction à $\Omega_{\delta} \setminus \bar{\omega}$ appartienne à $\mathbf{H}^m(\Omega_{\delta} \setminus \bar{\omega})$.

I.1 The Stokes case

I.1.1 The Dirichlet case

Let $\omega \in \mathcal{O}_{\delta}$, where $\mathcal{O}_{\delta}$ is defined by (I.0.1) and O be a non-empty open subset of $\partial\Omega$. Let us consider $\mathbf{g} \in \mathbf{H}^{1/2}(\partial\Omega)$ such that $\mathbf{g} \neq \mathbf{0}$ and satisfying the following compatibility condition:

$$\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n} = 0. \quad (\text{I.1.1})$$

Let $(\mathbf{u}, p) \in \mathbf{H}^1(\Omega \setminus \bar{\omega}) \times L^2(\Omega \setminus \bar{\omega})$ be the solution of the following Stokes problem

$$\begin{cases} -\nu \Delta \mathbf{u} + \nabla p = \mathbf{0} & \text{in } \Omega \setminus \bar{\omega} \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega \setminus \bar{\omega} \\ \mathbf{u} = \mathbf{g} & \text{on } \partial\Omega \\ \mathbf{u} = \mathbf{0} & \text{on } \partial\omega \end{cases}$$

such that $\langle m^2(\sigma(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{n}, 1 \rangle_O = c_n$, where $c_n \in \mathbb{R}$ is a normalization constant and where the function $m \in C_c^\infty(\partial\Omega)$ is such that $\text{supp}(m) = \overline{O}$. Obviously, we can consider another normalization condition but in order to be close to the sequel, we choose this one because it will be imposed in Chapter II.

The results of existence, uniqueness and regularity of the solution of the Stokes system with Dirichlet boundary conditions are classical and we do not recall it here. We refer for example to [13, 31, 44, 56, 61, 97]. In particular, a local regularity result can be found in [56, Theorem IV.5.1].

I.1.1.1 The results

Let $\boldsymbol{\theta} \in \mathcal{U}$, where $\mathcal{U}$ is defined by (I.0.2). We set $(\mathbf{u}_\theta, p_\theta) \in \mathbf{H}^1(\Omega \setminus \overline{\omega_\theta}) \times L^2(\Omega \setminus \overline{\omega_\theta})$ the unique solution of

$$\begin{cases} -\nu \Delta \mathbf{u}_\theta + \nabla p_\theta &= \mathbf{0} & \text{in } \Omega \setminus \overline{\omega_\theta} \\ \operatorname{div} \mathbf{u}_\theta &= 0 & \text{in } \Omega \setminus \overline{\omega_\theta} \\ \mathbf{u}_\theta &= \mathbf{g} & \text{on } \partial\Omega \\ \mathbf{u}_\theta &= \mathbf{0} & \text{on } \partial\omega_\theta \end{cases}$$

such that $\langle (\sigma(\mathbf{u}_\theta, p_\theta)\mathbf{n}) \cdot \mathbf{n}, 1 \rangle_O = c_n$. According to [14, Lemma 3.3], let us consider $\mathbf{G} \in \mathbf{H}^1(\Omega)$ such that

$$\mathbf{G} = \mathbf{g} \text{ on } \partial\Omega, \quad \operatorname{div} \mathbf{G} = 0 \text{ in } \Omega \quad \text{and} \quad \mathbf{G} = \mathbf{0} \text{ in } \Omega_\delta.$$

Thus $(\mathbf{z}_\theta := \mathbf{u}_\theta - \mathbf{G}, p_\theta) \in \mathbf{H}_0^1(\Omega \setminus \overline{\omega_\theta}) \times L^2(\Omega \setminus \overline{\omega_\theta})$ is such that

$$\begin{cases} \int_{\Omega \setminus \overline{\omega_\theta}} \sigma(\mathbf{z}_\theta, p_\theta) : \nabla \boldsymbol{\varphi}_\theta &= - \int_{\Omega \setminus \overline{\omega_\theta}} \nu \nabla \mathbf{G} : \nabla \boldsymbol{\varphi}_\theta & \forall \boldsymbol{\varphi}_\theta \in \mathbf{H}_0^1(\Omega \setminus \overline{\omega_\theta}) \\ \int_{\Omega \setminus \overline{\omega_\theta}} \xi_\theta \operatorname{div} \mathbf{z}_\theta &= 0 & \forall \xi_\theta \in L^2(\Omega \setminus \overline{\omega_\theta}) \\ \langle m^2(\sigma(\mathbf{z}_\theta, p_\theta)\mathbf{n}) \cdot \mathbf{n}, 1 \rangle_O &= c_n - \langle m^2(\sigma(\mathbf{G}, 0)\mathbf{n}) \cdot \mathbf{n}, 1 \rangle_O. \end{cases} \quad (\text{I.1.2})$$

Let us define the key objects of our differentiability proof:

$$\mathbf{v}_\theta := \mathbf{z}_\theta \circ (\mathbf{I} + \boldsymbol{\theta}) \in \mathbf{H}_0^1(\Omega \setminus \overline{\omega}) \quad \text{and} \quad q_\theta := p_\theta \circ (\mathbf{I} + \boldsymbol{\theta}) \in L^2(\Omega \setminus \overline{\omega}).$$

First order shape differentiability. To prove that $(\mathbf{u}, p)$ is differentiable with respect to the shape, we need the following three lemmas:

Lemma I.1.1 (Characterization of $(\mathbf{v}_\theta, q_\theta)$). *For $\boldsymbol{\theta} \in \mathcal{U}$, the pair $(\mathbf{v}_\theta, q_\theta)$ satisfies for all test functions $\boldsymbol{\varphi} \in \mathbf{H}_0^1(\Omega \setminus \overline{\omega})$ and $\xi \in L^2(\Omega \setminus \overline{\omega})$*

$$\begin{cases} \int_{\Omega \setminus \overline{\omega}} [(\nu \nabla \mathbf{v}_\theta A(\boldsymbol{\theta})) : \nabla \boldsymbol{\varphi} - q_\theta B(\boldsymbol{\theta}) : \nabla \boldsymbol{\varphi}] &= \int_{\Omega \setminus \overline{\omega}} -\nu \nabla \mathbf{G} : \nabla \boldsymbol{\varphi} \\ \int_{\Omega \setminus \overline{\omega}} (\nabla \mathbf{v}_\theta : B(\boldsymbol{\theta})) \xi &= 0 \\ \langle m^2(\sigma(\mathbf{v}_\theta, q_\theta)\mathbf{n}) \cdot \mathbf{n}, 1 \rangle_O &= c_n - \langle m^2(\sigma(\mathbf{G}, 0)\mathbf{n}) \cdot \mathbf{n}, 1 \rangle_O, \end{cases} \quad (\text{I.1.3})$$

with

$$\begin{aligned} J_\theta &:= \det(\mathbf{I} + \nabla \boldsymbol{\theta}) \in W^{2,\infty}(\overline{\Omega_\delta}), \\ A(\boldsymbol{\theta}) &:= J_\theta (\mathbf{I} + \nabla \boldsymbol{\theta})^{-1} (\mathbf{I} + {}^t \nabla \boldsymbol{\theta})^{-1} \in W^{2,\infty}(\overline{\Omega_\delta}, \mathcal{M}_{d,d}), \\ B(\boldsymbol{\theta}) &:= J_\theta (\mathbf{I} + {}^t \nabla \boldsymbol{\theta})^{-1} \in W^{2,\infty}(\overline{\Omega_\delta}, \mathcal{M}_{d,d}). \end{aligned}$$

Lemma I.1.2 (Differentiability of $\boldsymbol{\theta} \mapsto (\mathbf{v}_\theta, q_\theta)$). *The function*

$$\boldsymbol{\theta} \in \mathcal{U} \mapsto (\mathbf{v}_\theta, q_\theta) \in \mathbf{X}^{1,2}(\Omega \setminus \overline{\omega}, \Omega_\delta \setminus \overline{\omega}) \times X^{0,1}(\Omega \setminus \overline{\omega}, \Omega_\delta \setminus \overline{\omega})$$

is differentiable in a neighborhood of $\mathbf{0}$ (and even C^∞).

Remark I.1.3. *This lemma implies that there exists a constant $c > 0$ such that*

$$\|\mathbf{u}_\theta \circ (\mathbf{I} + \theta) - \mathbf{u}\|_{\mathbf{X}^{1,2}(\Omega \setminus \bar{\omega}, \Omega_\delta \setminus \bar{\omega})} + \|p_\theta \circ (\mathbf{I} + \theta) - p\|_{X^{0,1}(\Omega \setminus \bar{\omega}, \Omega_\delta \setminus \bar{\omega})} \leq c \|\theta\|_{\mathcal{U}}.$$

Lemma I.1.4 (Differentiability of $\theta \mapsto (\mathbf{u}_\theta, p_\theta)$). *There exists $\tilde{\mathbf{u}}_\theta, \tilde{p}_\theta$ some respective extensions in Ω of $\mathbf{u}_\theta \in \mathbf{H}^1(\Omega \setminus \bar{\omega}_\theta)$, $p_\theta \in L^2(\Omega \setminus \bar{\omega}_\theta)$ such that the functions*

$$\theta \in \mathcal{U} \mapsto \tilde{\mathbf{u}}_\theta \in \mathbf{H}^1(\Omega) \quad \text{and} \quad \theta \in \mathcal{U} \mapsto \tilde{p}_\theta \in L^2(\Omega)$$

are differentiable at $\mathbf{0}$ (and even C^1 in a neighborhood of $\mathbf{0}$).

Remark I.1.5. *We will prove these three lemmas under weaker assumptions: ω with a $C^{1,1}$ boundary and $\theta \in \mathbf{W}^{2,\infty}(\mathbb{R}^d)$.*

Higher order differentiability. Obvious adaptation of the proof of the first order differentiability (see Lemma I.1.2) with

$$\begin{aligned} \mathbf{E}_1 &:= \{(\mathbf{v}, q) \in \mathbf{H}_0^1(\Omega \setminus \bar{\omega}) \times L^2(\Omega \setminus \bar{\omega}); (\Phi \mathbf{v}, \Phi q) \in \mathbf{H}^3(\Omega \setminus \bar{\omega}) \times \mathbf{H}^2(\Omega \setminus \bar{\omega})\}, \\ \mathbf{E}_2 &:= \{\mathbf{f} \in \mathbf{H}^{-1}(\Omega \setminus \bar{\omega}); \Phi \mathbf{f} \in \mathbf{H}^1(\Omega \setminus \bar{\omega})\}, \\ \mathbf{E}_3 &:= \left\{ \psi \in L^2(\Omega \setminus \bar{\omega}), \Phi \psi \in \mathbf{H}^2(\Omega \setminus \bar{\omega}), \int_{\Omega \setminus \bar{\omega}} \psi = 0 \right\}, \end{aligned}$$

permits to prove the following lemma which is used to check that $(\mathbf{u}, p)$ is twice differentiable with respect to the shape:

Lemma I.1.6. *The function*

$$\theta \in \mathcal{U} \mapsto (\mathbf{v}_\theta, q_\theta) \in \mathbf{X}^{1,3}(\Omega \setminus \bar{\omega}, \Omega_\delta \setminus \bar{\omega}) \times X^{0,2}(\Omega \setminus \bar{\omega}, \Omega_\delta \setminus \bar{\omega})$$

is twice differentiable in a neighborhood of $\mathbf{0}$ (and even C^∞).

Remark I.1.7. *This lemma implies that there exists a constant $c > 0$ such that*

$$\|\mathbf{u}_\theta \circ (\mathbf{I} + \theta) - \mathbf{u}\|_{\mathbf{X}^{1,3}(\Omega \setminus \bar{\omega}, \Omega_\delta \setminus \bar{\omega})} + \|p_\theta \circ (\mathbf{I} + \theta) - p\|_{X^{0,2}(\Omega \setminus \bar{\omega}, \Omega_\delta \setminus \bar{\omega})} \leq c \|\theta\|_{\mathcal{U}}.$$

Finally, we prove that $(\mathbf{u}, p)$ is twice differentiable with respect to the domain:

Lemma I.1.8 (Second order shape differentiability). *The solution $(\mathbf{u}, p)$ is twice differentiable with respect to the domain.*

I.1.1.2 The proofs

Proof of Lemma I.1.1: characterization of $(\mathbf{v}_\theta, q_\theta)$. The main idea is to make the change of variables $x = (\mathbf{I} + \theta)y$ in (I.1.2). First, notice that, since $\operatorname{div} \mathbf{z}_\theta = 0$ in $\Omega \setminus \bar{\omega}_\theta$,

$$\int_{\Omega \setminus \bar{\omega}_\theta} \sigma(\mathbf{z}_\theta, p_\theta) : \nabla \varphi_\theta = \int_{\Omega \setminus \bar{\omega}_\theta} (\nu \nabla \mathbf{z}_\theta : \nabla \varphi_\theta - p_\theta \operatorname{div} \varphi_\theta), \quad \forall \varphi_\theta \in \mathbf{H}_0^1(\Omega \setminus \bar{\omega}_\theta).$$

Let $\varphi \in \mathbf{H}_0^1(\Omega \setminus \bar{\omega})$, $\xi \in L^2(\Omega \setminus \bar{\omega})$ and $\theta \in \mathcal{U}$. Using

$$\varphi_\theta := \varphi \circ (\mathbf{I} + \theta)^{-1} \in \mathbf{H}_0^1(\Omega \setminus \bar{\omega}_\theta) \quad \text{and} \quad \xi_\theta := \xi \circ (\mathbf{I} + \theta)^{-1} \in L^2(\Omega \setminus \bar{\omega}_\theta)$$

as test functions in the variational formulation (I.1.2), we have

$$\left\{ \begin{array}{l} \int_{\Omega \setminus \overline{\omega_\theta}} \left\{ \nu [(\nabla \mathbf{v}_\theta (\mathbf{I} + \nabla \boldsymbol{\theta})^{-1}) \circ (\mathbf{I} + \boldsymbol{\theta})^{-1}] : [(\nabla \boldsymbol{\varphi} (\mathbf{I} + \nabla \boldsymbol{\theta})^{-1}) \circ (\mathbf{I} + \boldsymbol{\theta})^{-1}] \right. \\ \quad \left. - (q_\theta \circ (\mathbf{I} + \boldsymbol{\theta})^{-1}) \operatorname{div} (\boldsymbol{\varphi} \circ (\mathbf{I} + \boldsymbol{\theta})^{-1}) \right\} = - \int_{\Omega \setminus \overline{\omega_\theta}} \nu \nabla \mathbf{G} : \nabla \boldsymbol{\varphi} \\ \int_{\Omega \setminus \overline{\omega_\theta}} [\xi \circ (\mathbf{I} + \boldsymbol{\theta})^{-1}] \operatorname{div} (\mathbf{v}_\theta \circ (\mathbf{I} + \boldsymbol{\theta})^{-1}) = 0 \\ \langle m^2 (\sigma(\mathbf{v}_\theta, q_\theta) \mathbf{n}) \cdot \mathbf{n}, 1 \rangle_O = c_n - \langle m^2 (\sigma(\mathbf{G}, 0) \mathbf{n}) \cdot \mathbf{n}, 1 \rangle_O, \end{array} \right.$$

noticing that $\boldsymbol{\theta} \equiv \mathbf{0}$ in $\Omega \setminus \overline{\Omega_\delta}$ (and therefore on $O \subset \partial\Omega$) and that $\mathbf{G} \equiv \mathbf{0}$ in Ω_δ . However, we notice that $\operatorname{div} (\boldsymbol{\varphi} \circ (\mathbf{I} + \boldsymbol{\theta})^{-1}) = (\nabla \boldsymbol{\varphi} \circ (\mathbf{I} + \boldsymbol{\theta})^{-1}) : ((\mathbf{I} + {}^t \nabla \boldsymbol{\theta})^{-1} \circ (\mathbf{I} + \boldsymbol{\theta})^{-1})$, and a similar equality holds with $\mathbf{v}_\theta$ instead of $\boldsymbol{\varphi}$. Thus, the conclusion follows after the change of variables $x = (\mathbf{I} + \boldsymbol{\theta})y$. $\square$

Proof of Lemma I.1.2: differentiability of $\boldsymbol{\theta} \mapsto (\mathbf{v}_\theta, q_\theta)$. In this proof, we use the classical implicit functions theorem.

Step 1: notations. We need some additional tools: a fourth domain $\tilde{\Omega}_\delta$ which is an open set with a C^∞ boundary such that $\Omega_\delta \subset \subset \tilde{\Omega}_\delta \subset \subset \Omega$ and a truncation function $\Phi \in C_c^\infty(\tilde{\Omega}_\delta)$ such that $\Phi \equiv 1$ in Ω_δ . We define the spaces:

$$\begin{aligned} \mathbf{E}_1 &:= \{(\mathbf{v}, q) \in \mathbf{H}_0^1(\Omega \setminus \overline{\omega}) \times L^2(\Omega \setminus \overline{\omega}); (\Phi \mathbf{v}, \Phi q) \in \mathbf{H}^2(\Omega \setminus \overline{\omega}) \times H^1(\Omega \setminus \overline{\omega})\}, \\ \mathbf{E}_2 &:= \{\mathbf{f} \in \mathbf{H}^{-1}(\Omega \setminus \overline{\omega}); \Phi \mathbf{f} \in \mathbf{L}^2(\Omega \setminus \overline{\omega})\}, \\ E_3 &:= \left\{ \psi \in L^2(\Omega \setminus \overline{\omega}); \Phi \psi \in H^1(\Omega \setminus \overline{\omega}), \int_{\Omega \setminus \overline{\omega}} \psi = 0 \right\}. \end{aligned}$$

Note that $\mathbf{E}_1$, $\mathbf{E}_2$ and E_3 are Hilbert spaces with respective norms

$$\begin{aligned} \|(\mathbf{v}, q)\|_{\mathbf{E}_1}^2 &:= \|\mathbf{v}\|_{\mathbf{H}^1(\Omega \setminus \overline{\omega})}^2 + \|q\|_{L^2(\Omega \setminus \overline{\omega})}^2 + \|\Phi \mathbf{v}\|_{\mathbf{H}^2(\Omega \setminus \overline{\omega})}^2 + \|\Phi q\|_{H^1(\Omega \setminus \overline{\omega})}^2, \\ \|\mathbf{f}\|_{\mathbf{E}_2}^2 &:= \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega \setminus \overline{\omega})}^2 + \|\Phi \mathbf{f}\|_{\mathbf{L}^2(\Omega \setminus \overline{\omega})}^2, \\ \|\psi\|_{E_3}^2 &:= \|\psi\|_{L^2(\Omega \setminus \overline{\omega})}^2 + \|\Phi \psi\|_{H^1(\Omega \setminus \overline{\omega})}^2. \end{aligned}$$

Moreover, the following continuous embeddings hold: $\mathbf{E}_1 \hookrightarrow \mathbf{X}^{1,2}(\Omega \setminus \overline{\omega}, \Omega_\delta \setminus \overline{\omega}) \times X^{0,1}(\Omega \setminus \overline{\omega}, \Omega_\delta \setminus \overline{\omega})$, $\mathbf{E}_2 \hookrightarrow \mathbf{X}^{-1,0}(\Omega \setminus \overline{\omega}, \Omega_\delta \setminus \overline{\omega})$ and $E_3 \hookrightarrow X^{0,1}(\Omega \setminus \overline{\omega}, \Omega_\delta \setminus \overline{\omega})$. Using the notations introduced in Lemma I.1.1, we also define, for $\boldsymbol{\theta} \in \mathcal{U}$ and $(\mathbf{v}, q) \in \mathbf{E}_1$, the following functions:

– $\mathbf{f}_1(\boldsymbol{\theta}) \in \mathbf{H}^{-1}(\Omega \setminus \overline{\omega})$ by $\forall \boldsymbol{\varphi} \in \mathbf{H}_0^1(\Omega \setminus \overline{\omega})$,

$$\langle \mathbf{f}_1(\boldsymbol{\theta}), \boldsymbol{\varphi} \rangle_{\Omega \setminus \overline{\omega}} = - \int_{\Omega \setminus \overline{\omega}} \nu \nabla \mathbf{G} : \nabla \boldsymbol{\varphi} = - \int_{\Omega \setminus \overline{\Omega_\delta}} \nu \nabla \mathbf{G} : \nabla \boldsymbol{\varphi},$$

– $\mathbf{F}_1(\boldsymbol{\theta}, (\mathbf{v}, q)) \in \mathbf{H}^{-1}(\Omega \setminus \overline{\omega})$ by $\forall \boldsymbol{\varphi} \in \mathbf{H}_0^1(\Omega \setminus \overline{\omega})$,

$$\langle \mathbf{F}_1(\boldsymbol{\theta}, (\mathbf{v}, q)), \boldsymbol{\varphi} \rangle_{\Omega \setminus \overline{\omega}} = \int_{\Omega \setminus \overline{\omega}} \{[\nu \nabla \mathbf{v} A(\boldsymbol{\theta})] : \nabla \boldsymbol{\varphi} - q B(\boldsymbol{\theta}) : \nabla \boldsymbol{\varphi}\},$$

– $\mathbf{F} : \mathcal{U} \times \mathbf{E}_1 \rightarrow \mathbf{E}_2 \times E_3 \times \mathbb{R}$ by

$$\mathbf{F}(\boldsymbol{\theta}, (\mathbf{v}, q)) = \left(\mathbf{F}_1(\boldsymbol{\theta}, (\mathbf{v}, q)) - \mathbf{f}_1(\boldsymbol{\theta}), \nabla \mathbf{v} : B(\boldsymbol{\theta}), \langle m^2 (\sigma(\mathbf{v}, q) \mathbf{n} - c_n \mathbf{n} + \sigma(\mathbf{G}, 0) \mathbf{n}), \mathbf{n} \rangle_O \right).$$

Note that $\mathbf{F}$ is well-defined. In particular $\nabla \mathbf{v} : B(\boldsymbol{\theta})$ belongs to E_3 because, using the change of variables $x = (\mathbf{I} + \boldsymbol{\theta})^{-1}y$, we check that

$$\int_{\Omega \setminus \overline{\omega}} \nabla \mathbf{v} : B(\boldsymbol{\theta}) = \int_{\Omega \setminus \overline{\omega_\theta}} \operatorname{div} (\mathbf{v} \circ (\mathbf{I} + \boldsymbol{\theta})^{-1}) = 0,$$

since $\mathbf{v} \in \mathbf{H}_0^1(\Omega \setminus \bar{\omega})$ (and then $\mathbf{v} \circ (\mathbf{I} + \boldsymbol{\theta})^{-1} \in \mathbf{H}_0^1(\Omega \setminus \bar{\omega}_\theta)$).

Step 2: for $\boldsymbol{\theta} \equiv \mathbf{0}$. We have $\mathbf{F}(\mathbf{0}, (\mathbf{v}_0, q_0)) = \mathbf{F}(\mathbf{0}, (\mathbf{z}_0, p)) = (\mathbf{0}, 0, 0)$.

Step 3: differentiability of $\mathbf{F}$. We use the chain rule. Since the functions

$$\boldsymbol{\theta} \in \mathcal{U} \mapsto J_\theta \in W^{1,\infty}(\bar{\Omega}) \quad \text{and} \quad \boldsymbol{\theta} \in \mathcal{U} \mapsto (\mathbf{I} + \nabla \boldsymbol{\theta})^{-1} \in W^{1,\infty}(\bar{\Omega}, \mathcal{M}_{d \times d})$$

are C^∞ , the functions

$$\boldsymbol{\theta} \in \mathcal{U} \mapsto A(\boldsymbol{\theta}) \in W^{1,\infty}(\bar{\Omega}, \mathcal{M}_{d \times d}) \quad \text{and} \quad \boldsymbol{\theta} \in \mathcal{U} \mapsto B(\boldsymbol{\theta}) \in W^{1,\infty}(\bar{\Omega}, \mathcal{M}_{d \times d})$$

are C^∞ . Moreover, the following functions are C^∞ :

$$\begin{aligned} L^\infty(\bar{\Omega}, \mathcal{M}_{d \times d}) \times \{\mathbf{v} \in \mathbf{H}^1(\Omega \setminus \bar{\omega}); \Phi \mathbf{v} \in \mathbf{H}^2(\Omega \setminus \bar{\omega})\} &\rightarrow E_3 \\ (B, \mathbf{v}) &\mapsto \nabla \mathbf{v} : B, \end{aligned}$$

$$\mathbf{F}_1 : (\boldsymbol{\theta}, (\mathbf{v}, q)) \in \mathcal{U} \times \mathbf{E}_1 \mapsto \mathbf{F}_1(\boldsymbol{\theta}, (\mathbf{v}, q)) \in \mathbf{E}_2.$$

Finally, $\mathbf{f}_1 : \mathcal{U} \times \mathbf{E}_1 \rightarrow \mathbf{E}_2$ and $(\mathbf{v}, q) \in \mathbf{E}_1 \mapsto \langle m^2(\sigma(\mathbf{v}, q)\mathbf{n} - c_n\mathbf{n} + \sigma(\mathbf{G}, 0)\mathbf{n}), \mathbf{n} \rangle_O \in \mathbb{R}$ are clearly C^∞ . By composition, we conclude that $\mathbf{F}$ is C^∞ .

Step 4: $D_{(\mathbf{z}_0, p)}\mathbf{F}(\mathbf{0}, (\mathbf{z}_0, p))$ is an isomorphism. We compute for all $(\mathbf{v}, q) \in \mathbf{E}_1$

$$D_{(\mathbf{z}_0, p)}\mathbf{F}(\mathbf{0}, (\mathbf{z}_0, p))(\mathbf{v}, q) = \left(-\nu \Delta \mathbf{v} + \nabla q, \operatorname{div} \mathbf{v}, \langle m^2(\sigma(\mathbf{v}, q)\mathbf{n}), \mathbf{n} \rangle_O \right).$$

Let $(\chi, \eta, r) \in \mathbf{E}_2 \times E_3 \times \mathbb{R}$. Let us prove there exists a unique pair $(\mathbf{v}, q) \in \mathbf{E}_1$ such that

$$\begin{cases} -\nu \Delta \mathbf{v} + \nabla q &= \chi & \text{in } \Omega \setminus \bar{\omega} \\ \operatorname{div} \mathbf{v} &= \eta & \text{in } \Omega \setminus \bar{\omega} \\ \mathbf{v} &= \mathbf{0} & \text{on } \partial\Omega \\ \mathbf{v} &= \mathbf{0} & \text{on } \partial\omega \\ \langle m^2(\nu \partial_n \mathbf{v} - q \mathbf{n}), \mathbf{n} \rangle_O &= r. \end{cases} \quad (\text{I.1.4})$$

Note that the compatibility condition $\int_{\Omega \setminus \bar{\omega}} \eta = 0$ is satisfied because $\eta \in E_3$. Then there exists a unique solution $(\mathbf{v}, q) \in \mathbf{H}_0^1(\Omega \setminus \bar{\omega}) \times L_0^2(\Omega \setminus \bar{\omega})$ of Problem (I.1.4) and using a local regularity argument, the pair $(\mathbf{v}, q)$ belongs to $\mathbf{H}^2(\Omega_\delta \setminus \bar{\omega}) \times H^1(\Omega_\delta \setminus \bar{\omega})$. More precisely, we obtain

$$(\Phi \mathbf{v}, \Phi q) \in \mathbf{H}^2(\Omega \setminus \bar{\omega}) \times H^1(\Omega \setminus \bar{\omega}),$$

by multiplying (I.1.4) by Φ (see Remark A.2.10).

Step 5: conclusion. The implicit function theorem applies and we know that there exists a C^∞ function $\boldsymbol{\theta} \in \mathcal{U} \mapsto (\mathbf{v}(\boldsymbol{\theta}), q(\boldsymbol{\theta})) \in \mathbf{E}_1$ such that $\mathbf{F}(\boldsymbol{\theta}, (\mathbf{v}(\boldsymbol{\theta}), q(\boldsymbol{\theta}))) = (\mathbf{0}, 0, 0)$ in a neighborhood of $\mathbf{0}$. Using the uniqueness of the solution of such a problem, we conclude using the fact that $\mathbf{E}_1$ is continuously imbedded in $\mathbf{X}^{1,2}(\Omega \setminus \bar{\omega}, \Omega_\delta \setminus \bar{\omega}) \times X^{0,1}(\Omega \setminus \bar{\omega}, \Omega_\delta \setminus \bar{\omega})$. $\square$

Proof of Lemma I.1.4: differentiability of $\boldsymbol{\theta} \mapsto (\mathbf{u}_\theta, p_\theta)$. We use the chain rule. We only prove the result on $\tilde{\mathbf{u}}_\theta$ because the ideas are exactly the same for the result on $\tilde{p}_\theta$.

For $\boldsymbol{\theta} \in \mathcal{U}$, $\mathbf{z}_\theta = \mathbf{v}_\theta \circ (\mathbf{I} + \boldsymbol{\theta})^{-1} \in \mathbf{H}^2(\Omega_\delta \setminus \bar{\omega}_\theta)$. According to Lemma I.1.2 and Stein's extension Theorem (see [2, Theorem 5.24] for example), there exists $\tilde{\mathbf{v}}_\theta$, an extension of $\mathbf{v}_\theta$, such that the mapping $\boldsymbol{\theta} \in \mathcal{U} \mapsto \tilde{\mathbf{v}}_\theta \in \mathbf{H}^2(\Omega_\delta)$ is differentiable at $\mathbf{0}$. Moreover,

$$\boldsymbol{\theta} \in \mathcal{U} \mapsto (\mathbf{I} + \boldsymbol{\theta})^{-1} - \mathbf{I} \in \mathbf{W}^{2,\infty}(\bar{\Omega})$$

is differentiable at $\mathbf{0}$. Thus,

$$\varphi_1 : \boldsymbol{\theta} \in \mathcal{U} \mapsto (\tilde{\mathbf{v}}_{\boldsymbol{\theta}}, (\mathbf{I} + \boldsymbol{\theta})^{-1} - \mathbf{I}) \in \mathbf{H}^2(\Omega_{\delta}) \times \mathbf{W}^{2,\infty}(\overline{\Omega})$$

is differentiable at $\mathbf{0}$. We apply Lemma 5.3.9 in [72], to get that

$$\varphi_2 : (\mathbf{g}, \boldsymbol{\mu}) \in \mathbf{H}^2(\mathbb{R}^d) \times \mathbf{W}^{2,\infty}(\overline{\Omega}) \mapsto \mathbf{g} \circ (\mathbf{I} + \boldsymbol{\mu}) \in \mathbf{H}^1(\mathbb{R}^d)$$

is C^1 in a neighborhood of $\mathbf{0}$. By composition, $\varphi_2 \circ \varphi_1$ is differentiable at $\mathbf{0}$. Then, we define an extension of $\mathbf{z}_{\boldsymbol{\theta}}$ as

$$\widetilde{\mathbf{z}}_{\boldsymbol{\theta}} := \begin{cases} \varphi_2 \circ \varphi_1(\boldsymbol{\theta}) & \text{in } \Omega_{\delta}, \\ \mathbf{z}_{\boldsymbol{\theta}} & \text{in } \Omega \setminus \overline{\Omega_{\delta}}. \end{cases}$$

Using Lemma I.1.2, the function

$$\boldsymbol{\theta} \in \mathcal{U} \mapsto \mathbf{z}_{\boldsymbol{\theta}}|_{\Omega \setminus \overline{\Omega_{\delta}}} = \mathbf{v}_{\boldsymbol{\theta}} \circ (\mathbf{I} + \boldsymbol{\theta})^{-1}|_{\Omega \setminus \overline{\Omega_{\delta}}} = \mathbf{v}_{\boldsymbol{\theta}}|_{\Omega \setminus \overline{\Omega_{\delta}}} \in \mathbf{H}^1(\Omega \setminus \overline{\Omega_{\delta}})$$

is differentiable at $\mathbf{0}$. Moreover, on a neighborhood $\mathcal{V}(\partial\Omega_{\delta})$ of $\partial\Omega_{\delta}$, $\widetilde{\mathbf{z}}_{\boldsymbol{\theta}} = \mathbf{z}_{\boldsymbol{\theta}} = \mathbf{v}_{\boldsymbol{\theta}}$. Thus, using Lemma I.1.2, $\boldsymbol{\theta} \in \mathcal{U} \mapsto \widetilde{\mathbf{z}}_{\boldsymbol{\theta}}|_{\mathcal{V}(\partial\Omega_{\delta})} \in \mathbf{H}^1(\mathcal{V}(\partial\Omega_{\delta}))$ is differentiable at $\mathbf{0}$. Therefore, it follows that the mapping $\boldsymbol{\theta} \in \mathcal{U} \mapsto \widetilde{\mathbf{z}}_{\boldsymbol{\theta}} \in \mathbf{H}^1(\Omega)$ is differentiable at $\mathbf{0}$ and we conclude. $\square$

Proof of Lemma I.1.8: second order shape derivatives. Let us consider the function ϕ defined by (I.0.3). Then, let us set $\mathbf{v}_t := \mathbf{z}_t \circ \phi(t)$ and $q_t := p_t \circ \phi(t)$, where $\mathbf{z}_t$ and p_t denote respectively $\mathbf{z}_{t\mathbf{V}}$ and $p_{t\mathbf{V}}$. Using the Fréchet differentiability Lemma I.1.6, we obtain the Gâteaux differentiability in the direction $\mathbf{V}$: the functions

$$t \in [0, T) \mapsto \mathbf{v}_t|_{\Omega_{\delta} \setminus \overline{\omega}} \in \mathbf{H}^3(\Omega_{\delta} \setminus \overline{\omega}) \quad \text{and} \quad t \in [0, T) \mapsto q_t|_{\Omega_{\delta} \setminus \overline{\omega}} \in \mathbf{H}^2(\Omega_{\delta} \setminus \overline{\omega})$$

are two times differentiable at 0. Using Stein's extension Theorem, there exists

$$P : \mathbf{H}^3(\Omega_{\delta} \setminus \overline{\omega}) \rightarrow \mathbf{H}^3(\Omega_{\delta}) \quad \text{and} \quad Q : \mathbf{H}^2(\Omega_{\delta} \setminus \overline{\omega}) \rightarrow \mathbf{H}^2(\Omega_{\delta}),$$

two linear continuous extension operators. Thus, using Lemma 5.3.9 in [72], the functions

$$t \in [0, T) \mapsto P(\mathbf{v}_t) \circ (\mathbf{I} + t\mathbf{V})^{-1} =: \widetilde{\mathbf{z}}_t \in \mathbf{H}^1(\Omega_{\delta})$$

and

$$t \in [0, T) \mapsto Q(q_t) \circ (\mathbf{I} + t\mathbf{V})^{-1} =: \widetilde{p}_t \in L^2(\Omega_{\delta})$$

are twice differentiable in a neighborhood of 0. Moreover, the functions

$$t \in [0, T) \mapsto \mathbf{z}_t|_{\Omega \setminus \overline{\Omega_{\delta}}} = \mathbf{v}_t|_{\Omega \setminus \overline{\Omega_{\delta}}} \in \mathbf{H}^1(\Omega \setminus \overline{\Omega_{\delta}})$$

and

$$t \in [0, T) \mapsto p_t|_{\Omega \setminus \overline{\Omega_{\delta}}} = q_t|_{\Omega \setminus \overline{\Omega_{\delta}}} \in L^2(\Omega \setminus \overline{\Omega_{\delta}})$$

are twice differentiable in a neighborhood of 0 by composition. Thus, defining

$$\widetilde{\mathbf{z}}_t := \begin{cases} \widetilde{\mathbf{z}}_t & \text{in } \Omega_{\delta} \\ \mathbf{z}_t & \text{in } \Omega \setminus \overline{\Omega_{\delta}} \end{cases} \quad \text{and} \quad \widetilde{p}_t := \begin{cases} \widetilde{p}_t & \text{in } \Omega_{\delta} \\ p_t & \text{in } \Omega \setminus \overline{\Omega_{\delta}}, \end{cases}$$

we conclude as in the proof of Lemma I.1.4. $\square$

I.1.2 The Neumann case

Let $\omega \in \mathcal{O}_\delta$, where $\mathcal{O}_\delta$ is defined by (I.0.1) and O be a non-empty open subset of $\partial\Omega$. Let us consider $\mathbf{f} = (f_1, \dots, f_d) \in [\mathbf{H}^1(\Omega)]'$ such that $\mathbf{f}|_{\Omega_\delta} \in \mathbf{H}^3(\Omega_\delta)$, $\mathbf{f} \neq \mathbf{0}$ and satisfying the following compatibility condition:

$$\langle f_i, 1 \rangle_{[\mathbf{H}^1(\Omega)]', \mathbf{H}^1(\Omega)} = 0 \quad \forall i = 1, \dots, d. \quad (\text{I.1.5})$$

Let $(\mathbf{u}, p) \in \mathbf{H}^1(\Omega \setminus \bar{\omega}) \times L^2(\Omega \setminus \bar{\omega})$ the solution of the following Stokes problem

$$\begin{cases} -\nu \Delta \mathbf{u} + \nabla p &= \mathbf{f} & \text{in } \Omega \setminus \bar{\omega} \\ \operatorname{div} \mathbf{u} &= 0 & \text{in } \Omega \setminus \bar{\omega} \\ -\nu \partial_{\mathbf{n}} \mathbf{u} + p \mathbf{n} &= \mathbf{0} & \text{on } \partial\Omega \\ -\nu \partial_{\mathbf{n}} \mathbf{u} + p \mathbf{n} &= \mathbf{0} & \text{on } \partial\omega \end{cases}$$

such that $\int_O \mathbf{u} = c_n$, where $c_n \in \mathbb{R}^d$ is a normalization constant. Obviously, we can consider another normalization condition but in order to be close to the sequel, we choose this one because it will be imposed in Chapter II.

The results of existence, uniqueness and regularity of the solution of the Stokes system with Neumann boundary conditions are recalled in Appendix A.1. Hence, the existence and uniqueness of the pair $(\mathbf{u}, p)$ is guaranteed by Theorem A.1.2 since we imposed the compatibility condition (I.1.5).

Notice that working with or without a right hand side $\mathbf{f}$ and with homogeneous or inhomogeneous boundary conditions on the exterior boundary $\partial\Omega$ does not change the results. In the case where we work with a right hand side $\mathbf{f}$, we just need that $\mathbf{f}$ is smooth enough in a neighborhood of $\partial\omega$. However, the condition $\mathbf{f}|_{\Omega_\delta} \in \mathbf{H}^3(\Omega_\delta)$ will be automatically satisfied in the inverse problem considered in Chapter II since we will have to impose that $\mathbf{f} \equiv \mathbf{0}$ in Ω_δ in order to have the identifiability result.

Finally, contrary to the Dirichlet case presented above, we need here a generalization of the implicit function theorem. We first recall this result proved by Simon in [92, Theorem 6] for reader's convenience.

I.1.2.1 A preliminary result

Theorem I.1.9 (Simon, [92]). *We give us*

- an open set $\mathcal{U}$ in a Banach space U , $u_0 \in \mathcal{U}$, two reflexive Banach spaces E_1 and E_2 ,
- a map $F : \mathcal{U} \times E_1 \rightarrow E_2$, such that $F(u, \cdot) \in \mathcal{L}(E_1, E_2)$ for all $u \in \mathcal{U}$,
- a function $m : \mathcal{U} \rightarrow E_1$ and a function $f : \mathcal{U} \rightarrow E_2$ such that

$$F(u, m(u)) = f(u) \quad \forall u \in \mathcal{U}.$$

(i) *Assume that*

- $u \mapsto F(u, \cdot)$ is differentiable at u_0 into $\mathcal{L}(E_1, E_2)$,
- f is differentiable at u_0 ,
- $\|F(u_0, x)\|_{E_2} \geq \alpha \|x\|_{E_1} \quad \forall x \in E_1$, for some $\alpha > 0$.

Then, the map $u \mapsto m(u)$ is differentiable at u_0 . Its derivative $m'(u_0, \cdot)$ is the unique solution of

$$F(u_0, m'(u_0, v)) = f'(u_0, v) - \partial_u F(u_0, m(u_0), v) \quad \forall v \in U.$$

(ii) *In addition, assume that for some integer $k \geq 1$, $u \mapsto F(u, \cdot)$ and f are k times differentiable at u_0 . Then, the map $u \mapsto m(u)$ is k times differentiable at u_0 .*

I.1.2.2 The results

Let $\boldsymbol{\theta} \in \mathcal{U}$, where $\mathcal{U}$ is defined by (I.0.2). We set $(\mathbf{u}_\theta, p_\theta) \in \mathbf{H}^1(\Omega \setminus \overline{\omega_\theta}) \times L^2(\Omega \setminus \overline{\omega_\theta})$ the unique solution of

$$\begin{cases} -\nu \Delta \mathbf{u}_\theta + \nabla p_\theta &= \mathbf{f} & \text{in } \Omega \setminus \overline{\omega_\theta} \\ \operatorname{div} \mathbf{u}_\theta &= 0 & \text{in } \Omega \setminus \overline{\omega_\theta} \\ -\nu \partial_{\mathbf{n}} \mathbf{u}_\theta + p_\theta \mathbf{n} &= \mathbf{0} & \text{on } \partial \Omega \\ -\nu \partial_{\mathbf{n}_\theta} \mathbf{u}_\theta + p_\theta \mathbf{n}_\theta &= \mathbf{0} & \text{on } \partial \omega_\theta \end{cases} \quad (\text{I.1.6})$$

such that $\int_O \mathbf{u}_\theta = \mathbf{c}_n$. Here, $\mathbf{n}_\theta$ denotes the exterior unit normal on $\partial \omega_\theta$ with respect to the domain $\Omega \setminus \overline{\omega_\theta}$. This boundary values problem has the following variational formulation:

$$\begin{cases} \int_{\Omega \setminus \overline{\omega_\theta}} \{ \nu \nabla \mathbf{u}_\theta : \nabla \boldsymbol{\varphi}_\theta - p_\theta \operatorname{div} \boldsymbol{\varphi}_\theta \} &= \int_{\Omega \setminus \overline{\omega_\theta}} \mathbf{f} \cdot \boldsymbol{\varphi}_\theta & \forall \boldsymbol{\varphi}_\theta \in \mathbf{H}^1(\Omega \setminus \overline{\omega_\theta}) \\ \int_{\Omega \setminus \overline{\omega_\theta}} \xi_\theta \operatorname{div} \mathbf{u}_\theta &= 0 & \forall \xi_\theta \in L^2(\Omega \setminus \overline{\omega_\theta}) \\ \int_O \mathbf{u}_\theta &= \mathbf{c}_n. \end{cases} \quad (\text{I.1.7})$$

Let us define the key objects of our differentiability proof:

$$\mathbf{v}_\theta := \mathbf{u}_\theta \circ (\mathbf{I} + \boldsymbol{\theta}) \in \mathbf{H}^1(\Omega \setminus \overline{\omega}) \quad \text{and} \quad q_\theta := p_\theta \circ (\mathbf{I} + \boldsymbol{\theta}) \in L^2(\Omega \setminus \overline{\omega}).$$

First order shape differentiability. To prove that $(\mathbf{u}, p)$ is differentiable with respect to the shape, we need the following three lemmas:

Lemma I.1.10 (Characterization of $(\mathbf{v}_\theta, q_\theta)$). *For $\boldsymbol{\theta} \in \mathcal{U}$, the pair $(\mathbf{v}_\theta, q_\theta)$ satisfies for all test functions $\boldsymbol{\varphi} \in \mathbf{H}^1(\Omega \setminus \overline{\omega})$ and $\xi \in L^2(\Omega \setminus \overline{\omega})$*

$$\begin{cases} \int_{\Omega \setminus \overline{\omega}} \{ [\nabla \mathbf{v}_\theta A(\boldsymbol{\theta})] : \nabla \boldsymbol{\varphi} - q_\theta B(\boldsymbol{\theta}) : \nabla \boldsymbol{\varphi} \} &= \int_{\Omega \setminus \overline{\omega}} [\mathbf{f} \circ (\mathbf{I} + \boldsymbol{\theta})] \cdot \boldsymbol{\varphi} J_\theta \\ \int_{\Omega \setminus \overline{\omega}} (\nabla \mathbf{v}_\theta : B(\boldsymbol{\theta})) \xi &= 0 \\ \int_O \mathbf{v}_\theta &= \mathbf{c}_n, \end{cases}$$

with

$$\begin{aligned} J_\theta &:= \det(\mathbf{I} + \nabla \boldsymbol{\theta}) \in W^{2,\infty}(\overline{\Omega_\delta}), \\ A(\boldsymbol{\theta}) &:= \nu J_\theta (\mathbf{I} + \nabla \boldsymbol{\theta})^{-1} (\mathbf{I} + {}^t \nabla \boldsymbol{\theta})^{-1} \in W^{2,\infty}(\overline{\Omega_\delta}, \mathcal{M}_{d,d}), \\ B(\boldsymbol{\theta}) &:= J_\theta (\mathbf{I} + {}^t \nabla \boldsymbol{\theta})^{-1} \in W^{2,\infty}(\overline{\Omega_\delta}, \mathcal{M}_{d,d}). \end{aligned}$$

Lemma I.1.11 (Differentiability of $\boldsymbol{\theta} \mapsto (\mathbf{v}_\theta, q_\theta)$). *The function*

$$\boldsymbol{\theta} \in \mathcal{U} \mapsto (\mathbf{v}_\theta, q_\theta) \in \mathbf{X}^{1,2}(\Omega \setminus \overline{\omega}, \Omega_\delta \setminus \overline{\omega}) \times X^{0,1}(\Omega \setminus \overline{\omega}, \Omega_\delta \setminus \overline{\omega})$$

is differentiable in a neighborhood of $\mathbf{0}$.

Remark I.1.12. *This lemma implies that there exists a constant $c > 0$ such that*

$$\|\mathbf{u}_\theta \circ (\mathbf{I} + \boldsymbol{\theta}) - \mathbf{u}\|_{\mathbf{X}^{1,2}(\Omega \setminus \overline{\omega}, \Omega_\delta \setminus \overline{\omega})} + \|p_\theta \circ (\mathbf{I} + \boldsymbol{\theta}) - p\|_{X^{0,1}(\Omega \setminus \overline{\omega}, \Omega_\delta \setminus \overline{\omega})} \leq c \|\boldsymbol{\theta}\|_{\mathcal{U}}.$$

We then prove the following lemma in exactly the same way than the analogous lemma in the Dirichlet case (see Lemma I.1.4).

Lemma I.1.13 (Differentiability of $\theta \mapsto (\mathbf{u}_\theta, p_\theta)$). *There exists $\tilde{\mathbf{u}}_\theta, \tilde{p}_\theta$ some respective extension in Ω of $\mathbf{u}_\theta \in \mathbf{H}^1(\Omega \setminus \overline{\omega_\theta})$, $p_\theta \in L^2(\Omega \setminus \overline{\omega_\theta})$ such that the functions*

$$\theta \in \mathcal{U} \mapsto \tilde{\mathbf{u}}_\theta \in \mathbf{H}^1(\Omega) \quad \text{and} \quad \theta \in \mathcal{U} \mapsto \tilde{p}_\theta \in L^2(\Omega)$$

are differentiable at $\mathbf{0}$.

Remark I.1.14. *We will prove this three lemmas under weaker assumptions: $\mathbf{f}|_{\Omega_\delta} \in \mathbf{H}^1(\Omega_\delta)$, ω with a $C^{1,1}$ boundary and $\theta \in \mathbf{W}^{2,\infty}(\mathbb{R}^d)$.*

Higher order differentiability. To prove that $(\mathbf{u}, p)$ is twice differentiable with respect to the shape, we use the second order differentiability of $\theta \mapsto (\mathbf{v}_\theta, q_\theta)$. Therefore, we first prove the following lemma:

Lemma I.1.15. *The function*

$$\theta \in \mathcal{U} \mapsto (\mathbf{v}_\theta, q_\theta) \in \mathbf{X}^{1,3}(\Omega \setminus \overline{\omega}, \Omega_\delta \setminus \overline{\omega}) \times X^{0,2}(\Omega \setminus \overline{\omega}, \Omega_\delta \setminus \overline{\omega})$$

is twice differentiable in a neighborhood of $\mathbf{0}$.

Remark I.1.16. *This lemma implies that there exists a constant $c > 0$ such that*

$$\|\mathbf{u}_\theta \circ (\mathbf{I} + \theta) - \mathbf{u}\|_{\mathbf{X}^{1,3}(\Omega \setminus \overline{\omega}, \Omega_\delta \setminus \overline{\omega})} + \|p_\theta \circ (\mathbf{I} + \theta) - p\|_{X^{0,2}(\Omega \setminus \overline{\omega}, \Omega_\delta \setminus \overline{\omega})} \leq c \|\theta\|_{\mathcal{U}}.$$

We then prove the following lemma in exactly the same way than the analogous lemma in the Dirichlet case (see Lemma I.1.8).

Lemma I.1.17 (Second order shape differentiability). *The solution $(\mathbf{u}, p)$ is twice differentiable with respect to the domain.*

I.1.2.3 The proofs

Proof of Lemma I.1.10: characterization of $(\mathbf{v}_\theta, q_\theta)$. Let $\varphi \in \mathbf{H}^1(\Omega \setminus \overline{\omega})$, $\xi \in L^2(\Omega \setminus \overline{\omega})$ and $\theta \in \mathcal{U}$. Using

$$\varphi_\theta := \varphi \circ (\mathbf{I} + \theta)^{-1} \in \mathbf{H}^1(\Omega \setminus \overline{\omega_\theta}) \quad \text{and} \quad \xi_\theta := \xi \circ (\mathbf{I} + \theta)^{-1} \in L^2(\Omega \setminus \overline{\omega_\theta})$$

as test functions in the variational formulation (I.1.7), we have

$$\left\{ \begin{array}{l} \int_{\Omega \setminus \overline{\omega_\theta}} \left\{ \nu [(\nabla \mathbf{v}_\theta (\mathbf{I} + \nabla \theta)^{-1}) \circ (\mathbf{I} + \theta)^{-1}] : [(\nabla \varphi (\mathbf{I} + \nabla \theta)^{-1}) \circ (\mathbf{I} + \theta)^{-1}] \right. \\ \quad \left. - (q_\theta \circ (\mathbf{I} + \theta)^{-1}) \operatorname{div} (\varphi \circ (\mathbf{I} + \theta)^{-1}) \right\} = \int_{\Omega \setminus \overline{\omega_\theta}} \mathbf{f} \cdot [\varphi \circ (\mathbf{I} + \theta)^{-1}] \\ \int_{\Omega \setminus \overline{\omega_\theta}} [\xi \circ (\mathbf{I} + \theta)^{-1}] \operatorname{div} (\mathbf{v}_\theta \circ (\mathbf{I} + \theta)^{-1}) = 0 \\ \int_O \mathbf{v}_\theta \circ (\mathbf{I} + \theta)^{-1} = \mathbf{c}_n. \end{array} \right.$$

However, $\operatorname{div} (\varphi \circ (\mathbf{I} + \theta)^{-1}) = (\nabla \varphi \circ (\mathbf{I} + \theta)^{-1}) : ((\mathbf{I} + {}^t \nabla \theta)^{-1} \circ (\mathbf{I} + \theta)^{-1})$, and a similar equality holds with $\mathbf{v}_\theta$ instead of φ . Thus, the conclusion follows after the change of variables $x = (\mathbf{I} + \theta)y$ and by noticing that

$$\int_O \mathbf{v}_\theta J_\theta = \mathbf{c}_n \iff \int_O \mathbf{v}_\theta = \mathbf{c}_n,$$

since $\theta \equiv \mathbf{0}$ on $O \subset \partial\Omega$. □

Proof of Lemma I.1.11: differentiability of $\theta \mapsto (v_\theta, q_\theta)$. Let us check the assumptions of Simon's Theorem.

First step: notations. We need some additional tools: a fourth domain $\tilde{\Omega}_\delta$ which is an open set with a C^∞ boundary and such that $\Omega_\delta \subset \subset \tilde{\Omega}_\delta \subset \subset \Omega$ and a truncation function $\Phi \in C_c^\infty(\tilde{\Omega}_\delta)$ such that $\Phi \equiv 1$ in Ω_δ . Using the notations introduced in Lemma I.1.10, we define the spaces:

$$\begin{aligned} \mathbf{E}_1 &:= \{ (v, q) \in \mathbf{H}^1(\Omega \setminus \bar{\omega}) \times \mathbf{L}^2(\Omega \setminus \bar{\omega}) ; (\Phi v, \Phi q) \in \mathbf{H}^2(\Omega \setminus \bar{\omega}) \times \mathbf{H}^1(\Omega \setminus \bar{\omega}) \}, \\ \mathbf{E}_2 &:= \left\{ (f, g) \in [\mathbf{H}^1(\Omega \setminus \bar{\omega})]' \times \mathbf{L}^2(\Omega \setminus \bar{\omega}) ; (\Phi f, \Phi g) \in \mathbf{L}^2(\Omega \setminus \bar{\omega}) \times \mathbf{H}^1(\Omega \setminus \bar{\omega}) \right\} \times \mathbb{R}^d. \end{aligned}$$

Note that $\mathbf{E}_1$ and $\mathbf{E}_2$ are Hilbert spaces with respective norms

$$\begin{aligned} \|(v, q)\|_{\mathbf{E}_1}^2 &:= \|v\|_{\mathbf{H}^1(\Omega \setminus \bar{\omega})}^2 + \|q\|_{\mathbf{L}^2(\Omega \setminus \bar{\omega})}^2 + \|\Phi v\|_{\mathbf{H}^2(\Omega \setminus \bar{\omega})}^2 + \|\Phi q\|_{\mathbf{H}^1(\Omega \setminus \bar{\omega})}^2, \\ \|((f, g), r)\|_{\mathbf{E}_2}^2 &:= \|f\|_{[\mathbf{H}^1(\Omega \setminus \bar{\omega})]'}^2 + \|g\|_{\mathbf{L}^2(\Omega \setminus \bar{\omega})}^2 + \|\Phi f\|_{\mathbf{L}^2(\Omega \setminus \bar{\omega})}^2 + \|\Phi g\|_{\mathbf{H}^1(\Omega \setminus \bar{\omega})}^2 + |r|^2. \end{aligned}$$

Moreover, the following continuous embeddings hold: $\mathbf{E}_1 \hookrightarrow \mathbf{X}^{1,2}(\Omega \setminus \bar{\omega}, \Omega_\delta \setminus \bar{\omega}) \times \mathbf{X}^{0,1}(\Omega \setminus \bar{\omega}, \Omega_\delta \setminus \bar{\omega})$ and $\mathbf{E}_2 \hookrightarrow \mathbf{X}^{*,0}(\Omega \setminus \bar{\omega}, \Omega_\delta \setminus \bar{\omega})$. We also define the functions:

$$- f_1(\theta, (v, q)) \in [\mathbf{H}^1(\Omega \setminus \bar{\omega})]' \text{ by } \forall \varphi \in \mathbf{H}^1(\Omega \setminus \bar{\omega}),$$

$$\langle f_1(\theta, (v, q)), \varphi \rangle_{\Omega \setminus \bar{\omega}} := \int_{\Omega \setminus \bar{\omega}} [f \circ (\mathbf{I} + \theta)] \cdot J_\theta \varphi, \quad ,$$

$$- F_1(\theta, (v, q)) \in [\mathbf{H}^1(\Omega \setminus \bar{\omega})]' \text{ by } \forall \varphi \in \mathbf{H}^1(\Omega \setminus \bar{\omega}),$$

$$\langle F_1(\theta, (v, q)), \varphi \rangle_{\Omega \setminus \bar{\omega}} := \int_{\Omega \setminus \bar{\omega}} \{[\nabla v A(\theta)] : \nabla \varphi - q B(\theta) : \nabla \varphi\},$$

$$- m(\theta) := (v_\theta, q_\theta) \quad \text{and} \quad \mathcal{F}(\theta) := \left(f_1(\theta, (v, q)), 0, \int_O f_b \right),$$

$$- F(\theta, (v, q)) := \left(F_1(\theta, (v, q)), \nabla v : B(\theta), \int_O v \right).$$

By the characterization of (v_θ, q_θ) obtained in Lemma I.1.10,

$$F(\theta, m(\theta)) = \mathcal{F}(\theta) \quad \forall \theta \in \mathcal{U}.$$

Second step: differentiability of F and f at $\mathbf{0}$. We use the chain rule. Since the functions

$$\theta \in \mathcal{U} \mapsto J_\theta \in W^{1,\infty}(\bar{\Omega}) \quad \text{and} \quad \theta \in \mathcal{U} \mapsto (\mathbf{I} + \nabla \theta)^{-1} \in W^{1,\infty}(\bar{\Omega}, \mathcal{M}_{d \times d})$$

are C^∞ , the functions

$$\theta \in \mathcal{U} \mapsto A(\theta) \in W^{1,\infty}(\bar{\Omega}, \mathcal{M}_{d \times d}) \quad \text{and} \quad \theta \in \mathcal{U} \mapsto B(\theta) \in W^{1,\infty}(\bar{\Omega}, \mathcal{M}_{d \times d})$$

are C^∞ . Moreover, the following functions are C^∞ :

$$\begin{aligned} \mathbf{L}^\infty(\bar{\Omega}, \mathcal{M}_{d \times d}) \times \{ v \in \mathbf{H}^1(\Omega \setminus \bar{\omega}) ; \Phi v \in \mathbf{H}^2(\Omega \setminus \bar{\omega}) \} &\rightarrow \{ g \in \mathbf{L}^2(\Omega \setminus \bar{\omega}) ; \Phi g \in \mathbf{H}^1(\Omega \setminus \bar{\omega}) \} \\ (B, v) &\mapsto \nabla v : B, \end{aligned}$$

$$v \in \mathbf{X}^{1,2}(\Omega \setminus \bar{\omega}, \Omega_\delta \setminus \bar{\omega}) \mapsto \int_O v \in \mathbb{R}^d,$$

$$F_1 : (\theta, (v, q)) \in \mathcal{U} \times \mathbf{E}_1 \mapsto F_1(\theta, (v, q)) \in \left\{ f \in [\mathbf{H}^1(\Omega \setminus \bar{\omega})]' ; \Phi f \in \mathbf{L}^2(\Omega \setminus \bar{\omega}) \right\}.$$

Finally, since $\mathbf{f}|_{\Omega_\delta} \in \mathbf{H}^1(\Omega_\delta)$ by assumption, the function

$$\boldsymbol{\theta} \in \mathcal{U} \mapsto \mathbf{f} \circ (\mathbf{I} + \boldsymbol{\theta})|_{\Omega_\delta} \in \mathbf{L}^2(\Omega_\delta)$$

is C^1 in a neighborhood of $\mathbf{0}$ by Lemma 5.3.3 in [72]. Since $\text{supp } \boldsymbol{\theta} \subset \Omega_\delta$, we deduce that the function

$$\boldsymbol{\theta} \in \mathcal{U} \mapsto \mathbf{f} \circ (\mathbf{I} + \boldsymbol{\theta}) \in \mathbf{L}^2(\Omega)$$

is C^1 in a neighborhood of $\mathbf{0}$. Then, the function

$$\mathbf{f}_1 : (\boldsymbol{\theta}, (\mathbf{v}, q)) \in \mathcal{U} \times \mathbf{E}_1 \mapsto \mathbf{f}_1(\boldsymbol{\theta}, (\mathbf{v}, q)) \in \left\{ \mathbf{f} \in [\mathbf{H}^1(\Omega \setminus \bar{\omega})]'; \Phi \mathbf{f} \in \mathbf{L}^2(\Omega \setminus \bar{\omega}) \right\}$$

is C^1 . By composition, $\mathbf{F}$ is C^∞ and $\mathcal{F}$ is C^1 in a neighborhood of $\mathbf{0}$.

Third step: existence of $\alpha > 0$ such that $\|\mathbf{F}(\mathbf{0}, (\mathbf{v}, q))\|_{\mathbf{E}_2} \geq \alpha \|(\mathbf{v}, q)\|_{\mathbf{E}_1}$. We consider a pair $(\mathbf{v}, q) \in \mathbf{E}_1$ and we define $(\boldsymbol{\xi}, \eta, \mathbf{r}) \in \mathbf{E}_2$ by $\mathbf{F}(\mathbf{0}, (\mathbf{v}, q)) = (\boldsymbol{\xi}, \eta, \mathbf{r})$. Then, we have, for all $\boldsymbol{\varphi} \in \mathbf{H}^1(\Omega \setminus \bar{\omega})$,

$$\begin{cases} \int_{\Omega \setminus \bar{\omega}} \{\nu \nabla \mathbf{v} : \nabla \boldsymbol{\varphi} - q \operatorname{div} \boldsymbol{\varphi}\} &= \langle \boldsymbol{\xi}, \boldsymbol{\varphi} \rangle_{\Omega \setminus \bar{\omega}} \\ \operatorname{div} \mathbf{v} &= \eta \\ \int_O \mathbf{v} &= \mathbf{r}. \end{cases} \quad \text{in } \Omega \setminus \bar{\omega}$$

Setting $\boldsymbol{\varphi}_i := {}^t(0, \dots, 0, 1, 0, \dots, 0)$, ($i = 1, \dots, N$), as test functions in the first line of the previous system, we check the compatibility condition of Theorem A.1.2. Thus, according to Theorem A.1.2, there exists $\mathbf{w} \in \mathbf{H}^1(\Omega \setminus \bar{\omega})$ such that $\int_{\Omega \setminus \bar{\omega}} \mathbf{w} = \mathbf{0}$ and such that, for all $\boldsymbol{\varphi} \in \mathbf{H}^1(\Omega \setminus \bar{\omega})$,

$$\begin{cases} \int_{\Omega \setminus \bar{\omega}} \{\nu \nabla \mathbf{w} : \nabla \boldsymbol{\varphi} - q \operatorname{div} \boldsymbol{\varphi}\} &= \langle \boldsymbol{\xi}, \boldsymbol{\varphi} \rangle_{\Omega \setminus \bar{\omega}} \\ \operatorname{div} \mathbf{w} &= \eta \end{cases} \quad \text{in } \Omega \setminus \bar{\omega},$$

and there exists a constant $\alpha_1 > 0$ such that

$$\|\mathbf{w}\|_{\mathbf{H}^1(\Omega \setminus \bar{\omega})} + \|q\|_{L^2(\Omega \setminus \bar{\omega})} \leq \alpha_1 (\|\boldsymbol{\xi}\|_{[\mathbf{H}^1(\Omega \setminus \bar{\omega})]'} + \|\eta\|_{L^2(\Omega \setminus \bar{\omega})}). \quad (\text{I.1.8})$$

Such a problem has a unique solution, we then get that

$$\mathbf{v} = \mathbf{w} - \frac{1}{|O|} \int_O \mathbf{w} + \mathbf{r}.$$

Thus, by computation, there exists a constant $\alpha_2 > 0$ such that

$$\|\mathbf{v}\|_{\mathbf{H}^1(\Omega \setminus \bar{\omega})} - |\Omega \setminus \bar{\omega}|^{-1/2} |\mathbf{r}| \leq \alpha_2 \|\mathbf{w}\|_{\mathbf{H}^1(\Omega \setminus \bar{\omega})}.$$

Hence, using the above inequality and (I.1.8), there exists a constant $\alpha_3 > 0$, such that

$$\|\mathbf{v}\|_{\mathbf{H}^1(\Omega \setminus \bar{\omega})} + \|q\|_{L^2(\Omega \setminus \bar{\omega})} \leq \alpha_3 (\|\boldsymbol{\xi}\|_{[\mathbf{H}^1(\Omega \setminus \bar{\omega})]'} + \|\eta\|_{L^2(\Omega \setminus \bar{\omega})} + |\mathbf{r}|). \quad (\text{I.1.9})$$

Mimicking the proof of the local regularity result A.1.4 (see Inequality (A.1.9)), we use the argument stated above to prove the existence of a constant α_4 such that

$$\begin{aligned} & \|\Phi \mathbf{v}\|_{\mathbf{H}^2(\tilde{\Omega}_\delta \setminus \bar{\omega})} + \|\Phi q\|_{\mathbf{H}^1(\tilde{\Omega}_\delta \setminus \bar{\omega})} \\ & \leq \alpha_4 \left(\|\Phi \boldsymbol{\xi}\|_{\mathbf{L}^2(\tilde{\Omega}_\delta \setminus \bar{\omega})} + \|\Phi \eta\|_{\mathbf{H}^1(\tilde{\Omega}_\delta \setminus \bar{\omega})} + \|\mathbf{v}\|_{\mathbf{H}^1(\Omega \setminus \bar{\omega})} + \|q\|_{L^2(\Omega \setminus \bar{\omega})} + |\mathbf{r}| \right). \end{aligned}$$

Thus, since Φ is compactly supported in $\tilde{\Omega}_\delta$,

$$\begin{aligned} & \|\Phi \mathbf{v}\|_{\mathbf{H}^2(\Omega \setminus \bar{\omega})} + \|\Phi q\|_{\mathbf{H}^1(\Omega \setminus \bar{\omega})} \\ & \leq \alpha_4 \left(\|\Phi \boldsymbol{\xi}\|_{\mathbf{L}^2(\Omega \setminus \bar{\omega})} + \|\Phi \eta\|_{\mathbf{H}^1(\Omega \setminus \bar{\omega})} + \|\mathbf{v}\|_{\mathbf{H}^1(\Omega \setminus \bar{\omega})} + \|q\|_{\mathbf{L}^2(\Omega \setminus \bar{\omega})} + |\mathbf{r}| \right). \end{aligned} \quad (\text{I.1.10})$$

Gathering (I.1.9) and (I.1.10), there exists a constant $\alpha > 0$ such that

$$\|\mathbf{F}(\mathbf{0}, (\mathbf{v}, q))\|_{\mathbf{E}_2} \geq \alpha \|(\mathbf{v}, q)\|_{\mathbf{E}_1}.$$

Fourth step: conclusion. By Simon's Theorem,

$$\boldsymbol{\theta} \in \mathcal{U} \mapsto (\mathbf{v}_\theta, q_\theta) \in \mathbf{E}_1$$

is differentiable in a neighborhood of $\mathbf{0}$. We conclude using the fact that $\mathbf{E}_1$ is continuously embedded in $\mathbf{X}^{1,2}(\Omega \setminus \bar{\omega}, \Omega_\delta \setminus \bar{\omega}) \times \mathbf{X}^{0,1}(\Omega \setminus \bar{\omega}, \Omega_\delta \setminus \bar{\omega})$. $\square$

Remark I.1.18. Notice that the regularity of the function

$$\boldsymbol{\theta} \in \mathcal{U} \mapsto (\mathbf{v}_\theta, q_\theta) \in \mathbf{X}^{1,2}(\Omega \setminus \bar{\omega}, \Omega_\delta \setminus \bar{\omega}) \times \mathbf{X}^{0,1}(\Omega \setminus \bar{\omega}, \Omega_\delta \setminus \bar{\omega})$$

is limited by the regularity of the function $\mathcal{F}$ and then by the regularity of the right hand side $\mathbf{f}$ in Ω_δ . In Chapter II, we will have to impose $\mathbf{f}|_{\Omega_\delta} \equiv \mathbf{0}$. Hence, this mapping will be C^∞ in a neighborhood of $\mathbf{0}$.

Proof of Lemma I.1.13: differentiability of $\boldsymbol{\theta} \mapsto (\mathbf{u}_\theta, p_\theta)$. The proof is exactly the same than in the Dirichlet case detailed before (see the proof of Lemma I.1.4). We refer to this one for details. The idea is to use the chain rule, the differentiability result I.1.11 and the composition by $(\mathbf{I} + \boldsymbol{\theta})^{-1}$ (see [72, Lemma 5.3.9]). $\square$

Proof of Lemma I.1.15. We use Simon's Theorem I.1.9. Let the open set $\tilde{\Omega}_\delta$ defined in the proof of Lemma I.1.11 and $\Phi \in C_c^\infty(\tilde{\Omega}_\delta)$ such that $\Phi \equiv 1$ in Ω_δ . We consider the functions $\mathbf{F}$ and $\mathcal{F}$ defined in the proof of Lemma I.1.11 with modified sets of departure and destination:

$$\begin{aligned} \mathbf{E}_1 &:= \{(\mathbf{v}, q) \in \mathbf{H}^1(\Omega \setminus \bar{\omega}) \times \mathbf{L}^2(\Omega \setminus \bar{\omega}); (\Phi \mathbf{v}, \Phi q) \in \mathbf{H}^3(\Omega \setminus \bar{\omega}) \times \mathbf{H}^2(\Omega \setminus \bar{\omega})\}, \\ \mathbf{E}_2 &:= \left\{(\mathbf{f}, g) \in [\mathbf{H}^1(\Omega \setminus \bar{\omega})]' \times \mathbf{L}^2(\Omega \setminus \bar{\omega}); (\Phi \mathbf{f}, \Phi g) \in \mathbf{H}^1(\Omega \setminus \bar{\omega}) \times \mathbf{H}^2(\Omega \setminus \bar{\omega})\right\} \times \mathbb{R}^d. \end{aligned}$$

We adapt the proof of Lemma I.1.11 to obtain that $\mathbf{F}$ and $\mathcal{F}$ are C^2 in a neighborhood of $\mathbf{0}$. Indeed, since $\mathbf{f}|_{\Omega_\delta} \in \mathbf{H}^3(\Omega_\delta)$, then the function

$$\boldsymbol{\theta} \in \mathcal{U} \mapsto \mathbf{f} \circ (\mathbf{I} + \boldsymbol{\theta}) \in \mathbf{H}^1(\Omega)$$

is C^2 in a neighborhood of $\mathbf{0}$ (see [72, Lemma 5.3.9]). Moreover, we check last assumption of Theorem I.1.9 (the estimation) like in the proof of Lemma I.1.11. We use in particular the local regularity result A.1.4. $\square$

I.2 The Navier-Stokes (and linearized Navier-Stokes) case

Let $\omega \in \mathcal{O}_\delta$, where $\mathcal{O}_\delta$ is defined by (I.0.1) and O be a non-empty open subset of $\partial\Omega$. We denote by $\{\Gamma_i\}_{i=1, \dots, I}$ the connected components of $\partial\Omega$. Let $\mathbf{g} \in \mathbf{H}^{1/2}(\partial\Omega)$, $\mathbf{g} \neq \mathbf{0}$ such that

$$\int_{\Gamma_i} \mathbf{g} \cdot \mathbf{n} = 0, \quad \forall i = 1, \dots, I. \quad (\text{I.2.1})$$

For $\omega \in \mathcal{O}_\delta$, we consider the Navier-Stokes problem

$$\begin{cases} -\nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p &= \mathbf{0} & \text{in } \Omega \setminus \bar{\omega} \\ \operatorname{div} \mathbf{u} &= 0 & \text{in } \Omega \setminus \bar{\omega} \\ \mathbf{u} &= \mathbf{g} & \text{on } \partial\Omega \\ \mathbf{u} &= \mathbf{0} & \text{on } \partial\omega. \end{cases} \quad (\text{I.2.2})$$

The results of existence, uniqueness and regularity of the solution of the Navier-Stokes equations are recalled in Appendix A.2 (we also refer to [31, 44, 61, 97]). Hence the existence of a solution of Problem (I.2.2) is guaranteed by Theorem A.2.1 since we imposed the compatibility condition (I.2.1). Note also that the solution is not necessarily unique. To overcome this difficulty, we assume ν big enough and independent of the object $\omega \in \mathcal{O}_\delta$ (see Theorem A.2.5) in order to guarantee the uniqueness of the solution of the previous problem (once a normalization condition on the pressure p is imposed). More precisely, we assume that

$$\nu > c_b(\Omega) \left(2\|\Delta \mathbf{G}\|_{\mathbf{W}'} + \frac{2}{\nu} \|(\mathbf{G} \cdot \nabla) \mathbf{G}\|_{\mathbf{W}'} + \|\mathbf{G}\|_{\mathbf{H}^1(\Omega \setminus \bar{\Omega}_\delta)} \right) \quad (\text{I.2.3})$$

where $\mathbf{W}'$ is the dual space of

$$\mathbf{W} := \{ \boldsymbol{\varphi} \in \mathbf{H}^1(\Omega \setminus \bar{\Omega}_\delta), \operatorname{div} \boldsymbol{\varphi} = 0 \text{ in } \Omega \setminus \bar{\Omega}_\delta \text{ and } \boldsymbol{\varphi} = \mathbf{0} \text{ on } \partial\Omega \},$$

where, according to [14, Lemma 3.3], $\mathbf{G} \in \mathbf{H}^1(\Omega)$ satisfies

$$\mathbf{G} = \mathbf{g} \text{ on } \partial\Omega, \quad \operatorname{div} \mathbf{G} = 0 \text{ in } \Omega \quad \text{and} \quad \mathbf{G} = \mathbf{0} \text{ in } \Omega_\delta, \quad (\text{I.2.4})$$

and where

$$c_b(\Omega) := \begin{cases} \frac{|\Omega|^{1/2}}{2} & \text{if } d = 2 \\ \frac{2\sqrt{2}|\Omega|^{1/6}}{3} & \text{if } d = 3, \end{cases} \quad (\text{I.2.5})$$

(see [56, Vol. 2, Chap. VIII, page 6]). Moreover, for $c_n \in \mathbb{R}$, we choose the normalization:

$$\langle m^2(\nu \partial_{\mathbf{n}} \mathbf{u} - p \mathbf{n}), \mathbf{n} \rangle_O = c_n, \quad (\text{I.2.6})$$

where the function $m \in C_c^\infty(\partial\Omega)$ is such that $\operatorname{supp}(m) = \bar{O}$. Hence, according to Theorem A.2.5, this solution $(\mathbf{u}, p)$ is unique and is referred as the state. Obviously, we can consider another normalization condition but in order to be close to the sequel, we choose this one because it will be imposed in Chapter II.

Remark I.2.1. *The estimate (I.2.3) implies that*

$$\nu > c_b(\Omega \setminus \bar{\omega}) \|\mathbf{u}\|_{\mathbf{H}^1(\Omega \setminus \bar{\omega})}, \quad (\text{I.2.7})$$

where $c_b(\Omega \setminus \bar{\omega})$ is defined analogously to (I.2.5). Indeed, if $\nu \geq 2c_b(\Omega) \|\mathbf{G}\|_{\mathbf{H}^1(\Omega \setminus \bar{\Omega}_\delta)}$, we know that $\|\mathbf{u} - \mathbf{G}\|_{\mathbf{H}^1(\Omega)} \leq \frac{2}{\nu} \|\nu \Delta \mathbf{G} - (\mathbf{G} \cdot \nabla) \mathbf{G}\|_{\mathbf{W}'}$ (see [97, Ch. II, eq. (1.89)]) and obviously $c_b(\Omega) \geq c_b(\Omega \setminus \bar{\omega})$. Moreover, Inequality (I.2.3) implies the uniqueness of the solution of all the perturbed problems we will consider.

Finally, for an open set $\mathcal{D} \subset \Omega$, $b_{\mathcal{D}}(\cdot, \cdot, \cdot)$ denotes the trilinear continuous form defined, for $(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathbf{H}^1(\mathcal{D}) \times \mathbf{H}^1(\mathcal{D}) \times \mathbf{H}^1(\mathcal{D})$, by

$$b_{\mathcal{D}}(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \sum_{i,j=1}^N \int_{\mathcal{D}} u_i \frac{\partial v_j}{\partial x_i} w_j.$$

I.2.1 The results

Let $\theta \in \mathcal{U}$, where $\mathcal{U}$ is defined by (I.0.2). We set $(\mathbf{u}_\theta, p_\theta) \in \mathbf{H}^1(\Omega \setminus \overline{\omega_\theta}) \times L^2(\Omega \setminus \overline{\omega_\theta})$ solution of

$$\begin{cases} -\nu \Delta \mathbf{u}_\theta + (\mathbf{u}_\theta \cdot \nabla) \mathbf{u}_\theta + \nabla p_\theta &= \mathbf{0} & \text{in } \Omega \setminus \overline{\omega_\theta} \\ \operatorname{div} \mathbf{u}_\theta &= 0 & \text{in } \Omega \setminus \overline{\omega_\theta} \\ \mathbf{u}_\theta &= \mathbf{g} & \text{on } \partial\Omega \\ \mathbf{u}_\theta &= \mathbf{0} & \text{on } \partial\omega_\theta \end{cases} \quad (\text{I.2.8})$$

with

$$\langle m^2(\nu \partial_{\mathbf{n}} \mathbf{u}_\theta - p_\theta \mathbf{n}), \mathbf{n} \rangle_O = c_n.$$

Notice that, since we assumed (I.2.3), we have, for all $\theta \in \mathcal{U}$,

$$\nu > c_b(\Omega \setminus \overline{\omega_\theta}) \|\mathbf{u}_\theta\|_{\mathbf{H}^1(\Omega \setminus \overline{\omega_\theta})},$$

where $c_b(\Omega \setminus \overline{\omega_\theta})$ is defined analogously to (I.2.5). Hence, using Theorem A.2.5, the couple $(\mathbf{u}_\theta, p_\theta)$ is unique.

Let $\mathbf{G} \in \mathbf{H}^1(\Omega)$ satisfying (I.2.4). Then $(\mathbf{w}_\theta := \mathbf{u}_\theta - \mathbf{G}, p_\theta) \in \mathbf{H}_0^1(\Omega \setminus \overline{\omega_\theta}) \times L^2(\Omega \setminus \overline{\omega_\theta})$ is such that for all $\varphi_\theta \in \mathbf{H}_0^1(\Omega \setminus \overline{\omega_\theta})$ and for all $\xi_\theta \in L^2(\Omega \setminus \overline{\omega_\theta})$

$$\begin{cases} \int_{\Omega \setminus \overline{\omega_\theta}} \nu \nabla \mathbf{w}_\theta : \nabla \varphi_\theta + b_{\Omega \setminus \overline{\omega_\theta}}(\mathbf{w}_\theta, \mathbf{w}_\theta, \varphi_\theta) + b_{\Omega \setminus \overline{\omega_\theta}}(\mathbf{w}_\theta, \mathbf{G}, \varphi_\theta) \\ \quad + b_{\Omega \setminus \overline{\omega_\theta}}(\mathbf{G}, \mathbf{w}_\theta, \varphi_\theta) - \int_{\Omega \setminus \overline{\omega_\theta}} p_\theta \operatorname{div} \varphi_\theta &= - \int_{\Omega \setminus \overline{\omega_\theta}} \nu \nabla \mathbf{G} : \nabla \varphi_\theta \\ \quad - b_{\Omega \setminus \overline{\omega_\theta}}(\mathbf{G}, \mathbf{G}, \varphi_\theta) \\ \int_{\Omega \setminus \overline{\omega_\theta}} \xi_\theta \operatorname{div} \mathbf{w}_\theta &= 0 \\ \langle m^2(\nu \partial_{\mathbf{n}} \mathbf{w}_\theta - p_\theta \mathbf{n}), \mathbf{n} \rangle_O &= c_n - \langle \nu m^2 \partial_{\mathbf{n}} \mathbf{G}, \mathbf{n} \rangle_O. \end{cases} \quad (\text{I.2.9})$$

Let us define the key objects of our differentiability proof:

$$\mathbf{v}_\theta := \mathbf{w}_\theta \circ (\mathbf{I} + \theta) \in \mathbf{H}_0^1(\Omega \setminus \overline{\omega}) \quad \text{and} \quad q_\theta := p_\theta \circ (\mathbf{I} + \theta) \in L^2(\Omega \setminus \overline{\omega}).$$

First order differentiability. To prove that $(\mathbf{u}, p)$ is differentiable with respect to the shape, we need the following three lemmas:

Lemma I.2.2 (Characterization of $(\mathbf{v}_\theta, q_\theta)$). *For $\theta \in \mathcal{U}$, the pair $(\mathbf{v}_\theta, q_\theta)$ satisfies for all $\varphi \in \mathbf{H}_0^1(\Omega \setminus \overline{\omega})$ and all $\xi \in L^2(\Omega \setminus \overline{\omega})$*

$$\begin{cases} \int_{\Omega \setminus \overline{\omega}} [(\nu \nabla \mathbf{v}_\theta A(\theta)) : \nabla \varphi - q_\theta B(\theta) : \nabla \varphi] \\ \quad + b_J(\theta, \mathbf{v}_\theta, \mathbf{v}_\theta, \varphi) + b_{\Omega \setminus \overline{\omega}}(\mathbf{G}, \mathbf{v}_\theta, \varphi) \\ \quad + b_{\Omega \setminus \overline{\omega}}(\mathbf{v}_\theta, \mathbf{G}, \varphi) &= - \int_{\Omega \setminus \overline{\omega}} \nu \nabla \mathbf{G} : \nabla \varphi - b_{\Omega \setminus \overline{\omega}}(\mathbf{G}, \mathbf{G}, \varphi) \\ \int_{\Omega \setminus \overline{\omega}} (\nabla \mathbf{v}_\theta : B(\theta)) \xi &= 0 \\ \langle m^2(\nu \partial_{\mathbf{n}} \mathbf{v}_\theta - q_\theta \mathbf{n}), \mathbf{n} \rangle_O &= c_n - \langle \nu m^2 \partial_{\mathbf{n}} \mathbf{G}, \mathbf{n} \rangle_O, \end{cases}$$

with

$$\begin{aligned} J_\theta &:= \det(\mathbf{I} + \nabla \theta) \in W^{2,\infty}(\overline{\Omega_\delta}), \\ A(\theta) &:= \nu J_\theta (\mathbf{I} + \nabla \theta)^{-1} (\mathbf{I} + {}^t \nabla \theta)^{-1} \in W^{2,\infty}(\overline{\Omega_\delta}, \mathcal{M}_{d,d}), \\ B(\theta) &:= J_\theta (\mathbf{I} + {}^t \nabla \theta)^{-1} \in W^{2,\infty}(\overline{\Omega_\delta}, \mathcal{M}_{d,d}), \\ b_J(\theta, \mathbf{u}, \mathbf{v}, \mathbf{w}) &:= \sum_{i,j=1}^d \int_{\Omega \setminus \overline{\omega}} u_i [\nabla v_j \cdot (\partial_{x_i} ((\mathbf{I} + \theta)^{-1}) \circ (\mathbf{I} + \theta))] w_j J_\theta. \end{aligned}$$

Lemma I.2.3 (Differentiability of $\theta \mapsto (v_\theta, q_\theta)$). *The function*

$$\theta \in \mathcal{U} \mapsto (v_\theta, q_\theta) \in \mathbf{X}^{1,2}(\Omega \setminus \bar{\omega}, \Omega_\delta \setminus \bar{\omega}) \times \mathbf{X}^{0,1}(\Omega \setminus \bar{\omega}, \Omega_\delta \setminus \bar{\omega})$$

is differentiable in a neighborhood of $\mathbf{0}$ (and even C^∞).

Remark I.2.4. *This lemma implies that there exists a constant $c > 0$ such that*

$$\|u_\theta \circ (\mathbf{I} + \theta) - u\|_{\mathbf{X}^{1,2}(\Omega \setminus \bar{\omega}, \Omega_\delta \setminus \bar{\omega})} + \|p_\theta \circ (\mathbf{I} + \theta) - p\|_{\mathbf{X}^{0,1}(\Omega \setminus \bar{\omega}, \Omega_\delta \setminus \bar{\omega})} \leq c \|\theta\|_{\mathcal{U}}.$$

We then prove the following lemma in exactly the same way than the analogous Lemma in the Stokes case (see Lemma I.1.4).

Lemma I.2.5 (Differentiability of $\theta \mapsto (u_\theta, p_\theta)$). *There exists $\tilde{u}_\theta, \tilde{p}_\theta$ some respective extensions in Ω of $u_\theta \in \mathbf{H}^1(\Omega \setminus \bar{\omega}_\theta), p_\theta \in L^2(\Omega \setminus \bar{\omega}_\theta)$ such that the functions*

$$\theta \in \mathcal{U} \mapsto \tilde{u}_\theta \in \mathbf{H}^1(\Omega) \quad \text{and} \quad \theta \in \mathcal{U} \mapsto \tilde{p}_\theta \in L^2(\Omega)$$

are differentiable at $\mathbf{0}$ (and even C^1 in a neighborhood of $\mathbf{0}$).

Remark I.2.6. *We will prove these three lemmas under weaker assumptions: ω with a $C^{1,1}$ boundary and $\theta \in \mathbf{W}^{2,\infty}(\mathbb{R}^d)$.*

Higher order differentiability Obvious adaptation of the proof of the first order differentiability (see Lemma I.2.3) with

$$\begin{aligned} E_1 &:= \{ (v, q) \in \mathbf{H}_0^1(\Omega \setminus \bar{\omega}) \times L^2(\Omega \setminus \bar{\omega}); (\Phi v, \Phi q) \in \mathbf{H}^3(\Omega \setminus \bar{\omega}) \times \mathbf{H}^2(\Omega \setminus \bar{\omega}) \}, \\ E_2 &:= \{ f \in \mathbf{H}^{-1}(\Omega \setminus \bar{\omega}); \Phi f \in \mathbf{H}^1(\Omega \setminus \bar{\omega}) \}, \\ E_3 &:= \left\{ \psi \in L^2(\Omega \setminus \bar{\omega}), \Phi \psi \in \mathbf{H}^2(\Omega \setminus \bar{\omega}), \int_{\Omega \setminus \bar{\omega}} \psi = 0 \right\}, \end{aligned}$$

permits to prove the following lemma which is used to check that (u, p) is twice differentiable with respect to the shape:

Lemma I.2.7 (Higher order differentiability of $\theta \mapsto (v_\theta, q_\theta)$). *The function*

$$\theta \in \mathcal{U} \mapsto (v_\theta, q_\theta) \in \mathbf{X}^{1,3}(\Omega \setminus \bar{\omega}, \Omega_\delta \setminus \bar{\omega}) \times \mathbf{X}^{0,2}(\Omega \setminus \bar{\omega}, \Omega_\delta \setminus \bar{\omega})$$

is twice differentiable in a neighborhood of $\mathbf{0}$ (and even C^∞).

Remark I.2.8. *This lemma implies that there exists a constant $c > 0$ such that*

$$\|u_\theta \circ (\mathbf{I} + \theta) - u\|_{\mathbf{X}^{1,3}(\Omega \setminus \bar{\omega}, \Omega_\delta \setminus \bar{\omega})} + \|p_\theta \circ (\mathbf{I} + \theta) - p\|_{\mathbf{X}^{0,2}(\Omega \setminus \bar{\omega}, \Omega_\delta \setminus \bar{\omega})} \leq c \|\theta\|_{\mathcal{U}}.$$

We then prove the following lemma in exactly the same way than the analogous Lemma in the Stokes case (see Lemma I.1.8).

Lemma I.2.9 (Second order shape differentiability). *The solution (u, p) is twice differentiable with respect to the domain.*

Differentiability result for the associated linearized Navier-Stokes problem. In order to be close to the problem considered in the sequel, we assume that there exists an open subset $\tilde{O} \subset \partial\Omega$ of class $C^{1,1}$ such that $O \subset\subset \tilde{O}$ and that $\mathbf{g}|_{\tilde{O}} \in \mathbf{H}^{3/2}(\tilde{O})$. Then, using a local regularity result similar to Theorem A.2.6, we check that $(\nu\partial_{\mathbf{n}}\mathbf{u} - p\mathbf{n})|_O \in \mathbf{H}^{1/2}(O)$ (because $(\mathbf{u}, p)$ belongs to $\mathbf{H}^2 \times \mathbf{H}^1$ in a neighborhood of O , since $\tilde{O}$ is $C^{1,1}$). Hence, the following problem (in particular the boundary condition) is well defined.

We want now to study the differentiability with respect to the shape of the solution of the linearized Navier-Stokes problem

$$\left\{ \begin{array}{ll} -\nu\Delta\mathbf{w} + {}^t\nabla\mathbf{u}\mathbf{w} - \nabla\mathbf{w}\mathbf{u} + \nabla\pi &= \mathbf{0} & \text{in } \Omega \setminus \bar{\omega} \\ \operatorname{div} \mathbf{w} &= 0 & \text{in } \Omega \setminus \bar{\omega} \\ \mathbf{w} &= 2m^2(\nu\partial_{\mathbf{n}}\mathbf{u} - p\mathbf{n} - \mathbf{f}_b) & \text{on } \partial\Omega \\ \mathbf{w} &= \mathbf{0} & \text{on } \partial\omega, \end{array} \right. \quad (\text{I.2.10})$$

where $\mathbf{u} \in \mathbf{H}^1(\Omega \setminus \bar{\omega})$ solves the Navier-Stokes equations (I.2.2) and with $\mathbf{f}_b \in \mathbf{H}^{1/2}(O)$. By Theorem A.2.7, this problem admits a unique solution $(\mathbf{w}, \pi) \in \mathbf{H}^1(\Omega \setminus \bar{\omega}) \times L^2(\Omega \setminus \bar{\omega})$ with the normalization condition:

$$\langle m^2(\nu\partial_{\mathbf{n}}\mathbf{w} - \pi\mathbf{n}), \mathbf{n} \rangle_O = 0,$$

since we assume that ν satisfies (I.2.7). In the same way as we defined Problem (I.2.8), we defined the previous problem (I.2.10) on $\Omega \setminus \bar{\omega}_\theta$ and we denote by $(\mathbf{w}_\theta, \pi_\theta)$ its solution such that

$$\langle m^2(\nu\partial_{\mathbf{n}}\mathbf{w}_\theta - \pi_\theta\mathbf{n}), \mathbf{n} \rangle_O = 0.$$

According to [14, Lemma 3.3], we consider $\mathbf{H} \in \mathbf{H}^1(\Omega)$ such that

$$\mathbf{H} = \nu\partial_{\mathbf{n}}\mathbf{u} - p\mathbf{n} - \mathbf{f}_b \text{ on } \partial\Omega, \quad \operatorname{div} \mathbf{H} = 0 \text{ in } \Omega \quad \text{and} \quad \mathbf{H} = \mathbf{0} \text{ in } \Omega_\delta.$$

Thus $(\mathbf{z}_\theta := \mathbf{w}_\theta - \mathbf{H}, \pi_\theta) \in \mathbf{H}_0^1(\Omega \setminus \bar{\omega}_\theta) \times L^2(\Omega \setminus \bar{\omega}_\theta)$ is such that for all $\varphi_\theta \in \mathbf{H}_0^1(\Omega \setminus \bar{\omega}_\theta)$ and for all $\xi_\theta \in L^2(\Omega \setminus \bar{\omega}_\theta)$

$$\left\{ \begin{array}{l} \int_{\Omega \setminus \bar{\omega}_\theta} \nu \nabla \mathbf{z}_\theta : \nabla \varphi_\theta + b_{\Omega \setminus \bar{\omega}_\theta}(\varphi_\theta, \mathbf{u}_\theta, \mathbf{z}_\theta) \\ - b_{\Omega \setminus \bar{\omega}_\theta}(\mathbf{u}_\theta, \mathbf{z}_\theta, \varphi_\theta) - \int_{\Omega \setminus \bar{\omega}_\theta} \pi_\theta \operatorname{div} \varphi_\theta = - \int_{\Omega \setminus \bar{\omega}_\theta} \nu \nabla \mathbf{H} : \nabla \varphi_\theta - b_{\Omega \setminus \bar{\omega}_\theta}(\varphi_\theta, \mathbf{u}_\theta, \mathbf{H}) \\ + b_{\Omega \setminus \bar{\omega}_\theta}(\mathbf{u}_\theta, \mathbf{H}, \varphi_\theta) \\ \int_{\Omega \setminus \bar{\omega}_\theta} \xi_\theta \operatorname{div} \mathbf{z}_\theta = 0 \\ \langle m^2(\nu\partial_{\mathbf{n}}\mathbf{z}_\theta - \pi_\theta\mathbf{n}), \mathbf{n} \rangle_O = - \langle \nu m^2 \partial_{\mathbf{n}} \mathbf{H}, \mathbf{n} \rangle_O. \end{array} \right.$$

We then proceed in the same way as the proof of the differentiability of the solution $(\mathbf{u}, p)$ of the Navier-Stokes to prove the following lemma:

Lemma I.2.10 (Differentiability of $\theta \mapsto (\mathbf{w}_\theta, \pi_\theta)$). *There exists $\tilde{\mathbf{w}}_\theta, \tilde{\pi}_\theta$ some respective extensions of $\mathbf{w}_\theta \in \mathbf{H}^1(\Omega \setminus \bar{\omega})$, $\pi_\theta \in L^2(\Omega \setminus \bar{\omega})$ such that the functions*

$$\theta \in \mathcal{U} \mapsto \tilde{\mathbf{w}}_\theta \in \mathbf{H}^1(\Omega) \quad \text{and} \quad \theta \in \mathcal{U} \mapsto \tilde{\pi}_\theta \in L^2(\Omega)$$

are differentiable at $\mathbf{0}$. In particular, the solution $(\mathbf{w}, \pi)$ is differentiable with respect to the domain.

We precise that, in the proof of this lemma, we define $\mathbf{F}$ in a similar way as in the proof of Lemma I.2.3 and have

$$\mathbf{D}_{(\mathbf{z}, \pi)} \mathbf{F}(\mathbf{0}, (\mathbf{z}, \pi))(\mathbf{v}, q) = \left(-\nu\Delta\mathbf{v} + {}^t\nabla\mathbf{u}\mathbf{v} - \nabla\mathbf{v}\mathbf{u} + \nabla q, \operatorname{div} \mathbf{v}, \langle m^2(\nu\partial_{\mathbf{n}}\mathbf{v} - q\mathbf{n}), \mathbf{n} \rangle_O \right).$$

We then use Remark A.2.9.

I.2.2 The proofs

Proof of Lemma I.2.2: characterization of $(\mathbf{v}_\theta, q_\theta)$. The main idea is to make the change of variables $x = (\mathbf{I} + \boldsymbol{\theta})y$ in the variational formulation (I.2.9). This proof is similar to the proof of the corresponding lemma I.1.1 in the Stokes case and we refer to it for details. We here only focus on the trilinear form $b(\cdot, \cdot, \cdot)$.

Let $\boldsymbol{\varphi} \in \mathbf{H}_0^1(\Omega \setminus \overline{\omega})$ and let us define $\boldsymbol{\varphi}_\theta := \boldsymbol{\varphi} \circ (\mathbf{I} + \boldsymbol{\theta})^{-1} \in \mathbf{H}_0^1(\Omega \setminus \overline{\omega_\theta})$. We have

$$\begin{aligned} b_{\Omega \setminus \overline{\omega_\theta}}(\mathbf{w}_\theta, \mathbf{w}_\theta, \boldsymbol{\varphi}_\theta) &= \sum_{i,j=1}^d \int_{\Omega \setminus \overline{\omega_\theta}} w_\theta^i \frac{\partial w_\theta^j}{\partial x_i} \varphi_\theta^j \\ &= \sum_{i,j=1}^d \int_{\Omega \setminus \overline{\omega_\theta}} (v_\theta^i \circ (\mathbf{I} + \boldsymbol{\theta})^{-1}) \left[\left(\nabla v_\theta^j \circ (\mathbf{I} + \boldsymbol{\theta})^{-1} \right) \cdot \frac{\partial((\mathbf{I} + \boldsymbol{\theta})^{-1})}{\partial x_i} \right] \\ &\quad (\varphi^j \circ (\mathbf{I} + \boldsymbol{\theta})^{-1}) \\ &= \sum_{i,j=1}^d \int_{\Omega \setminus \overline{\omega}} v_\theta^i \left[\nabla v_\theta^j \cdot \left(\frac{\partial((\mathbf{I} + \boldsymbol{\theta})^{-1})}{\partial x_i} \circ (\mathbf{I} + \boldsymbol{\theta}) \right) \right] \varphi^j J_\theta. \end{aligned}$$

We conclude as in the Stokes case using the fact that $\boldsymbol{\theta} \equiv \mathbf{0}$ in $\Omega \setminus \overline{\Omega_\delta}$ and $\mathbf{G} \equiv \mathbf{0}$ in Ω_δ . $\square$

Proof of Lemma I.2.3: differentiability of $\boldsymbol{\theta} \mapsto (\mathbf{v}_\theta, q_\theta)$. In this proof, we use the classical implicit functions theorem.

Step 1: notations. We need some additional tools: a fourth domain $\tilde{\Omega}_\delta$ which is an open set with a C^∞ boundary such that $\Omega_\delta \subset \subset \tilde{\Omega}_\delta \subset \subset \Omega$ and a truncation function $\Phi \in C_c^\infty(\tilde{\Omega}_\delta)$ such that $\Phi \equiv 1$ in Ω_δ . We define the spaces:

$$\begin{aligned} \mathbf{E}_1 &:= \{(\mathbf{v}, q) \in \mathbf{H}_0^1(\Omega \setminus \overline{\omega}) \times L^2(\Omega \setminus \overline{\omega}); (\Phi \mathbf{v}, \Phi q) \in \mathbf{H}^2(\Omega \setminus \overline{\omega}) \times H^1(\Omega \setminus \overline{\omega})\}, \\ \mathbf{E}_2 &:= \{\mathbf{f} \in \mathbf{H}^{-1}(\Omega \setminus \overline{\omega}); \Phi \mathbf{f} \in \mathbf{L}^2(\Omega \setminus \overline{\omega})\}, \\ E_3 &:= \left\{ \psi \in L^2(\Omega \setminus \overline{\omega}); \Phi \psi \in H^1(\Omega \setminus \overline{\omega}), \int_{\Omega \setminus \overline{\omega}} \psi = 0 \right\}. \end{aligned}$$

Note that $\mathbf{E}_1$, $\mathbf{E}_2$ and E_3 are Hilbert spaces with respective norms

$$\begin{aligned} \|(\mathbf{v}, q)\|_{\mathbf{E}_1}^2 &:= \|\mathbf{v}\|_{\mathbf{H}^1(\Omega \setminus \overline{\omega})}^2 + \|q\|_{L^2(\Omega \setminus \overline{\omega})}^2 + \|\Phi \mathbf{v}\|_{\mathbf{H}^2(\Omega \setminus \overline{\omega})}^2 + \|\Phi q\|_{H^1(\Omega \setminus \overline{\omega})}^2, \\ \|\mathbf{f}\|_{\mathbf{E}_2}^2 &:= \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega \setminus \overline{\omega})}^2 + \|\Phi \mathbf{f}\|_{\mathbf{L}^2(\Omega \setminus \overline{\omega})}^2, \\ \|\psi\|_{E_3}^2 &:= \|\psi\|_{L^2(\Omega \setminus \overline{\omega})}^2 + \|\Phi \psi\|_{H^1(\Omega \setminus \overline{\omega})}^2. \end{aligned}$$

Moreover, the following continuous embeddings hold: $\mathbf{E}_1 \hookrightarrow \mathbf{X}^{1,2}(\Omega \setminus \overline{\omega}, \Omega_\delta \setminus \overline{\omega}) \times X^{0,1}(\Omega \setminus \overline{\omega}, \Omega_\delta \setminus \overline{\omega})$, $\mathbf{E}_2 \hookrightarrow \mathbf{X}^{-1,0}(\Omega \setminus \overline{\omega}, \Omega_\delta \setminus \overline{\omega})$ and $E_3 \hookrightarrow X^{0,1}(\Omega \setminus \overline{\omega}, \Omega_\delta \setminus \overline{\omega})$. Using the notations introduced in Lemma I.2.2, we also define:

– $\mathbf{F}_1(\boldsymbol{\theta}, (\mathbf{v}, q)) \in \mathbf{H}^{-1}(\Omega \setminus \overline{\omega})$ by $\forall \boldsymbol{\varphi} \in \mathbf{H}_0^1(\Omega \setminus \overline{\omega})$,

$$\langle \mathbf{F}_1(\boldsymbol{\theta}, (\mathbf{v}, q)), \boldsymbol{\varphi} \rangle_{\Omega \setminus \overline{\omega}} := \int_{\Omega \setminus \overline{\omega}} [(\nu \nabla v A(\boldsymbol{\theta})) : \nabla \boldsymbol{\varphi} - q B(\boldsymbol{\theta}) : \nabla \boldsymbol{\varphi}],$$

– $\mathbf{F}_2(\boldsymbol{\theta}, (\mathbf{v}, q)) \in \mathbf{H}^{-1}(\Omega \setminus \overline{\omega})$ by $\forall \boldsymbol{\varphi} \in \mathbf{H}_0^1(\Omega \setminus \overline{\omega})$,

$$\langle \mathbf{F}_2(\boldsymbol{\theta}, (\mathbf{v}, q)), \boldsymbol{\varphi} \rangle_{\Omega \setminus \overline{\omega}} := b_J(\boldsymbol{\theta}, \mathbf{v}, \mathbf{v}, \boldsymbol{\varphi}) + b_{\Omega \setminus \overline{\omega}}(\mathbf{v}, \mathbf{G}, \boldsymbol{\varphi}) + b_{\Omega \setminus \overline{\omega}}(\mathbf{G}, \mathbf{v}, \boldsymbol{\varphi}),$$

– $\mathbf{F}_3 \in \mathbf{H}^{-1}(\Omega \setminus \bar{\omega})$ by $\forall \varphi \in \mathbf{H}_0^1(\Omega \setminus \bar{\omega})$,

$$\langle \mathbf{F}_3, \varphi \rangle_{\Omega \setminus \bar{\omega}} := \int_{\Omega \setminus \bar{\omega}} \nu \nabla \mathbf{G} : \nabla \varphi + b_{\Omega \setminus \bar{\omega}}(\mathbf{G}, \mathbf{G}, \varphi),$$

– $\bar{\mathbf{F}}(\boldsymbol{\theta}, (\mathbf{v}, q)) \in \mathbf{H}^{-1}(\Omega \setminus \bar{\omega})$ by $\forall \varphi \in \mathbf{H}_0^1(\Omega \setminus \bar{\omega})$,

$$\langle \bar{\mathbf{F}}(\boldsymbol{\theta}, (\mathbf{v}, q)), \varphi \rangle_{\Omega \setminus \bar{\omega}} := \langle \mathbf{F}_1(\boldsymbol{\theta}, (\mathbf{v}, q)) + \mathbf{F}_2(\boldsymbol{\theta}, (\mathbf{v}, q)) + \mathbf{F}_3, \varphi \rangle_{\Omega \setminus \bar{\omega}},$$

– $\mathbf{F} : \mathcal{U} \times \mathbf{E}_1 \rightarrow \mathbf{E}_2 \times \mathbf{E}_3 \times \mathbb{R}$ by

$$\mathbf{F}(\boldsymbol{\theta}, (\mathbf{v}, q)) = \left(\bar{\mathbf{F}}(\boldsymbol{\theta}, (\mathbf{v}, q)), \nabla \mathbf{v} : B(\boldsymbol{\theta}), \langle m^2(\nu \partial_{\mathbf{n}} \mathbf{v} - q \mathbf{n} - c_n \mathbf{n} + \nu \partial_{\mathbf{n}} \mathbf{G}), \mathbf{n} \rangle_O \right).$$

Note that $\mathbf{F}$ is well-defined. In particular $\nabla \mathbf{v} : B(\boldsymbol{\theta})$ belongs to E_3 because, using the change of variables $x = (\mathbf{I} + \boldsymbol{\theta})^{-1}y$, we check that

$$\int_{\Omega \setminus \bar{\omega}} \nabla \mathbf{v} : B(\boldsymbol{\theta}) = \int_{\Omega \setminus \bar{\omega}_{\bar{\theta}}} \operatorname{div}(\mathbf{v} \circ (\mathbf{I} + \boldsymbol{\theta})^{-1}) = 0,$$

since $\mathbf{v} \in \mathbf{H}_0^1(\Omega \setminus \bar{\omega})$ (and then $\mathbf{v} \circ (\mathbf{I} + \boldsymbol{\theta})^{-1} \in \mathbf{H}_0^1(\Omega \setminus \bar{\omega}_{\bar{\theta}})$).

Step 2: for $\boldsymbol{\theta} \equiv \mathbf{0}$. We have $\mathbf{F}(\mathbf{0}, (\mathbf{v}_0, q_0)) = \mathbf{F}(\mathbf{0}, (\mathbf{w}, p)) = (\mathbf{0}, 0, 0)$.

Step 3: differentiability of $\mathbf{F}$. In the same way as what is done in the Stokes case (see the proof of Lemma I.1.2), we prove that

$$(\boldsymbol{\theta}, (\mathbf{v}, q)) \in \mathcal{U} \times \mathbf{E}_1 \mapsto \bar{\mathbf{F}}(\boldsymbol{\theta}, (\mathbf{v}, q)) \in \mathbf{E}_2 \quad \text{and} \quad (\boldsymbol{\theta}, (\mathbf{v}, q)) \in \mathcal{U} \times \mathbf{E}_1 \mapsto \nabla \mathbf{v} : B(\boldsymbol{\theta}) \in E_3$$

are C^∞ . Moreover, the function $(\mathbf{v}, q) \in \mathbf{E}_1 \mapsto \langle m^2(\nu \partial_{\mathbf{n}} \mathbf{v} - q \mathbf{n} - c_n \mathbf{n} + \nu \partial_{\mathbf{n}} \mathbf{G}), \mathbf{n} \rangle_O \in \mathbb{R}$ is C^∞ . We then conclude that $\mathbf{F}$ is C^∞ .

Step 4: $D_{(\mathbf{w}, p)} \mathbf{F}(\mathbf{0}, (\mathbf{w}, p))$ is an isomorphism. In order to simplify the notations, we use $\mathbf{w}$ for $\mathbf{w}_0$ in the end of this proof. We compute for all $(\mathbf{v}, q) \in \mathbf{E}_1$

$$D_{(\mathbf{w}, p)} \mathbf{F}(\mathbf{0}, (\mathbf{w}, p))(\mathbf{v}, q) = \left(-\nu \Delta \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{w} + (\mathbf{w} \cdot \nabla) \mathbf{v} + \nabla q, \operatorname{div} \mathbf{v}, \langle m^2(\nu \partial_{\mathbf{n}} \mathbf{v} - q \mathbf{n}), \mathbf{n} \rangle_O \right).$$

Let $(\chi, \eta, r) \in \mathbf{E}_2 \times E_3 \times \mathbb{R}$. Let us prove there exists a unique pair $(\mathbf{v}, q) \in \mathbf{E}_1$ such that

$$\begin{cases} -\nu \Delta \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{w} + (\mathbf{w} \cdot \nabla) \mathbf{v} + \nabla q &= \chi & \text{in } \Omega \setminus \bar{\omega} \\ \operatorname{div} \mathbf{v} &= \eta & \text{in } \Omega \setminus \bar{\omega} \\ \mathbf{v} &= \mathbf{0} & \text{on } \partial \Omega \\ \mathbf{v} &= \mathbf{0} & \text{on } \partial \omega \\ \langle m^2(\nu \partial_{\mathbf{n}} \mathbf{v} - q \mathbf{n}), \mathbf{n} \rangle_O &= r. \end{cases} \quad (\text{I.2.11})$$

Note that the compatibility condition $\int_{\Omega \setminus \bar{\omega}} \eta = 0$ is satisfied because η belongs to E_3 . Moreover, since $\nu > c_b(\Omega \setminus \bar{\omega}) \|\mathbf{w}\|_{\mathbf{H}^1(\Omega \setminus \bar{\omega})}$ by Assumption (I.2.7), Theorem A.2.7 guarantees that there exists a unique solution $(\mathbf{v}, q) \in \mathbf{H}_0^1(\Omega \setminus \bar{\omega}) \times L^2(\Omega \setminus \bar{\omega})$ of Problem (I.2.11) and the local regularity Theorem A.2.8 proves $(\mathbf{v}, q)$ belongs to $\mathbf{H}^2(\Omega_\delta \setminus \bar{\omega}) \times H^1(\Omega_\delta \setminus \bar{\omega})$. More precisely, we obtain

$$(\Phi \mathbf{v}, \Phi q) \in \mathbf{H}^2(\Omega \setminus \bar{\omega}) \times H^1(\Omega \setminus \bar{\omega}),$$

by multiplying (I.2.11) by Φ (see Remark A.2.10). Note that assumptions of Theorem A.2.7 are satisfied since $\mathbf{w}|_{\Omega_\delta \setminus \bar{\omega}} = \mathbf{u}|_{\Omega_\delta \setminus \bar{\omega}} \in \mathbf{H}^2(\Omega_\delta \setminus \bar{\omega})$ by the local regularity Theorem A.2.6.

Step 5: conclusion. The implicit function Theorem applies and we know that there exists a C^∞ function $\boldsymbol{\theta} \in \mathcal{U} \mapsto (\mathbf{v}(\boldsymbol{\theta}), q(\boldsymbol{\theta})) \in \mathbf{E}_1$ such that $\mathbf{F}(\boldsymbol{\theta}, (\mathbf{v}(\boldsymbol{\theta}), q(\boldsymbol{\theta}))) = (\mathbf{0}, 0, 0)$ in a neighborhood of $\mathbf{0}$. Using the uniqueness of the solution of such a problem, we conclude using the fact that $\mathbf{E}_1$ is continuously embedded in $\mathbf{X}^{1,2}(\Omega \setminus \bar{\omega}, \Omega_\delta \setminus \bar{\omega}) \times X^{0,1}(\Omega \setminus \bar{\omega}, \Omega_\delta \setminus \bar{\omega})$. $\square$

Chapitre II

Analyse du problème de détection : identifiabilité, dérivation par rapport à la forme et instabilité

Dans ce chapitre, nous analysons plus concrètement le problème de détection d'un objet immergé dans un fluide visqueux dans le cas où le mouvement du fluide est régi par les équations de Stokes ou Navier-Stokes incompressibles stationnaires. Nous démontrons un résultat d'identifiabilité puis étudions ce problème inverse comme un problème d'optimisation géométrique : nous minimisons une fonctionnelle de forme à l'aide du calcul de son gradient de forme. La hessienne de forme est finalement caractérisée afin de montrer que ce problème est sévèrement mal posé : la fonctionnelle est dégénérée pour les hautes fréquences.

Nous voulons ici reconstruire l'objet (ou les objets) supposé(s) immobile(s), c'est-à-dire détecter sa (leur) position et sa (leur) forme. L'obstacle ω est situé dans un domaine plus grand Ω dans lequel s'écoule un fluide visqueux, incompressible et stationnaire. Nous effectuons alors une mesure sur une partie O du bord extérieur (c'est-à-dire à la surface du fluide) et nous voulons reconstruire l'objet à partir de cette mesure. Par exemple, si des conditions de Dirichlet sont imposées sur le bord extérieur, nous mesurons les forces de Cauchy sur une partie de ce bord comme nous le résumons dans la figure II.1 ci-dessous. Pour reconstruire l'obstacle, nous définissons des fonctionnelles de forme

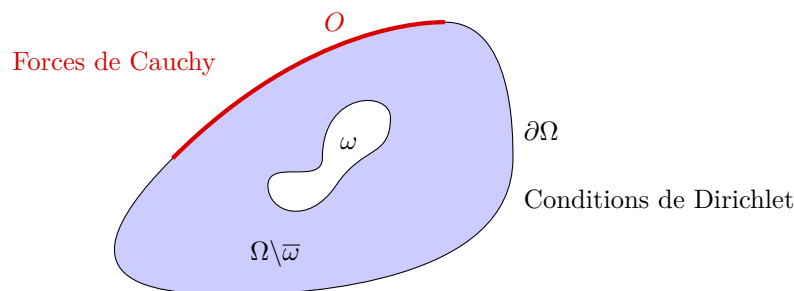


Figure II.1 – Configuration type de notre étude

coûts permettant d'évaluer l'erreur entre la solution réelle (au travers de la mesure effectuée) et la solution approchée. L'objectif est d'étudier ce problème inverse comme un problème de minimisation de fonctionnelle à l'aide d'un algorithme d'optimisation de descente utilisant leur gradient de forme afin de se rapprocher de la solution réelle (voir par exemple [3, 18, 54, 75, 76]). Nous travaillons ici avec deux types de fonctionnelles : une fonctionnelle de type moindres carrés et une de type Kohn-Vogelius.

Le premier résultat à étudier dans ce type de problème de reconstruction d'un obstacle à partir d'une mesure est un résultat d'identifiabilité : il faut garantir l'unicité de la solution en montrant que deux mesures identiques correspondent à deux objets identiques. Autrement dit, ce résultat nous assure que ce problème est bien posé au sens où il admet un unique minimum : on peut alors espérer reconstruire l'objet à partir de cette mesure du fait que cette dernière est liée de façon unique à l'obstacle inconnu.

Ensuite, nous nous intéressons à la résolution de ce problème inverse par minimisation d'une fonctionnelle de forme à l'aide du gradient de forme (voir par exemple [3, 18, 54, 75, 76]). Nous avons ici deux approches : une approche utilisant une fonctionnelle de type moindres carrés et une utilisant une fonctionnelle de type Kohn-Vogelius. Afin de minimiser ces fonctionnelles et de réaliser des simulations numériques, nous caractérisons leur gradient de forme.

Avant d'effectuer des tests numériques, nous étudions la stabilité du problème. Ainsi, nous caractérisons la Hessienne de forme des fonctionnelles considérées en un point critique. À l'aide de cette analyse d'ordre deux, nous montrons alors que ce problème est sévèrement mal posé et qu'une méthode de régularisation sera ainsi nécessaire pour le résoudre numériquement. En effet, nous montrons que le gradient de forme n'a pas une sensibilité uniforme par rapport aux directions de déformation : la fonctionnelle est dégénérée pour les hautes fréquences.

Pour finir, nous étudions plus concrètement la dégénérescence de la fonctionnelle. Pour cela, nous effectuons des calculs explicites de la Hessienne de forme dans le cas des équations de Stokes avec une approche de Kohn-Vogelius. Nous nous plaçons dans le cadre de géométries particulières, à savoir dans le cas de deux cercles concentriques

Ce chapitre se décompose en quatre parties. La première présente l'approche des moindres carrés dans le cas des équations de Stokes avec des conditions de Dirichlet puis des conditions de Neumann sur le bord intérieur $\partial\omega$. Les conditions de Neumann compliquent les démonstrations, notamment la dérivation du problème et de la fonctionnelle par rapport à la forme. La deuxième partie analyse l'approche utilisant la fonctionnelle de Kohn-Vogelius dans le cas des conditions de Dirichlet. Nous complétons celle-ci par des calculs explicites de la Hessienne de forme soulignant la dégénérescence de la fonctionnelle pour les hautes fréquences. La troisième partie traite des équations de Navier-Stokes avec des conditions de Dirichlet et une approche des moindres carrés. Nous suivons la même recette que dans le cas des équations de Stokes mais nous soulignons les difficultés qu'entraînent la non-linéarité. Finalement, nous terminons sur une courte remarque concernant l'instabilité du problème. Nous expliquons ce résultat en donnant des pistes pour réaliser les futures simulations numériques présentées au Chapitre III.

Le résultat d'identifiabilité dans le cadre des équations Stokes et Navier-Stokes (stationnaires et instationnaires) est dû à Alvarez et al. dans [6]. Il a également été traité par Doubova et al. dans [51]. Cependant, nous re-démontrons ce résultat en détaillant la preuve. En effet, les preuves faites dans [6] et [51] ne traitent pas le cas où $\omega_0 \setminus \overline{\omega_1}$ n'est pas Lipschitz et connexe (où ω_0 et ω_1 sont deux objets inclus dans Ω). Nous proposons donc ici une preuve complète en utilisant entre autres un résultat de densité pour le cas où $\omega_0 \setminus \overline{\omega_1}$ n'est pas Lipschitz. Nous soulignons également le fait que l'on peut mesurer $-\nu \partial_{\mathbf{n}} \mathbf{u} + p \mathbf{n}$ à la place de $\sigma(\mathbf{u}, p) \mathbf{n}$ (avec $(\mathbf{u}, p)$ solutions des équations de Stokes ou Navier-Stokes). Finalement, nous soulignons le fait que les mêmes techniques (à savoir le résultat de densité) ne peuvent pas être utilisées si l'on impose des conditions de Neumann sur l'objet. Dans, ce cas, la preuve complète (prenant en compte le cas où $\omega_0 \setminus \overline{\omega_1}$ n'est pas Lipschitz) que nous proposons ici nécessite de supposer que les objets ω sont plus réguliers (à savoir de classe $C^{1,1}$).

La dérivation par rapport au domaine des équations de Stokes (et Navier-Stokes) dans le cas des conditions de Dirichlet est classique (voir par exemple [19–22, 52, 57, 58, 73, 74, 92]). Cependant, nous verrons que les conditions de Neumann compliquent très largement celle-ci ainsi que la caractérisation du gradient. Nous soulignons également les différences entre les deux approches (celle des moindres carrés et celle de Kohn-Vogelius). En particulier, l'approche des moindres carrés nécessite

l'introduction d'un problème adjoint afin de faire apparaître explicitement la dépendance en la direction de perturbation $\mathbf{V}$ dans l'expression du gradient de forme. Cette dépendance explicite en $\mathbf{V}$ est importante en vue de réaliser des simulations numériques (voir Chapitre III).

L'analyse d'ordre deux d'un problème d'optimisation de forme est, à ma connaissance, peu étudié. Dans [92], Simon caractérise la dérivée d'ordre deux de la fonctionnelle coût dans le cadre de la minimisation de la traînée dans le cas des équations de Stokes. Nous pouvons également mentionner les travaux de Afraites et al. dans [3, 4] où ils montrent l'instabilité d'un problème de tomographie d'impédance électrique. Ils utilisent pour cela les opérateurs de simples et doubles couches. Le résultat d'instabilité de notre problème inverse est ici démontré à travers une analyse d'ordre deux en utilisant des arguments de régularité locale des solutions. Cette démonstration permet de mettre en avant le fait que plus l'objet est régulier, plus il sera difficile à détecter. Une preuve alternative pourrait être d'utiliser les opérateurs hydrodynamiques de simples et doubles couches (voir par exemple [77]).

Enfin, les calculs explicites de la Hessienne de forme dans le cas de deux cercles concentriques soulignent la dégénérescence exponentielle de la fonctionnelle. Ils permettent d'appréhender plus concrètement le caractère mal posé de ce problème inverse. Précisons que ces calculs ont, dans un premier temps, nécessité la résolution des équations de Stokes dans un anneau.

Nous nous intéressons ici à la dérivabilité d'ordre un et deux des solutions des problèmes de Stokes et Navier-Stokes stationnaires dans un ouvert borné lipschitzien connexe $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) par rapport à un domaine $\omega \subset \subset \Omega$. Afin de pouvoir montrer cette dérivabilité (d'ordre deux), nous introduisons un certain type de domaines perturbés appelés domaines admissibles. Pour $\delta > 0$ fixé (petit), on définit

$$\mathcal{O}_\delta := \left\{ \omega \subset \subset \Omega \text{ de classe } C^{2,1} \text{ tel que } d(x, \partial\Omega) > \delta \ \forall x \in \omega \text{ et tel que } \Omega \setminus \bar{\omega} \text{ est connexe} \right\}. \quad (\text{II.0.1})$$

Afin d'utiliser la régularité locale des solutions (des problèmes de Stokes et Navier-Stokes), nous définissons également un ouvert Ω_δ de classe C^∞ tel que

$$\{x \in \Omega; d(x, \partial\Omega) > \delta/2\} \subset \Omega_\delta \subset \{x \in \Omega; d(x, \partial\Omega) > \delta/3\}$$

(voir Figure II.2). Ainsi, on note

$$U := \{\theta \in \mathbf{W}^{3,\infty}(\mathbb{R}^d); \text{supp } \theta \subset \Omega_\delta\}$$

l'espace des déformations admissibles.

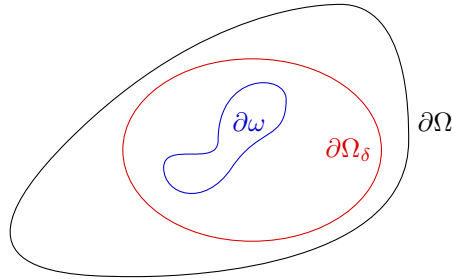


Figure II.2 – Le domaine Ω_δ

Remarque II.0.1. Les hypothèses de régularité présentées ici correspondent aux hypothèses concernant la dérivabilité par rapport à la forme d'ordre deux. Cependant, si l'on s'intéresse seulement à la dérivabilité d'ordre un, nous pouvons travailler avec $\theta \in \mathbf{W}^{2,\infty}(\mathbb{R}^d)$ et ω de classe $C^{1,1}$.

Chapitre II. Analyse du problème de détection : identifiabilité, dérivation par rapport à la forme et instabilité

Afin d'être plus précis, nous rappelons que nous utilisons la méthode des vitesses introduite par Murat et Simon en 1976 dans [84] afin de définir les dérivées de forme. Pour $\mathbf{V} \in \mathcal{U}$ et $t \in [0, T)$ (où $T > 0$ est un réel fixé suffisamment petit), on définit la fonction

$$\phi : t \in [0, T) \mapsto \mathbf{I} + t \mathbf{V} \in \mathbf{W}^{3,\infty}(\mathbb{R}^d). \quad (\text{II.0.2})$$

Remarquons que, pour t petit, $\phi(t)$ est un difféomorphisme de $\mathbb{R}^d$. Pour $t \in [0, T)$, on définit alors

$$\omega_t := \phi(t)(\omega).$$

Dans notre situation (i.e. pour $(\mathbf{u}, p)$ solution des équations de Stokes ou Navier-Stokes), on définit alors les dérivées de forme par rapport à l'obstacle $\omega \subset \subset \Omega$ comme suit (voir [72] pour plus de détails) :

- si l'application $\boldsymbol{\theta} \in \mathcal{U} \mapsto (\mathbf{u}_{\boldsymbol{\theta}} \circ (\mathbf{I} + \boldsymbol{\theta}), p_{\boldsymbol{\theta}} \circ (\mathbf{I} + \boldsymbol{\theta})) \in \mathbf{H}^1(\Omega \setminus \bar{\omega}) \times L^2(\Omega \setminus \bar{\omega})$ est Fréchet différentiable en $\mathbf{0}$, on dit que $\boldsymbol{\theta} \mapsto (\mathbf{u}_{\boldsymbol{\theta}}, p_{\boldsymbol{\theta}})$ possède une variation première totale (ou dérivée première totale) en $\mathbf{0}$. Dans ce cas, cette dérivée première totale en $\mathbf{0}$ dans la direction $\boldsymbol{\theta}$ est notée $(\dot{\mathbf{u}}_{\boldsymbol{\theta}}, \dot{p}_{\boldsymbol{\theta}})$;
- si, pour tout $\mathcal{D} \subset \subset \Omega \setminus \bar{\omega}$, l'application $\boldsymbol{\theta} \in \mathcal{U} \mapsto (\mathbf{u}_{\boldsymbol{\theta}|_{\mathcal{D}}}, p_{\boldsymbol{\theta}|_{\mathcal{D}}}) \in \mathbf{H}^1(\mathcal{D}) \times L^2(\mathcal{D})$ est Fréchet différentiable en $\mathbf{0}$, on dit que $\boldsymbol{\theta} \mapsto (\mathbf{u}_{\boldsymbol{\theta}}, p_{\boldsymbol{\theta}})$ possède une variation première locale (ou dérivée première locale) en $\mathbf{0}$. Dans ce cas, cette dérivée première locale en $\mathbf{0}$ dans la direction $\mathbf{V} \in \mathcal{U}$ est notée $(\mathbf{u}'_{\mathbf{V}}, p'_{\mathbf{V}})$ et est bien définie dans le domaine entier $\Omega \setminus \bar{\omega}$:

$$(\mathbf{u}'_{\mathbf{V}}, p'_{\mathbf{V}}) = \frac{d}{dt}(\mathbf{u}_{t\mathbf{V}|_{\mathcal{D}}}, p_{t\mathbf{V}|_{\mathcal{D}}})|_{t=0} \quad \text{dans tout } \mathcal{D} \subset \subset \Omega \setminus \bar{\omega}.$$

On définit de manière similaire les dérivées de forme d'ordres supérieurs. Dans la suite, pour $\mathbf{V} \in \mathcal{U}$, on note $(\mathbf{u}', p')$ la dérivée première locale $(\mathbf{u}'_{\mathbf{V}}, p'_{\mathbf{V}})$ qui sera appelée dérivée de forme de l'état. Pour des détails concernant la dérivation par rapport au domaine, nous renvoyons aux papiers de Simon [90, 91] et aux livres de Henrot et al. [72] ou de Sokołowski et al. [94].

Pour finir cette introduction, nous précisons que, dans la suite, $\mathbf{V}$ est nommée la direction de perturbation et on note

$$V_n := \mathbf{V} \cdot \mathbf{n}.$$

Remarquons que $\phi'(0) = \mathbf{V}$ s'annule sur $\partial\Omega$ et même dans le voisinage tubulaire $\Omega \setminus \bar{\Omega}_{\delta}$ de $\partial\Omega$. Précisons également que, dans la suite, l'indice " t " indiquera que la quantité est définie sur le domaine dépendant du pseudo-temps t . Par exemple, $\mathbf{n}_t$ est la normale unitaire extérieure à $\Omega \setminus \bar{\omega}_t$. De plus, nous définissons, pour $(\mathbf{u}, p) \in \mathbf{H}^1 \times L^2$ et $\nu > 0$ représentant le vitesse cinématique du fluide considéré, le tenseur des contraintes suivant :

$$\sigma(\mathbf{u}, p) := \nu(\nabla \mathbf{u} + {}^t\nabla \mathbf{u}) - p\mathbf{I},$$

où ${}^t\nabla \mathbf{u}$ représente la matrice transposée de $\nabla \mathbf{u}$. Enfin, nous précisons que les opérateurs différentiels tangentiels seront notés par un indice $_{\Gamma}$. En particulier, pour $\mathbf{w} \in \mathbf{W}^{1,1}(\partial\Omega)$ et $W \in W^{1,1}(\partial\Omega, \mathcal{M}_{d,d})$, les opérateurs suivants sont définis sur $\partial\Omega$:

- $\nabla_{\Gamma} \mathbf{w} := \nabla \mathbf{w} - (\nabla \mathbf{w} \mathbf{n}) \otimes \mathbf{n}$, où $\otimes$ représente le produit tensoriel ;
- $\text{div}_{\Gamma} W := \text{div } W - (\nabla W \mathbf{n}) \mathbf{n}$;
- $\Delta_{\Gamma} \mathbf{w} := \text{div}_{\Gamma}(\nabla_{\Gamma} \mathbf{w})$, l'opérateur Laplace-Beltrami.

Pour plus de détails sur des opérateurs différentiels tangentiels, nous renvoyons par exemple à [72, Section 5.4.3].

II.1 A least squares approach for the Stokes case

II.1.1 The results

II.1.1.1 The Dirichlet case

The problem setting Let O be a non-empty open subset of $\partial\Omega$. We assume that there exists an open subset $\tilde{O} \subset \partial\Omega$ of class $C^{1,1}$ such that $O \subset\subset \tilde{O}$. Let $\mathbf{g} \in \mathbf{H}^{1/2}(\partial\Omega)$ such that $\mathbf{g} \neq \mathbf{0}$, $\mathbf{g}|_{\tilde{O}} \in \mathbf{H}^{3/2}(\tilde{O})$ and satisfying the following condition:

$$\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n} = 0. \quad (\text{II.1.1})$$

Let us consider, for $\omega \in \mathcal{O}_\delta$ (with $\mathcal{O}_\delta$ defined by (II.0.1)) and a boundary measurement $\mathbf{f}_b \in \mathbf{H}^{1/2}(O)$, the following overdetermined Stokes boundary values problem:

$$\left\{ \begin{array}{ll} -\operatorname{div}(\sigma(\mathbf{u}, p)) &= \mathbf{0} \quad \text{in } \Omega \setminus \bar{\omega} \\ \operatorname{div} \mathbf{u} &= 0 \quad \text{in } \Omega \setminus \bar{\omega} \\ \mathbf{u} &= \mathbf{g} \quad \text{on } \partial\Omega \\ \mathbf{u} &= \mathbf{0} \quad \text{on } \partial\omega \\ \sigma(\mathbf{u}, p)\mathbf{n} &= \mathbf{f}_b \quad \text{on } O. \end{array} \right. \quad (\text{II.1.2})$$

We recall that, since $\operatorname{div} \mathbf{u} = 0$ in $\Omega \setminus \bar{\omega}$,

$$-\operatorname{div}(\sigma(\mathbf{u}, p)) = -\nu \Delta \mathbf{u} + \nabla p, \quad \text{in } \Omega,$$

and we will use indifferently both expressions.

We assume here that there exists $\omega \in \mathcal{O}_\delta$ such that (II.1.2) has a solution. This means that the measurement $\mathbf{f}_b$ is perfect, that is to say without error. Thus, we consider the following geometric inverse problem:

$$\text{find } \omega \in \mathcal{O}_\delta \text{ and a pair } (\mathbf{u}, p) \text{ which satisfies the overdetermined system (II.1.2).} \quad (\text{II.1.3})$$

To solve this inverse problem, we consider, for $\omega \in \mathcal{O}_\delta$, the least squares functional

$$J_D(\omega) := \frac{1}{2} \int_O m^2 |\sigma(\mathbf{u}(\omega), p(\omega)) \mathbf{n} - \mathbf{f}_b|^2,$$

where the function $m \in C_c^\infty(\partial\Omega)$ is such that $\operatorname{supp}(m) = \bar{O}$ and $(\mathbf{u}(\omega), p(\omega)) \in \mathbf{H}^1(\Omega \setminus \bar{\omega}) \times L^2(\Omega \setminus \bar{\omega})$ is a solution of the Stokes problem

$$\left\{ \begin{array}{ll} -\operatorname{div}(\sigma(\mathbf{u}, p)) &= \mathbf{0} \quad \text{in } \Omega \setminus \bar{\omega} \\ \operatorname{div} \mathbf{u} &= 0 \quad \text{in } \Omega \setminus \bar{\omega} \\ \mathbf{u} &= \mathbf{g} \quad \text{on } \partial\Omega \\ \mathbf{u} &= \mathbf{0} \quad \text{on } \partial\omega. \end{array} \right. \quad (\text{II.1.4})$$

Since we imposed the compatibility condition (II.1.1), problem (II.1.4) has a unique solution once a normalization condition on the pressure p is imposed (see for example [97, Chapter 1]). This solution will be called the state. Here, we choose the normalization

$$\int_O m^2 (\sigma(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{n} = \int_O m^2 (\mathbf{f}_b \cdot \mathbf{n}). \quad (\text{II.1.5})$$

Then, we try to minimize the least squares criterion J_D :

$$\omega^* = \operatorname{argmin}_{\omega \in \mathcal{O}_\delta} J_D(\omega). \quad (\text{II.1.6})$$

Indeed, if ω^* is solution of the inverse problem (II.1.3), then $J(\omega^*) = 0$ and (II.1.6) holds. Conversely, if ω^* solves (II.1.6) with $J(\omega^*) = 0$, then this domain ω^* is a solution of the inverse problem.

We refer, for example, to [13, 31, 44, 56, 61, 97], for the results of existence, uniqueness and regularity of the solution of the Stokes system with Dirichlet boundary conditions. In particular, a local regularity result can be found in [56, Theorem IV.5.1].

Remark II.1.1. Notice that using a local regularity result, we check that $\sigma(\mathbf{u}, p)\mathbf{n}|_O \in \mathbf{H}^{1/2}(O)$ (because $(\mathbf{u}, p)$ belongs to $\mathbf{H}^2 \times \mathbf{H}^1$ in a neighborhood of O , since $\tilde{O}$ is $C^{1,1}$ and $\mathbf{g}|_{\tilde{O}} \in \mathbf{H}^{3/2}(\tilde{O})$). Hence, the functional J_D and the integrals in (II.1.5) are well defined.

Identifiability result We first quote an identifiability result in the Dirichlet case (see [6, Theorem 1.2]) proved by Alvarez *et al.* (see also [51, Theorem 1.2] proved by Doubova *et al.*): it states that given a fixed $\mathbf{g}$, two different geometries ω_0 and ω_1 in $\mathcal{O}_\delta$ yield two different measures $\mathbf{f}_{b1}$ and $\mathbf{f}_{b2}$. Hence Problem (II.1.3) admits a unique solution.

Theorem II.1.2 (Alvarez *et al.* [6]). Let $\Omega \subseteq \mathbb{R}^d$, $d = 2$ or $d = 3$, be a bounded domain with a $C^{1,1}$ boundary, and O a non-empty open subset of $\partial\Omega$. Let

$$\omega_0, \omega_1 \in \mathcal{D}_{\text{ad}} := \{\omega \subset \subset \Omega; \omega \text{ is open, Lipschitz and } \Omega \setminus \bar{\omega} \text{ is connected}\}$$

and $\mathbf{g} \in \mathbf{H}^{3/2}(\partial\Omega)$ with $\mathbf{g} \neq \mathbf{0}$, satisfying the flux condition (II.1.1). For $\epsilon_* = 0$ or $\epsilon_* = 1$, let $(\mathbf{u}_j, p_j)$ for $j = 0, 1$, be a solution of

$$\begin{cases} -\operatorname{div}(\sigma(\mathbf{u}_j, p_j)) + \epsilon_* \operatorname{div}(\mathbf{u}_j \otimes \mathbf{u}_j) &= \mathbf{0} & \text{in } \Omega \setminus \bar{\omega}_j \\ \operatorname{div} \mathbf{u}_j &= 0 & \text{in } \Omega \setminus \bar{\omega}_j \\ \mathbf{u}_j &= \mathbf{g} & \text{on } \partial\Omega \\ \mathbf{u}_j &= \mathbf{0} & \text{on } \partial\omega_j. \end{cases}$$

Assume that $(\mathbf{u}_j, p_j)$ are such that

$$\sigma(\mathbf{u}_j, p_j) \mathbf{n} = \sigma(\mathbf{u}_j, p_j) \mathbf{n} \quad \text{on } O.$$

Then $\omega_0 \equiv \omega_1$.

Notice that this result is true in the Stokes case $\epsilon_* = 0$ only assuming Ω Lipschitz and $\mathbf{g} \in \mathbf{H}^{1/2}(\partial\Omega)$ (see the proof of the identifiability result in the Neumann case).

Sensitivity with respect to the domain Secondly, we aim to make a sensitivity (with respect to the shape) analysis. The Stokes problem on $\Omega \setminus \bar{\omega}_t$

$$\begin{cases} -\operatorname{div}(\sigma(\mathbf{u}_t, p_t)) &= \mathbf{0} & \text{in } \Omega \setminus \bar{\omega}_t \\ \operatorname{div} \mathbf{u}_t &= 0 & \text{in } \Omega \setminus \bar{\omega}_t \\ \mathbf{u}_t &= \mathbf{g} & \text{on } \partial\Omega \\ \mathbf{u}_t &= \mathbf{0} & \text{on } \partial\omega_t, \end{cases} \quad (\text{II.1.7})$$

admits a unique solution $(\mathbf{u}_t, p_t) \in \mathbf{H}^1(\Omega \setminus \bar{\omega}_t) \times L^2(\Omega \setminus \bar{\omega}_t)$ satisfying the normalization condition

$$\int_O m^2(\sigma(\mathbf{u}_t, p_t)\mathbf{n}) \cdot \mathbf{n} = \int_O m^2(\mathbf{f}_b \cdot \mathbf{n}).$$

Proposition II.1.3 (First order shape derivatives of the state). *The solution $(\mathbf{u}, p)$ is differentiable with respect to the domain. Moreover, the derivatives $\mathbf{u}'$ and p' belong to $\mathbf{H}^2(\Omega_\delta \setminus \overline{\omega})$ and to $H^1(\Omega_\delta \setminus \overline{\omega})$. The pair $(\mathbf{u}', p') \in \mathbf{H}^1(\Omega \setminus \overline{\omega}) \times L^2(\Omega \setminus \overline{\omega})$ is the only solution of the following boundary values problem*

$$\begin{cases} -\operatorname{div}(\sigma(\mathbf{u}', p')) &= \mathbf{0} & \text{in } \Omega \setminus \overline{\omega} \\ \operatorname{div} \mathbf{u}' &= 0 & \text{in } \Omega \setminus \overline{\omega} \\ \mathbf{u}' &= \mathbf{0} & \text{on } \partial\Omega \\ \mathbf{u}' &= -\partial_{\mathbf{n}} \mathbf{u} V_n & \text{on } \partial\omega, \end{cases} \quad (\text{II.1.8})$$

with the normalization condition

$$\int_O m^2(\sigma(\mathbf{u}', p') \mathbf{n}) \cdot \mathbf{n} = 0.$$

Proposition II.1.4 (First order shape derivatives of the functional). *For $\mathbf{V}$ in $\mathbf{U}$, the least squares functional J_D is differentiable at ω in the direction $\mathbf{V}$ with*

$$D J_D(\omega) \cdot \mathbf{V} = - \int_{\partial\omega} [(\sigma(\mathbf{w}, q) \mathbf{n}) \cdot \partial_{\mathbf{n}} \mathbf{u}] V_n,$$

where $(\mathbf{w}, q) \in \mathbf{H}^1(\Omega \setminus \overline{\omega}) \times L^2(\Omega \setminus \overline{\omega})$ is the solution of the following Stokes boundary values problem:

$$\begin{cases} -\operatorname{div}(\sigma(\mathbf{w}, q)) &= \mathbf{0} & \text{in } \Omega \setminus \overline{\omega} \\ \operatorname{div} \mathbf{w} &= 0 & \text{in } \Omega \setminus \overline{\omega} \\ \mathbf{w} &= 2m^2(\sigma(\mathbf{u}, p) \mathbf{n} - \mathbf{f}_b) & \text{on } \partial\Omega \\ \mathbf{w} &= \mathbf{0} & \text{on } \partial\omega, \end{cases} \quad (\text{II.1.9})$$

with the normalization condition

$$\int_O m^2(\sigma(\mathbf{w}, q) \mathbf{n}) \cdot \mathbf{n} = 0.$$

Remark II.1.5. *The differentiability with respect to the domain of $\mathbf{u}$ and p as well as Proposition II.1.4 remain true under weaker assumptions. Indeed, the proof is still valid with $\mathbf{V} \in \mathbf{W}^{2,\infty}(\mathbb{R}^d)$ and with ω with a $C^{1,1}$ boundary (but in this case, the expression of $D J_D(\omega) \cdot \mathbf{V}$ has to be seen as a duality product $H^{-1/2} \times H^{1/2}$).*

Second order analysis: justification of the instability Finally, we want to study the stability of the optimization problem (II.1.6) at a critical point $\omega^* \in \mathcal{O}_\delta$ (with $\mathcal{O}_\delta$ defined by (II.0.1)).

Proposition II.1.6 (Characterization of the shape Hessian at a critical shape). *The solution $(\mathbf{u}, p)$ is twice differentiable with respect to the domain. Moreover, for $\mathbf{V} \in \mathbf{U}$, we have*

$$D^2 J_D(\omega^*) \cdot \mathbf{V} \cdot \mathbf{V} = \int_{\partial\omega^*} -(\sigma(\mathbf{w}', q') \mathbf{n}) \cdot \partial_{\mathbf{n}} \mathbf{u} V_n,$$

where $(\mathbf{w}', q') \in \mathbf{H}^1(\Omega \setminus \overline{\omega^*}) \times L^2(\Omega \setminus \overline{\omega^*})$ is the solution of the following problem:

$$\begin{cases} -\operatorname{div}(\sigma(\mathbf{w}', q')) &= \mathbf{0} & \text{in } \Omega \setminus \overline{\omega^*} \\ \operatorname{div} \mathbf{w}' &= 0 & \text{in } \Omega \setminus \overline{\omega^*} \\ \mathbf{w}' &= 2m^2 \sigma(\mathbf{u}', p') \mathbf{n} & \text{on } \partial\Omega \\ \mathbf{w}' &= \mathbf{0} & \text{on } \partial\omega^*, \end{cases} \quad (\text{II.1.10})$$

with the normalization condition

$$\int_O m^2(\sigma(\mathbf{w}', q') \mathbf{n}) \cdot \mathbf{n} = 0.$$

Proposition II.1.7 (Compactness at a critical point). *The Riesz operator corresponding to $D^2 J_D(\omega^*)$ defined from $\mathbf{H}^{1/2}(\partial\omega^*)$ to $\mathbf{H}^{-1/2}(\partial\omega^*)$ is compact.*

Remark II.1.8. *We recall that we refer to [56, Theorem IV-5-1] for the local regularity result for the solutions in the Dirichlet case. This point is crucial for proving Proposition II.1.7 with the method we adopt here.*

II.1.1.2 The Neumann case

The problem setting We mimic the study of the Dirichlet case. Let O be a non-empty open subset of $\partial\Omega$. Let $\mathbf{f} = (f_1, \dots, f_d) \in [\mathbf{H}^1(\mathbb{R}^d)]'$ such that $\mathbf{f} \neq \mathbf{0}$, $\mathbf{f} \equiv \mathbf{0}$ in Ω_δ and satisfying the condition:

$$\langle f_i, 1 \rangle_\Omega = 0 \quad \forall i = 1, \dots, d. \quad (\text{II.1.11})$$

For $\omega \in \mathcal{O}_\delta$ (with $\mathcal{O}_\delta$ defined by (II.0.1)) and a boundary measurement $\mathbf{f}_b \in \mathbf{H}^{1/2}(O)$, we consider the overdetermined Stokes boundary values problem:

$$\left\{ \begin{array}{ll} -\nu \Delta \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega \setminus \bar{\omega} \\ \operatorname{div} \mathbf{u} &= 0 \quad \text{in } \Omega \setminus \bar{\omega} \\ -\nu \partial_{\mathbf{n}} \mathbf{u} + p \mathbf{n} &= \mathbf{0} \quad \text{on } \partial\Omega \\ -\nu \partial_{\mathbf{n}} \mathbf{u} + p \mathbf{n} &= \mathbf{0} \quad \text{on } \partial\omega \\ \mathbf{u} &= \mathbf{f}_b \quad \text{on } O. \end{array} \right. \quad (\text{II.1.12})$$

We assume here that there exists $\omega \in \mathcal{O}_\delta$ such that (II.1.12) has a solution. This means that the measurement $\mathbf{f}_b$ is perfect. Thus, we consider the geometric inverse problem:

$$\text{find } \omega \in \mathcal{O}_\delta \text{ and a pair } (\mathbf{u}, p) \text{ which satisfies the overdetermined system (II.1.12).} \quad (\text{II.1.13})$$

To study this inverse problem, we introduce, for $\omega \in \mathcal{O}_\delta$, the least squares functional

$$J(\omega) := \int_O |\mathbf{u}(\omega) - \mathbf{f}_b|^2,$$

where $(\mathbf{u}(\omega), p(\omega)) \in \mathbf{H}^1(\Omega \setminus \bar{\omega}) \times L^2(\Omega \setminus \bar{\omega})$ satisfies the Stokes boundary values problem

$$\left\{ \begin{array}{ll} -\nu \Delta \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega \setminus \bar{\omega} \\ \operatorname{div} \mathbf{u} &= 0 \quad \text{in } \Omega \setminus \bar{\omega} \\ -\nu \partial_{\mathbf{n}} \mathbf{u} + p \mathbf{n} &= \mathbf{0} \quad \text{on } \partial\Omega \\ -\nu \partial_{\mathbf{n}} \mathbf{u} + p \mathbf{n} &= \mathbf{0} \quad \text{on } \partial\omega. \end{array} \right. \quad (\text{II.1.14})$$

Since $\mathbf{f}$ satisfies the condition (II.1.11) and since $\mathbf{f} \equiv \mathbf{0}$ in Ω_δ , the compatibility condition of Problem (II.1.14) is automatically satisfied. According to Theorem A.1.2, it has a unique solution once a normalization condition on $\mathbf{u}$ is imposed. Here, we impose

$$\int_O \mathbf{u} = \int_O \mathbf{f}_b.$$

Then, we try to minimize the least squares criterion J :

$$\omega^* = \operatorname{argmin}_{\omega \in \mathcal{O}_\delta} J(\omega). \quad (\text{II.1.15})$$

Indeed, if ω^* is solution of the inverse problem (II.1.13), then $J(\omega^*) = 0$ and (II.1.15) holds. Conversely, if ω^* solves (II.1.15) with $J(\omega^*) = 0$, then this domain ω^* is a solution of the inverse problem.

The results of existence, uniqueness and regularity of the solution of the Stokes system with Neumann boundary conditions are recalled in Appendix A.1.

Remark II.1.9. In this Neumann case, we can also consider a non-homogeneous Neumann data of the kind $-\nu \partial_{\mathbf{n}} \mathbf{u} + p \mathbf{n} = \mathbf{g}$ on $\partial\Omega$ instead of/ or in addition to the force $\mathbf{f}$. In the Dirichlet case, we can also consider a force $\mathbf{f}$ with homogeneous Dirichlet data on $\partial\Omega$. These modifications do not change the conclusions of the work but complicate the notations and computations.

Identifiability result We first present a new identifiability result for the Neumann case.

Theorem II.1.10 (Identifiability result). *Let $\Omega \subseteq \mathbb{R}^d$, $d = 2$ or $d = 3$, be a bounded Lipschitz domain and O a non-empty open subset of $\partial\Omega$. Let*

$$\omega_0, \omega_1 \in \{\omega \subset\subset \Omega; \omega \text{ is a } C^{1,1} \text{ open set and } \Omega \setminus \overline{\omega} \text{ is connected}\}$$

and $\mathbf{f} \in [\mathbf{H}^1(\Omega)]'$ with $\mathbf{f} \neq \mathbf{0}$ and $\mathbf{f} \equiv \mathbf{0}$ in Ω_δ satisfying the condition (II.1.11). Let $(\mathbf{u}_j, p_j)$, for $j = 0, 1$, be a solution of

$$\begin{cases} -\nu \Delta \mathbf{u}_j + \nabla p_j &= \mathbf{f} & \text{in } \Omega \setminus \overline{\omega_j} \\ \operatorname{div} \mathbf{u}_j &= 0 & \text{in } \Omega \setminus \overline{\omega_j} \\ -\nu \partial_{\mathbf{n}} \mathbf{u}_j + p_j \mathbf{n} &= \mathbf{0} & \text{on } \partial\Omega \\ -\nu \partial_{\mathbf{n}_j} \mathbf{u}_j + p_j \mathbf{n}_j &= \mathbf{0} & \text{on } \partial\omega_j. \end{cases}$$

Assume that $(\mathbf{u}_j, p_j)$ are such that $\mathbf{u}_0 = \mathbf{u}_1$ on O . Then $\omega_0 \equiv \omega_1$.

Sensitivity with respect to the domain. Secondly, we aim to make a sensitivity (with respect to the shape) analysis, as in the Dirichlet case. The Stokes problem on $\Omega \setminus \overline{\omega_t}$

$$\begin{cases} -\nu \Delta \mathbf{u}_t + \nabla p_t &= \mathbf{f} & \text{in } \Omega \setminus \overline{\omega_t} \\ \operatorname{div} \mathbf{u}_t &= 0 & \text{in } \Omega \setminus \overline{\omega_t} \\ -\nu \partial_{\mathbf{n}} \mathbf{u}_t + p_t \mathbf{n} &= \mathbf{0} & \text{on } \partial\Omega \\ -\nu \partial_{\mathbf{n}_t} \mathbf{u}_t + p_t \mathbf{n}_t &= \mathbf{0} & \text{on } \partial\omega_t, \end{cases} \quad (\text{II.1.16})$$

admits a unique solution $(\mathbf{u}_t, p_t) \in \mathbf{H}^1(\Omega \setminus \overline{\omega_t}) \times L^2(\Omega \setminus \overline{\omega_t})$ satisfying the normalization condition

$$\int_O \mathbf{u}_t = \int_O \mathbf{f}_b.$$

Here $\mathbf{n}_t$ denotes the exterior unit normal on $\partial\omega_t$ with respect to $\Omega \setminus \overline{\omega_t}$. Notice that the compatibility condition of problem (II.1.16) is automatically satisfied since $\mathbf{f}$ satisfies the condition (II.1.11) and since $\mathbf{f} \equiv \mathbf{0}$ in Ω_δ .

Proposition II.1.11 (First order shape derivatives of the state). *The solution $(\mathbf{u}, p)$ is differentiable with respect to the domain. Moreover, the shape derivatives $\mathbf{u}'$ and p' belong to $\mathbf{H}^2(\Omega_\delta \setminus \overline{\omega})$ and to $\mathbf{H}^1(\Omega_\delta \setminus \overline{\omega})$. The pair $(\mathbf{u}', p') \in \mathbf{H}^1(\Omega \setminus \overline{\omega}) \times L^2(\Omega \setminus \overline{\omega})$ is the only solution of the following boundary values problem*

$$\begin{cases} -\nu \Delta \mathbf{u}' + \nabla p' &= \mathbf{0} & \text{in } \Omega \setminus \overline{\omega} \\ \operatorname{div} \mathbf{u}' &= 0 & \text{in } \Omega \setminus \overline{\omega} \\ -\nu \partial_{\mathbf{n}} \mathbf{u}' + p' \mathbf{n} &= \mathbf{0} & \text{on } \partial\Omega \\ -\nu \partial_{\mathbf{n}} \mathbf{u}' + p' \mathbf{n} &= (\nu \partial_{\mathbf{nn}}^2 \mathbf{u} - \partial_{\mathbf{n}} p \mathbf{n}) V_{\mathbf{n}} + p \nabla_{\Gamma}(V_{\mathbf{n}}) - \nu \nabla \mathbf{u} \nabla_{\Gamma}(V_{\mathbf{n}}) & \text{on } \partial\omega, \end{cases} \quad (\text{II.1.17})$$

with the normalization condition

$$\int_O \mathbf{u}' = \mathbf{0}.$$

Remark II.1.12 (Meaning of high order derivatives on $\partial\omega$). *By the local regularity result A.1.4, $\mathbf{u} \in \mathbf{H}^3(\Omega_\delta \setminus \overline{\omega})$, $p \in \mathbf{H}^2(\Omega_\delta \setminus \overline{\omega})$, then the quantity $\nu \partial_{\mathbf{nn}}^2 \mathbf{u} - \partial_{\mathbf{n}} p \mathbf{n}$ belongs to $\mathbf{H}^{1/2}(\partial\omega)$.*

Proposition II.1.13 (First order shape derivatives of the functional). *For $\mathbf{V}$ in $\mathbf{U}$, the least squares functional J is differentiable at ω in the direction $\mathbf{V}$ with*

$$\mathrm{D} J(\omega) \cdot \mathbf{V} = \int_{\partial\omega} [(\nu \partial_{\mathbf{nn}}^2 \mathbf{u} - \partial_{\mathbf{n}} p \mathbf{n}) \cdot \mathbf{w} - \operatorname{div}_{\Gamma} ((p \mathbf{I} - \nu^t \nabla \mathbf{u}) \mathbf{w})] V_{\mathbf{n}},$$

where $(\mathbf{w}, q) \in \mathbf{H}^1(\Omega \setminus \bar{\omega}) \times L^2(\Omega \setminus \bar{\omega})$ is the solution of the following problem:

$$\begin{cases} -\nu \Delta \mathbf{w} + \nabla q &= \mathbf{0} & \text{in } \Omega \setminus \bar{\omega} \\ \operatorname{div} \mathbf{w} &= 0 & \text{in } \Omega \setminus \bar{\omega} \\ -\nu \partial_{\mathbf{n}} \mathbf{w} + q \mathbf{n} &= 2(\mathbf{u} - \mathbf{f}_b) \mathbb{1}_O & \text{on } \partial\Omega \\ -\nu \partial_{\mathbf{n}} \mathbf{w} + q \mathbf{n} &= \mathbf{0} & \text{on } \partial\omega, \end{cases} \quad (\text{II.1.18})$$

with the normalization condition

$$\int_O \mathbf{w} = \mathbf{0}.$$

Remark II.1.14 (Meaning of high order derivatives on $\partial\omega$). The expression of shape derivative involves high order derivatives of the couple $(\mathbf{u}, p)$ that have to be well defined. By the local regularity result A.1.4, $\mathbf{w} \in \mathbf{H}^3(\Omega_\delta \setminus \bar{\omega})$. It follows that $\mathbf{w} \in \mathbf{H}^{5/2}(\partial\omega)$ and $p\mathbf{I} - \nu^t \nabla \mathbf{u} \in \mathbf{H}^{3/2}(\partial\omega)$. Therefore, the function $(p\mathbf{I} - \nu^t \nabla \mathbf{u})\mathbf{w} \in \mathbf{W}^{1,1}(\partial\omega)$ has a tangential divergence on $\partial\omega$.

Moreover, using the Sobolev embeddings in dimension $d = 2$ or $d = 3$,

$$\mathbf{w} \in \mathbf{H}^3(\Omega_\delta \setminus \bar{\omega}) \hookrightarrow \mathbf{L}^\infty(\Omega_\delta \setminus \bar{\omega}) \quad \text{and} \quad \nabla \mathbf{w} \in \mathbf{H}^2(\Omega_\delta \setminus \bar{\omega}) \hookrightarrow \mathbf{L}^\infty(\Omega_\delta \setminus \bar{\omega}).$$

Thus, $\mathbf{w} \in \mathbf{W}^{2,\infty}(\Omega_\delta \setminus \bar{\omega})$, then $(p\mathbf{I} - \nu^t \nabla \mathbf{u})\mathbf{w} \in \mathbf{H}^1(\Omega_\delta \setminus \bar{\omega})$ and finally we get that $\operatorname{div}_\Gamma(p\mathbf{I} - \nu^t \nabla \mathbf{u})\mathbf{w}$ belongs to $\mathbf{H}^{-1/2}(\partial\omega)$.

Remark II.1.15. The differentiability with respect to the domain of $\mathbf{u}$ and p as well as Proposition II.1.13 remain true under weaker assumptions. Indeed, the proof is still valid with $\mathbf{V} \in \mathbf{W}^{2,\infty}(\mathbb{R}^d)$ and with ω with a $C^{1,1}$ boundary (but in this case, the expression of $\mathbf{D}J(\omega) \cdot \mathbf{V}$ has to be seen as a duality product $\mathbf{H}^{-1/2} \times \mathbf{H}^{1/2}$).

Second order analysis: justification of the instability. Finally, we study the stability of the optimization problem (II.1.15) at a critical point $\omega^* \in \mathcal{O}_\delta$ (with $\mathcal{O}_\delta$ defined by (II.0.1)).

Proposition II.1.16 (Characterization of the shape Hessian at a critical shape). The solution $(\mathbf{u}, p)$ is twice differentiable with respect to the domain. Moreover, for $\mathbf{V} \in \mathbf{U}$, we have

$$\mathbf{D}^2 J(\omega^*) \cdot \mathbf{V} \cdot \mathbf{V} = \int_{\partial\omega^*} [(\nu \partial_{\mathbf{nn}}^2 \mathbf{u} - \partial_{\mathbf{n}} p \mathbf{n}) V_{\mathbf{n}} + p \nabla_\Gamma(V_{\mathbf{n}}) - \nu \nabla \mathbf{u} \nabla_\Gamma(V_{\mathbf{n}})] \cdot \mathbf{w}',$$

where $(\mathbf{w}', q') \in \mathbf{H}^1(\Omega \setminus \bar{\omega}^*) \times L^2(\Omega \setminus \bar{\omega}^*)$ satisfies the following Stokes boundary values problem:

$$\begin{cases} -\nu \Delta \mathbf{w}' + \nabla q' &= \mathbf{0} & \text{in } \Omega \setminus \bar{\omega}^* \\ \operatorname{div} \mathbf{w}' &= 0 & \text{in } \Omega \setminus \bar{\omega}^* \\ -\nu \partial_{\mathbf{n}} \mathbf{w}' + q' \mathbf{n} &= 2 \mathbf{u}' \mathbb{1}_O & \text{on } \partial\Omega \\ -\nu \partial_{\mathbf{n}} \mathbf{w}' + q' \mathbf{n} &= \mathbf{0} & \text{on } \partial\omega^*, \end{cases} \quad (\text{II.1.19})$$

with the normalization condition

$$\int_O \mathbf{w}' = \mathbf{0}.$$

Proposition II.1.17 (Compactness at a critical point). The Riesz operator corresponding to $\mathbf{D}^2 J(\omega^*)$ defined from $\mathbf{H}^{1/2}(\partial\omega^*)$ to $\mathbf{H}^{-1/2}(\partial\omega^*)$ is compact.

II.1.2 The proofs

Now, we will prove the previous results. The derivation of the problem and of the functional in the Dirichlet case is quite classical, then we just give the sketch of the proofs. However, we detail the Neumann case which is more mathematically interesting.

II.1.2.1 The Dirichlet case

First order shape derivatives of the state. We consider the perturbed Stokes problem (II.1.7). The solution $(\mathbf{u}_t, p_t) \in \mathbf{H}^1(\Omega \setminus \overline{\omega_t}) \times L^2(\Omega \setminus \overline{\omega_t})$ are such that

$$\begin{cases} \int_{\Omega \setminus \overline{\omega_t}} \sigma(\mathbf{u}_t, p_t) : \nabla \boldsymbol{\varphi}_t = \mathbf{0} & \forall \boldsymbol{\varphi}_t \in \mathbf{H}_0^1(\Omega \setminus \overline{\omega_t}) \\ \int_{\Omega \setminus \overline{\omega_t}} \xi_t \operatorname{div} \mathbf{u}_t = 0 & \forall \xi_t \in L^2(\Omega \setminus \overline{\omega_t}) \\ \int_O m^2 (\sigma(\mathbf{u}_t, p_t) \mathbf{n}) \cdot \mathbf{n} = \int_O m^2 (\mathbf{f}_b \cdot \mathbf{n}). \end{cases} \quad (\text{II.1.20})$$

Proof of Proposition II.1.3. First step: first order shape differentiability. Using the Fréchet differentiability Lemma I.1.4, we obtain the Gâteaux differentiability in the direction $\mathbf{V}$: there exist $\mathbf{u}_t$ and p_t , respective extensions in Ω of $\mathbf{u}_t$ and p_t such that the functions, defined on $[0, T]$, $t \mapsto \mathbf{u}_t \in \mathbf{H}^1(\Omega)$ and $t \mapsto p_t \in L^2(\Omega)$ are differentiable at 0 by composition. We denote by $\mathbf{u}'$ and p' their respective derivative at 0.

Second step: derivative of the normalization condition. The function

$$\int_O m^2 (\sigma(\mathbf{u}_t, p_t) \mathbf{n}) \cdot \mathbf{n} = \int_O m^2 (\mathbf{f}_b \cdot \mathbf{n}).$$

is constant on $[0, T]$. Thus, its derivative is 0. By differentiation under the sum sign, we then get

$$\int_{\partial\Omega} m^2 (\sigma(\mathbf{u}', p') \mathbf{n}) \cdot \mathbf{n} = 0.$$

Third step: derivative of the equalities on $\Omega \setminus \overline{\omega_t}$. Let $\boldsymbol{\varphi} \in \mathcal{D}(\Omega \setminus \overline{\omega})$. As $\Omega \setminus \overline{\omega}$ is open, we have, for t small enough, $\boldsymbol{\varphi} \in \mathcal{D}(\Omega \setminus \overline{\omega_t})$. Therefore, using the first line of the variational formulation (II.1.20), we have for all $t \in [0, T]$

$$0 = \int_{\Omega \setminus \overline{\omega_t}} \sigma(\mathbf{u}_t, p_t) : \nabla \boldsymbol{\varphi} = \int_{\Omega \setminus \overline{\omega}} \sigma(\mathbf{u}_t, p_t) : \nabla \boldsymbol{\varphi}.$$

We differentiate with respect to t at $t = 0$ the equality $0 = \int_{\Omega \setminus \overline{\omega}} \sigma(\mathbf{u}_t, p_t) : \nabla \boldsymbol{\varphi}$ to obtain

$$0 = \int_{\Omega \setminus \overline{\omega}} \sigma(\mathbf{u}', p') : \nabla \boldsymbol{\varphi} = \langle -\operatorname{div} (\sigma(\mathbf{u}', p')), \boldsymbol{\varphi} \rangle_{\mathcal{D}'(\Omega \setminus \overline{\omega}), \mathcal{D}(\Omega \setminus \overline{\omega})}.$$

As it is true for all $\boldsymbol{\varphi} \in \mathcal{D}(\Omega \setminus \overline{\omega})$, we get

$$-\operatorname{div} (\sigma(\mathbf{u}', p')) = \mathbf{0} \quad \text{in } \mathcal{D}'(\Omega \setminus \overline{\omega}).$$

Proceeding with the second line of the variational formulation (II.1.20) as previously, we check that $\operatorname{div} \mathbf{u}' = 0$ in $\mathcal{D}'(\Omega \setminus \overline{\omega})$.

Fourth step: derivatives of the boundary values condition. We differentiate with respect to t the relation $\mathbf{u}_t = \mathbf{g}$ on $\partial\Omega$. At $t = 0$, we obtain on $\partial\Omega$ that $\mathbf{u}' = \mathbf{0}$.

Let us set $\mathbf{v}_t := \mathbf{u}_t \circ \phi(t)$. By differentiation with respect to t (see [72, Corollary 5.2.5]), we obtain $\mathbf{u}' = \mathbf{v}' - \nabla \mathbf{u} \mathbf{V}$ in $\Omega_\delta \setminus \overline{\omega}$. Since $\mathbf{v}_t = \mathbf{0}$ on $\partial\omega$ for all $t \in [0, T]$, $\mathbf{v}' = \mathbf{0}$ on $\partial\omega$ and $\mathbf{u}' = -\nabla \mathbf{u} \mathbf{V}$. Finally, since $\mathbf{u} = \mathbf{0}$ on $\partial\omega$, $\nabla \mathbf{u} = \partial_{\mathbf{n}} \mathbf{u} \otimes \mathbf{n}$, where $\otimes$ is the tensorial product, and $\mathbf{u}' = -\partial_{\mathbf{n}} \mathbf{u} V_n$. $\square$

First order shape derivatives of the functional. Define, for all $t \in [0, T]$,

$$J_D(\omega_t) := j_D(t) := \frac{1}{2} \int_O m^2 |\sigma(\mathbf{u}_t, p_t) \mathbf{n} - \mathbf{f}_b|^2.$$

Proof of Proposition II.1.4. First step: derivative of j_D and adjoint problem. By Proposition II.1.3, the solution $(\mathbf{u}, p)$ is differentiable with respect to the shape. We denote the shape derivative by $(\mathbf{u}', p')$. Differentiating j_D with respect to t at 0, we obtain

$$j'_D(0) = \int_O m^2 (\sigma(\mathbf{u}', p') \mathbf{n}) \cdot (\sigma(\mathbf{u}, p) \mathbf{n} - \mathbf{f}_b).$$

Then, we consider the adjoint problem (II.1.9). Since we choose the normalization condition (II.1.5), the compatibility condition of the adjoint problem is satisfied. Therefore it admits a unique solution $(\mathbf{w}, q) \in \mathbf{H}^1(\Omega \setminus \bar{\omega}) \times L^2(\Omega \setminus \bar{\omega})$ with $\langle \sigma(\mathbf{w}, q) \mathbf{n}, \mathbf{n} \rangle_O = 0$.

Second step: writing of $j'_D(0)$ as an integral on $\partial\omega$. We proceed by successive integrations by parts. We multiply the first equation of the adjoint problem (II.1.9) by $\mathbf{u}'$ to get:

$$\int_{\Omega \setminus \bar{\omega}} \nabla \mathbf{w} : \nabla \mathbf{u}' = - \langle -\sigma(\mathbf{w}, q) \mathbf{n}, \mathbf{u}' \rangle_{\partial(\Omega \setminus \bar{\omega})}, \quad (\text{II.1.21})$$

since $\text{div } \mathbf{u}' = 0$ in $\Omega \setminus \bar{\omega}$ (see Proposition II.1.3). Then, we multiply the first equation of Problem (II.1.8) by $\mathbf{w}$ to obtain

$$\int_{\Omega \setminus \bar{\omega}} \nabla \mathbf{u}' : \nabla \mathbf{w} = - \langle -\sigma(\mathbf{u}', p') \mathbf{n}, \mathbf{w} \rangle_{\partial(\Omega \setminus \bar{\omega})}, \quad (\text{II.1.22})$$

since $\text{div } \mathbf{w} = 0$ in $\Omega \setminus \bar{\omega}$. Therefore, gathering (II.1.21) and (II.1.22) and using the boundary conditions of $(\mathbf{u}', p')$ and $(\mathbf{w}, q)$ (see Problems (II.1.8) and (II.1.9)), we obtain

$$\int_O m^2 (\sigma(\mathbf{u}', p') \mathbf{n}) \cdot (\sigma(\mathbf{u}, p) \mathbf{n} - \mathbf{f}_b) = - \int_{\partial\omega} (\sigma(\mathbf{w}, q) \mathbf{n}) \cdot \partial_{\mathbf{n}} \mathbf{u} V_n.$$

□

Characterization of the shape Hessian at a critical point. We consider $\omega^* \in \mathcal{O}_\delta$ (with $\mathcal{O}_\delta$ defined by (II.0.1)) a critical shape of the functional J_D . Thanks to the local regularity results (see for example [56, Theorem IV.5.1]),

$$(\mathbf{u}, p) \in \mathbf{H}^3(\Omega_\delta \setminus \bar{\omega}^*) \times \mathbf{H}^2(\Omega_\delta \setminus \bar{\omega}^*) \quad \text{and} \quad (\mathbf{u}', p') \in \mathbf{H}^2(\Omega_\delta \setminus \bar{\omega}^*) \times \mathbf{H}^1(\Omega_\delta \setminus \bar{\omega}^*).$$

Proof of Proposition II.1.6. First step: second order shape differentiability. By Lemma I.1.8, $(\mathbf{u}, p)$ is twice differentiable with respect to the domain. We denote by $(\mathbf{u}'', p'')$ the second order shape derivatives.

Second step: second derivative of j_D and derivative of the adjoint problem. Consider $\mathbf{V} \in \mathcal{U}$. We differentiate the function j_D twice with respect to t . At $t = 0$, it holds

$$j''_D(0) = D^2 J_D(\omega) \cdot \mathbf{V} \cdot \mathbf{V} = \int_O m^2 [(\sigma(\mathbf{u}'', p'') \mathbf{n}) \cdot ((\sigma(\mathbf{u}, p) \mathbf{n}) - \mathbf{f}_b) + |\sigma(\mathbf{u}', p') \mathbf{n}|^2].$$

Since ω^* solves the inverse problem, $\sigma(\mathbf{u}, p) \mathbf{n} = \mathbf{f}_b$ on $\partial\Omega$. Therefore

$$D^2 J_D(\omega^*) \cdot \mathbf{V} \cdot \mathbf{V} = 2 \int_O m^2 |\sigma(\mathbf{u}', p') \mathbf{n}|^2.$$

We introduce $(\mathbf{w}, q) \in \mathbf{H}^1(\Omega \setminus \bar{\omega}) \times L^2(\Omega \setminus \bar{\omega})$ with $\langle m^2 \sigma(\mathbf{w}, q) \mathbf{n}, \mathbf{n} \rangle_O = 0$ the solution of the adjoint system (II.1.9). In the same manner that we characterized $\mathbf{u}'$ and p' (see Proposition II.1.3), we characterize $\mathbf{w}'$ and q' , the shape derivatives of $\mathbf{w}$ and q to get

$$\begin{cases} -\text{div}(\sigma(\mathbf{w}, q)) &= \mathbf{0} & \text{in } \Omega \setminus \bar{\omega} \\ \text{div } \mathbf{w}' &= 0 & \text{in } \Omega \setminus \bar{\omega} \\ \mathbf{w}' &= 2m^2 \sigma(\mathbf{u}', p') \mathbf{n} & \text{on } \partial\Omega \\ \mathbf{w}' &= -\partial_{\mathbf{n}} \mathbf{w} V_n & \text{on } \partial\omega, \end{cases}$$

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with $\langle m^2 \sigma(\mathbf{w}', q') \mathbf{n}, \mathbf{n} \rangle_O = 0$. In particular, for $\omega = \omega^*$, $\sigma(\mathbf{u}, p) \mathbf{n} = \mathbf{f}_b$ on O . Therefore, according to System (II.1.9) solved by $(\mathbf{w}, q)$, the uniqueness of the solution of the Stokes problem enforces that $\mathbf{w} = \mathbf{0}$. Therefore $\partial_{\mathbf{n}} \mathbf{w} (\mathbf{V} \cdot \mathbf{n}) = \mathbf{0}$ and we obtain System (II.1.10).

Third step: writing of $j_D''(0)$ as an integral on $\partial\omega$. We multiply the first equation of Problem (II.1.10) by $\mathbf{u}'$ to get

$$\int_{\Omega \setminus \overline{\omega^*}} \nabla \mathbf{w}' : \nabla \mathbf{u}' = - \langle -\sigma(\mathbf{w}', q') \mathbf{n}, \mathbf{u}' \rangle_{\partial\omega^*}. \quad (\text{II.1.23})$$

We multiply the first equation of problem (II.1.8) by $\mathbf{w}'$ to get

$$\int_{\Omega \setminus \overline{\omega^*}} \nu \nabla \mathbf{u}' : \nabla \mathbf{w}' = - \langle -\sigma(\mathbf{u}', p') \mathbf{n}, \mathbf{w}' \rangle_{\partial\Omega}. \quad (\text{II.1.24})$$

Therefore, gathering (II.1.23) and (II.1.24), we obtain

$$\int_O 2m^2 |\sigma(\mathbf{u}', p') \mathbf{n}|^2 = \int_{\partial\omega^*} -(\sigma(\mathbf{w}', q') \mathbf{n}) \cdot \partial_{\mathbf{n}} \mathbf{u} V_n.$$

□

Justifying the ill-posedness of the problem. To prove Proposition II.1.7, we first decompose the shape Hessian at a critical point: for $\mathbf{V} \in \mathbf{U}$,

$$D^2 J(\omega^*) \cdot \mathbf{V} \cdot \mathbf{V} = \langle M_2 \circ M_1(\mathbf{V}), T_{u,p}(\mathbf{V}) \rangle_{\partial\omega^*}.$$

Here, $T_{u,p} : \mathbf{H}^{1/2}(\partial\omega^*) \rightarrow \mathbf{H}^{1/2}(\partial\omega^*)$ is defined by

$$T_{u,p}(\mathbf{V}) := -\partial_{\mathbf{n}} \mathbf{u} V_n,$$

and $M_1 : \mathbf{H}^{1/2}(\partial\omega^*) \rightarrow \mathbf{H}^{1/2}(\partial\Omega)$ by $M_1(\mathbf{V}) := 2m^2 \sigma(\mathbf{u}', p') \mathbf{n}$, with $(\mathbf{u}', p')$ the solution of (II.1.8). Finally, $M_2 : \mathbf{H}^{1/2}(\partial\Omega) \rightarrow \mathbf{H}^{-1/2}(\partial\omega^*)$ satisfies $M_2(\boldsymbol{\varphi}) := \sigma(\boldsymbol{\psi}, \chi) \mathbf{n}$, where $(\boldsymbol{\psi}, \chi)$ is the solution of the following problem

$$\begin{cases} -\operatorname{div}(\sigma(\boldsymbol{\psi}, \chi)) &= \mathbf{0} & \text{in } \Omega \setminus \overline{\omega^*} \\ \operatorname{div} \boldsymbol{\psi} &= 0 & \text{in } \Omega \setminus \overline{\omega^*} \\ \boldsymbol{\psi} &= \boldsymbol{\varphi} & \text{on } \partial\Omega \\ \boldsymbol{\psi} &= \mathbf{0} & \text{on } \partial\omega^*. \end{cases} \quad (\text{II.1.25})$$

We study each operator in the following two lemmas: Lemma II.1.18 states that $T_{u,p}$ and M_1 are linear continuous and Lemma II.1.19 claims that M_2 is compact. Hence, compactness is obtained by composition of linear continuous operator with a compact one.

Lemma II.1.18. *The operators $T_{u,p}$ and M_1 are linear continuous.*

Proof. Since $\mathbf{u}$ does not depend on $\mathbf{V}$ and using the continuity of the gradient operator ∇ , the operator $T_{u,p}$ is linear continuous as multiplier by a smooth function (see [82]).

Let $\mathbf{V} \in \mathbf{H}^{1/2}(\partial\omega^*)$. Using a local regularity argument, the operator

$$\Phi : \mathbf{B} \in \mathbf{H}^{1/2}(\partial\omega^*) \mapsto 2m^2 \sigma(\mathbf{v}, \pi) \mathbf{n} \in \mathbf{H}^{1/2}(\partial\Omega),$$

where $(\mathbf{v}, \pi)$ is the solution of the problem

$$\begin{cases} -\operatorname{div}(\sigma(\mathbf{v}, \pi)) &= \mathbf{0} & \text{in } \Omega \setminus \overline{\omega^*} \\ \operatorname{div} \mathbf{v} &= 0 & \text{in } \Omega \setminus \overline{\omega^*} \\ \mathbf{v} &= \mathbf{0} & \text{on } \partial\Omega \\ \mathbf{v} &= \mathbf{B} & \text{on } \partial\omega^*, \end{cases}$$

with $\int_O m^2 (\sigma(\mathbf{v}, \pi) \mathbf{n}) \cdot \mathbf{n} = 0$ is linear continuous. Finally, by composition, $M_1 = \Phi \circ T_{u,p}$ is linear continuous from $\mathbf{H}^{1/2}(\partial\omega^*)$ into $\mathbf{H}^{1/2}(\partial\Omega)$. □

Lemma II.1.19. *The operator $M_2 : \mathbf{H}^{1/2}(\partial\Omega) \rightarrow \mathbf{H}^{-1/2}(\partial\omega^*)$ is compact.*

Proof. According to the local regularity, the operator

$$\varphi_1 : \varphi \in \mathbf{H}^{1/2}(\partial\Omega) \mapsto (\psi, \chi) \in \mathbf{H}^3(\Omega_\delta \setminus \overline{\omega^*}) \times \mathbf{H}^2(\Omega_\delta \setminus \overline{\omega^*}),$$

where (ψ, χ) is the solution of Problem (II.1.25), is linear continuous. Moreover,

$$\varphi_2 : (\psi, \chi) \in \mathbf{H}^3(\Omega_\delta \setminus \overline{\omega^*}) \times \mathbf{H}^2(\Omega_\delta \setminus \overline{\omega^*}) \mapsto \sigma(\psi, \chi) \mathbf{n} \in \mathbf{H}^2(\Omega_\delta \setminus \overline{\omega^*})$$

is linear continuous and, since the domain $\Omega_\delta \setminus \overline{\omega^*}$ is bounded, the embedding of $\mathbf{H}^2(\Omega_\delta \setminus \overline{\omega^*})$ in $\mathbf{H}^1(\Omega_\delta \setminus \overline{\omega^*})$, noted φ_3 , is compact. Thus, since the trace operator of $\partial\omega^*$, noted $\gamma_{\partial\omega^*}$, from $\mathbf{H}^1(\Omega_\delta \setminus \overline{\omega^*})$ to $\mathbf{H}^{-1/2}(\partial\omega^*)$ is linear continuous, we get that the operator $M_2 = \gamma_{\partial\omega^*} \circ \varphi_3 \circ \varphi_2 \circ \varphi_1$ is compact by composition. $\square$

Remark II.1.20. *We see in the proof that if $\sigma(\psi, \chi) \mathbf{n} \in \mathbf{H}^k(\Omega_\delta \setminus \overline{\omega^*})$, for a $k \in \mathbb{N}^*$, we use the compactness of the embedding of $\mathbf{H}^k(\Omega_\delta \setminus \overline{\omega^*})$ in $\mathbf{H}^{-1/2}(\partial\omega^*)$. Thus, the more ω^* is smooth (with a $C^{k+1,1}$ boundary), the more $\sigma(\psi, \chi) \mathbf{n}$ is regular (it belongs to $\mathbf{H}^k(\Omega_\delta \setminus \overline{\omega^*})$), the more the functional J_D is degenerate and the more the object is difficult to detect. For example, a circle (for $d = 2$) is very difficult to detect, if we do not know that we are looking for a circle.*

II.1.2.2 The Neumann case

The identifiability result for Neumann conditions It is directly adapted from Theorem 1.2 in [6] (see also [51, Theorem 1.2] proved by Doubova *et al.*). However, we detail the proof here since, in [6, 51], the case where $\omega_0 \setminus \overline{\omega_1}$ (with ω_0 and ω_1 two obstacles) is not Lipschitz is not treated. This case leads difficulties and, if Neumann boundary conditions are imposed on the objects, we assume that the objects have a $C^{1,1}$ boundary (and not only Lipschitz) in order to conclude properly. As in [6, 51], the main ingredient is the following theorem due to Fabre and Lebeau:

Theorem II.1.21 (C. Fabre, G. Lebeau, [55]). *Let $\Omega_0 \subseteq \mathbb{R}^d$, $d \geq 2$, be a bounded domain and D_0 an open subset of Ω_0 . If $\mathbf{a} \in \mathbf{L}_{\text{loc}}^\infty(\Omega_0)$ and $(\mathbf{u}, p)$ in $\mathbf{H}_{\text{loc}}^1(\Omega_0) \times \mathbf{L}_{\text{loc}}^2(\Omega_0)$ is a solution of*

$$\begin{cases} -\Delta \mathbf{u} + (\mathbf{a} \cdot \nabla) \mathbf{u} + \nabla p = \mathbf{0} & \text{in } \Omega_0 \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega_0, \end{cases}$$

with $\mathbf{u} = \mathbf{0}$ in D_0 , then $\mathbf{u} = \mathbf{0}$ in Ω_0 and p is constant in Ω_0 .

We deduce from the above theorem the following corollary:

Corollary II.1.22 (Unique continuation property). *Let $\Omega_0 \subseteq \mathbb{R}^d$, $N \geq 2$, be a connected open set and $O \subset \partial\Omega_0$ be a non-empty Lipschitz relatively open subset. If $(\mathbf{u}, p) \in \mathbf{H}^1(\Omega_0) \times \mathbf{L}^2(\Omega_0)$ is a solution of*

$$\begin{cases} -\nu \Delta \mathbf{u} + \nabla p = \mathbf{0} & \text{in } \Omega_0 \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega_0 \\ -\nu \partial_{\mathbf{n}} \mathbf{u} + p \mathbf{n} = \mathbf{0} & \text{on } O \\ \mathbf{u} = \mathbf{0} & \text{on } O, \end{cases} \quad (\text{II.1.26})$$

then $\mathbf{u} = \mathbf{0}$ and $p = 0$ in Ω_0 .

Proof of Corollary II.1.22. The idea of the proof is to enlarge the domain Ω_0 at a part of O and to use the unique continuation Theorem II.1.21 in this enlarged domain. We enlarge the domain by a

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ball: we fix a point y in O , we choose a radius $\rho \leq d(y, \partial O)/2$ (or $\rho \leq |\partial \Omega|/2$ if $O = \partial \Omega$) such that $\Omega_0 \cap \mathcal{B}(y, \rho)$ and $\mathcal{B}(y, \rho) \setminus \overline{\Omega_0}$ are Lipschitz. We then set $\Omega_\rho := \Omega_0 \cup \mathcal{B}(y, \rho)$. We define

$$\tilde{\mathbf{u}} = \begin{cases} \mathbf{u} & \text{in } \Omega_0 \\ \mathbf{0} & \text{in } \Omega_\rho \setminus \Omega_0 \end{cases} \quad \text{and} \quad \tilde{p} = \begin{cases} p & \text{in } \Omega_0 \\ 0 & \text{in } \Omega_\rho \setminus \Omega_0. \end{cases}$$

Since $\mathbf{u} \in \mathbf{H}^1(\Omega_0)$ and $\mathbf{u} = \mathbf{0}$ on O , we check that $\tilde{p} \in L^2(\Omega_\rho)$ and $\tilde{\mathbf{u}} \in \mathbf{H}^1(\Omega_\rho)$. By summing on Ω_0 and on $\Omega_\rho^{\text{ext}} := \{\Omega_\rho \cap \mathcal{B}(y, \rho)\} \setminus O$, we get that for all $\boldsymbol{\varphi}$ in $\mathbf{C}_c^\infty(\mathcal{B}(y, \rho))$:

$$\int_{\mathcal{B}(y, \rho)} (\nu \nabla \tilde{\mathbf{u}} : \nabla \boldsymbol{\varphi} - \tilde{p} \operatorname{div} \boldsymbol{\varphi}) = 0.$$

Then, $-\nu \Delta \tilde{\mathbf{u}} + \nabla \tilde{p} = \mathbf{0}$ in $\mathcal{D}'(\mathcal{B}(y, \rho))$ and then in $\mathcal{D}'(\Omega_\rho)$. Proceeding as above, we also get $\operatorname{div} \tilde{\mathbf{u}} = 0$ in $\mathcal{D}'(\Omega_\rho)$. Moreover, $\tilde{\mathbf{u}} = \mathbf{0}$ in $\Omega_\rho^{\text{ext}} \subset \Omega_\rho$. Thus, according to the unique continuation Theorem II.1.21, $\tilde{\mathbf{u}} = \mathbf{0}$ and $\tilde{p}$ is constant in Ω_ρ . Since $\tilde{p} = 0$ in Ω_ρ^{ext} , $\tilde{p} = 0$ in Ω_ρ then on Ω_0 by restriction. $\square$

The proof of Theorem II.1.10 is split into two steps. Firstly, we use the unique continuation Corollary II.1.22 to prove that $\mathbf{u}_0 = \mathbf{u}_1$ and $p_0 = p_1$ in $\Omega \setminus \overline{\omega}$ where ω is the smallest simply connected open set such that $(\omega_0 \cup \omega_1) \subset \omega$. Secondly, we proceed by contradiction: we assume $\omega_0 \neq \omega_1$ and we use the unique continuation Theorem II.1.21 to prove that $\mathbf{u}_1$ and p_1 are constant (respectively $\mathbf{u}_0$ and p_0) in $\Omega \setminus \overline{\omega_1}$ (respectively in $\Omega \setminus \overline{\omega_0}$) which contradicts the fact that $\mathbf{f} \neq \mathbf{0}$.

Proof of Theorem II.1.10. We define $\mathbf{u} := \mathbf{u}_0 - \mathbf{u}_1$, $p := p_0 - p_1$ and let ω be the smallest simply connected open set such that $(\omega_0 \cup \omega_1) \subset \omega$ (see Figure II.3). Notice that, in the particular case where

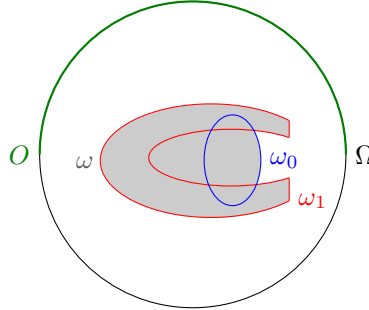


Figure II.3 – The set ω

$\omega_0 \cup \omega_1$ is simply connected, ω is exactly $\omega_0 \cup \omega_1$. Hence, the pair $(\mathbf{u}, p)$ satisfies

$$\begin{cases} -\nu \Delta \mathbf{u} + \nabla p = \mathbf{0} & \text{in } \Omega \setminus \overline{\omega} \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega \setminus \overline{\omega} \\ -\nu \partial_{\mathbf{n}} \mathbf{u} + p \mathbf{n} = \mathbf{0} & \text{on } O \\ \mathbf{u} = \mathbf{0} & \text{on } O. \end{cases}$$

Therefore, according to Corollary II.1.22, $\mathbf{u} = \mathbf{0}$ and $p = 0$ in $\Omega \setminus \overline{\omega}$.

We then proceed by contradiction assuming that $\omega_0 \setminus \overline{\omega_1}$ is non-empty. In the sequel, we assume that $\omega \setminus \overline{\omega_1}$ is connected. If not, it suffices to replace $\omega \setminus \overline{\omega_1}$ by one of its connected components in the following proof. Since $\mathbf{f} \equiv \mathbf{0}$ in Ω_δ by assumption, we know that

$$-\nu \Delta \mathbf{u}_1 + \nabla p_1 = \mathbf{0} \quad \text{in } \omega \setminus \overline{\omega_1}. \quad (\text{II.1.27})$$

In order to give a clear idea of the proof, let us first consider the particular case where $\omega = \omega_0 \cup \omega_1$.

Case where $\omega = \omega_0 \cup \omega_1$. Assume for the moment that $\omega_0 \setminus \overline{\omega_1}$ is Lipschitz. Then, multiplying Equation (II.1.27) by $\mathbf{u}_1$ and integrating by parts in $\omega_0 \setminus \overline{\omega_1}$, we obtain, since $-\nu \partial_{\mathbf{n}} \mathbf{u}_1 + p_1 \mathbf{n} = \mathbf{0}$ on $\partial \omega_1$,

$$\int_{\omega_0 \setminus \overline{\omega_1}} \nu |\nabla \mathbf{u}_1|^2 + \int_{(\partial \omega_0) \setminus \overline{\omega_1}} (-\nu \partial_{\mathbf{n}} \mathbf{u}_1 + p_1 \mathbf{n}) \cdot \mathbf{u}_1 = 0. \quad (\text{II.1.28})$$

Since $\mathbf{u}_0 = \mathbf{u}_1$ and $p_0 = p_1$ in $\Omega \setminus \overline{\omega}$, the boundary condition satisfied by $\mathbf{u}_0$ on $\partial \omega_0$ provides

$$-\nu \partial_{\mathbf{n}} \mathbf{u}_1 + p_1 \mathbf{n} = -\nu \partial_{\mathbf{n}} \mathbf{u}_0 + p_0 \mathbf{n} = \mathbf{0} \quad \text{on } (\partial \omega_0) \setminus \overline{\omega_1}.$$

Hence, equality (II.1.28) is simply

$$\int_{\omega_0 \setminus \overline{\omega_1}} \nu |\nabla \mathbf{u}_1|^2 = 0. \quad (\text{II.1.29})$$

Hence, $\mathbf{u}_1$ is constant in $\omega_0 \setminus \overline{\omega_1}$. By Theorem II.1.21, $\mathbf{u}_1$ and p_1 are then constant in $\Omega \setminus \overline{\omega_1}$ and $-\nu \Delta \mathbf{u}_1 + \nabla p_1 = \mathbf{0}$ in $\Omega \setminus \overline{\omega_1}$. This contradicts $\mathbf{f} \neq \mathbf{0}$ and then $\omega_0 \setminus \overline{\omega_1} = \emptyset$.

Let us now turn to the general case where $\omega_0 \setminus \overline{\omega_1}$ is not necessary Lipschitz. In such a case, the integration by parts yielding (II.1.28) is not justified. Then, to overcome this difficulty and get Equality (II.1.29), let us enlarge the domains by Lipschitz open sets (on which we will use the Green's formula). It is from here that we need to assume that ω_0 and ω_1 has a $C^{1,1}$ boundary (and not only Lipschitz as in the Dirichlet case). Indeed, we here need that $(\mathbf{u}_1, p_1)$ belongs to $\mathbf{H}^2(\Omega_\delta \setminus \overline{\omega_1}) \times \mathbf{H}^1(\Omega_\delta \setminus \overline{\omega_1})$ (using the local regularity Theorem A.1.4) in order to guarantee that we work with boundary integrals and not with duality products (to split it). We consider two Lipschitz open sets $O_0, O_1 \subset \Omega \setminus \overline{\omega_1}$ such that $O_0 \subset \Omega \setminus \overline{\omega_0}$, $\partial O_0 \setminus (\partial \omega_0 \cup \partial \omega_1) = \partial O_1 \setminus (\partial \omega_0 \cup \partial \omega_1)$ and $\partial \omega_1 \cap \omega_0 \subset \partial O_1$ (see Figure II.4). Then,

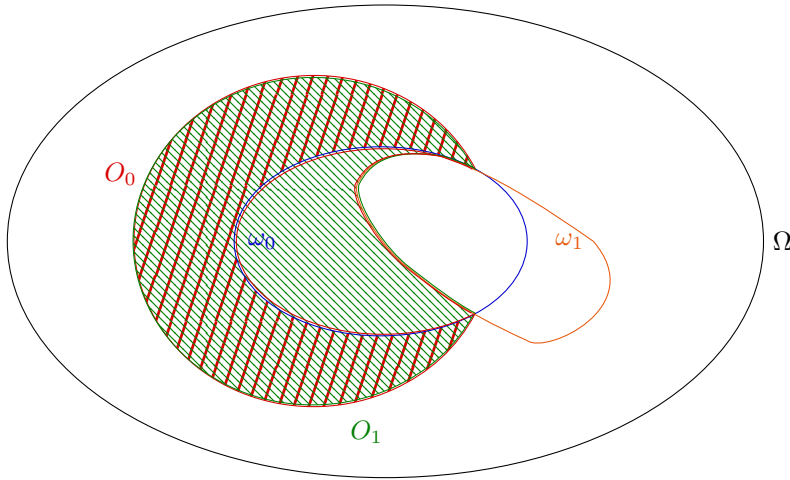


Figure II.4 – The sets O_0 and O_1

using the Green's formula on O_0 and O_1 , we obtain

$$\int_{O_0} \nu |\nabla \mathbf{u}_1|^2 - \int_{\partial O_0} (\nu \partial_{\mathbf{n}} \mathbf{u}_1 - p_1 \mathbf{n}) \cdot \mathbf{u}_1 = 0 \quad \text{and} \quad \int_{O_1} \nu |\nabla \mathbf{u}_1|^2 - \int_{\partial O_1} (\nu \partial_{\mathbf{n}} \mathbf{u}_1 - p_1 \mathbf{n}) \cdot \mathbf{u}_1 = 0.$$

By subtracting these two inequalities, we obtain Equality (II.1.29) since $\nu \partial_{\mathbf{n}} \mathbf{u}_1 - p_1 \mathbf{n} = \mathbf{0}$ on $\partial \omega_1$ and $\nu \partial_{\mathbf{n}} \mathbf{u}_1 - p_1 \mathbf{n} = \nu \partial_{\mathbf{n}} \mathbf{u}_0 - p_0 \mathbf{n} = \mathbf{0}$ on $\partial \omega_0$. We then conclude as previously.

Case where $\omega \neq \omega_0 \cup \omega_1$ (as in Figure II.3 for example). Let us now emphasize the difference of the proof in the general case where ω is not necessary equal to $\omega_0 \cup \omega_1$. We proceed in exactly the same way only replacing ω_0 by $\omega \setminus (\overline{\omega_1} \cup \overline{\omega_0})$. Indeed, we can not work with ω_0 since we have no information on $\mathbf{u}_1$ on $\partial \omega_0 \setminus \partial \omega$: we have only proved that $\mathbf{u}_1 = \mathbf{u}_0$ in $\Omega \setminus \overline{\omega}$.

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Conclusion. In conclusion, we have proved that $\omega_0 \setminus \overline{\omega_1} = \emptyset$. By symmetry, $\omega_1 \setminus \overline{\omega_0} = \emptyset$ and finally we obtain $\omega_0 = \omega_1$. $\square$

First order shape derivatives of the state. We recall the variational formulation of Problem (II.1.16) in $\Omega \setminus \overline{\omega_t}$: Find $(\mathbf{u}_t, p_t) \in \mathbf{H}^1(\Omega \setminus \overline{\omega_t}) \times L^2(\Omega \setminus \overline{\omega_t})$ such that

$$\left\{ \begin{array}{ll} \int_{\Omega \setminus \overline{\omega_t}} \{\nu \nabla \mathbf{u}_t : \nabla \boldsymbol{\varphi}_t - p_t \operatorname{div} \boldsymbol{\varphi}_t\} &= \int_{\Omega \setminus \overline{\omega_t}} \mathbf{f} \cdot \boldsymbol{\varphi}_t \quad \forall \boldsymbol{\varphi}_t \in \mathbf{H}^1(\Omega \setminus \overline{\omega_t}) \\ \int_{\Omega \setminus \overline{\omega_t}} \xi_t \operatorname{div} \mathbf{u}_t &= 0 \quad \forall \xi_t \in L^2(\Omega \setminus \overline{\omega_t}) \\ \int_O \mathbf{u}_t &= \int_O \mathbf{f}_b. \end{array} \right. \quad (\text{II.1.30})$$

Proof of Proposition II.1.11. First step: first order shape differentiability. Using the Fréchet differentiability Lemma I.1.13, we obtain the Gâteaux differentiability in the direction $\mathbf{V}$: there exist $\mathbf{u}_t$ and p_t , respective extensions in Ω of $\mathbf{u}_t$ and p_t such that the functions, defined on $[0, T)$, $t \mapsto \mathbf{u}_t \in \mathbf{H}^1(\Omega)$ and $t \mapsto p_t \in L^2(\Omega)$ are differentiable at 0 by composition. We denote by $\mathbf{u}'$ and p' their respective derivative at 0.

Second step: derivative of the normalization condition. The function

$$t \mapsto \int_O \mathbf{u}_t = \int_O \mathbf{f}_b$$

is constant on $[0, T)$. Thus, its derivative is 0. By differentiation under the sum sign, we get $\int_O \mathbf{u}' = 0$.

Third step: derivative of the equalities on $\Omega \setminus \overline{\omega_t}$. Let $\boldsymbol{\varphi} \in \mathcal{D}(\Omega \setminus \overline{\omega})$. As $\Omega \setminus \overline{\omega}$ is open, we have, for t small enough, $\boldsymbol{\varphi} \in \mathcal{D}(\Omega \setminus \overline{\omega_t})$. Therefore, using the first line of the variational formulation (II.1.30), we have for all $t \in [0, T)$

$$\begin{aligned} \int_{\Omega \setminus \overline{\omega}} \mathbf{f} \cdot \boldsymbol{\varphi} &= \int_{\Omega \setminus \overline{\omega_t}} \mathbf{f} \cdot \boldsymbol{\varphi} = \int_{\Omega \setminus \overline{\omega_t}} \{\nu \nabla \mathbf{u}_t : \nabla \boldsymbol{\varphi} - p_t \operatorname{div} \boldsymbol{\varphi}\} \\ &= \int_{\Omega \setminus \overline{\omega_t}} -\{\nu \mathbf{u}_t \cdot \Delta \boldsymbol{\varphi} - p_t \operatorname{div} \boldsymbol{\varphi}\} = \int_{\Omega \setminus \overline{\omega}} \{-\nu \mathbf{u}_t \cdot \Delta \boldsymbol{\varphi} - p_t \operatorname{div} \boldsymbol{\varphi}\}. \end{aligned}$$

We differentiate with respect to t at $t = 0$

$$\int_{\Omega \setminus \overline{\omega}} \mathbf{f} \cdot \boldsymbol{\varphi} = \int_{\Omega \setminus \overline{\omega}} \{-\nu \mathbf{u}_t \cdot \Delta \boldsymbol{\varphi} - p_t \operatorname{div} \boldsymbol{\varphi}\}$$

to obtain

$$0 = \int_{\Omega \setminus \overline{\omega}} \{-\nu \mathbf{u}' \cdot \Delta \boldsymbol{\varphi} - p' \operatorname{div} \boldsymbol{\varphi}\} = \langle -\nu \Delta \mathbf{u}' + \nabla p', \boldsymbol{\varphi} \rangle_{\mathcal{D}'(\Omega \setminus \overline{\omega}), \mathcal{D}(\Omega \setminus \overline{\omega})}.$$

As it is true for all $\boldsymbol{\varphi} \in \mathcal{D}(\Omega \setminus \overline{\omega})$, we get

$$-\nu \Delta \mathbf{u}' + \nabla p' = \mathbf{0} \quad \text{in } \mathcal{D}'(\Omega \setminus \overline{\omega}). \quad (\text{II.1.31})$$

Proceeding with the second line of the variational formulation (II.1.30) as previously, we check that $\operatorname{div} \mathbf{u}' = 0$ in $\mathcal{D}'(\Omega \setminus \overline{\omega})$.

Fourth step: variational characterization of $\mathbf{u}'$ and p' . Let $\boldsymbol{\varphi} \in \mathbf{H}^2(\Omega \setminus \overline{\omega})$. Using Stein's extension Theorem, $\boldsymbol{\varphi}$ admits an extension in $\mathbf{H}^2(\mathbb{R}^d)$ still noted $\boldsymbol{\varphi}$. This extension is in particular in $\mathbf{H}^2(\Omega \setminus \overline{\omega_t})$. Using $\boldsymbol{\varphi}$ as a test function in the first line of the variational formulation (II.1.30), we have

$$\int_{\Omega \setminus \overline{\omega_t}} \{\nu \nabla \mathbf{u}_t : \nabla \boldsymbol{\varphi} - p_t \operatorname{div} \boldsymbol{\varphi}\} = \int_{\Omega \setminus \overline{\omega_t}} \mathbf{f} \cdot \boldsymbol{\varphi}. \quad (\text{II.1.32})$$

To compute

$$I := \frac{d}{dt} \left(\int_{\Omega \setminus \overline{\omega_t}} \{ \nu \nabla \mathbf{u}_t : \nabla \boldsymbol{\varphi} - p_t \operatorname{div} \boldsymbol{\varphi} \} \right) \Big|_{t=0},$$

we use Hadamard's formula (see [72, Corollary 5.2.5]). Let us check its assumptions. Set

$$\eta(t) := \nu [\nabla \mathbf{v}_t (\mathbf{I} + \nabla \mathbf{V})^{-1}] : [(\nabla \boldsymbol{\varphi}) \circ \phi(t)] - q_t [(\operatorname{div} \boldsymbol{\varphi}) \circ \phi(t)]$$

so that for t in a neighborhood of 0,

$$\nu \nabla \mathbf{u}_t : \nabla \boldsymbol{\varphi} - p_t \operatorname{div} \boldsymbol{\varphi} = \eta(t) \circ \phi(t)^{-1},$$

(since $p_t = q_t \circ \phi(t)^{-1}$ and $\nabla \mathbf{u}_t = [\nabla \mathbf{v}_t (\mathbf{I} + \nabla \mathbf{V})^{-1}] \circ \phi(t)^{-1}$). Then, we have

- $\nu \nabla \mathbf{u}_t : \nabla \boldsymbol{\varphi} - p_t \operatorname{div} \boldsymbol{\varphi} \in \mathbf{L}^1(\Omega_\delta \setminus \overline{\omega_t})$ since $\boldsymbol{\varphi}$ belongs to $\mathbf{H}^2(\mathbb{R}^d)$, the couple $(\mathbf{u}_t, p_t)$ belongs to $\mathbf{H}^2(\Omega_\delta \setminus \overline{\omega_t}) \times \mathbf{H}^1(\Omega_\delta \setminus \overline{\omega_t})$ and $\Omega_\delta \setminus \overline{\omega_t}$ is bounded,
- $t \mapsto \eta(t) \in \mathbf{L}^1(\Omega_\delta \setminus \overline{\omega})$ is differentiable at 0 by composition since $t \mapsto \phi(t)$ is differentiable at 0 and $\boldsymbol{\theta} \mapsto (\mathbf{v}_\boldsymbol{\theta}, q_\boldsymbol{\theta})$ is differentiable at $\mathbf{0}$ using Lemma I.1.11,
- $\nu \nabla \mathbf{u}_0 : \nabla \boldsymbol{\varphi} - p_0 \operatorname{div} \boldsymbol{\varphi} = \nu \nabla \mathbf{u} : \nabla \boldsymbol{\varphi} - p \operatorname{div} \boldsymbol{\varphi} \in \mathbf{W}^{1,1}(\Omega_\delta \setminus \overline{\omega})$ since $\boldsymbol{\varphi}$ belongs to $\mathbf{H}^2(\mathbb{R}^d)$, $(\mathbf{u}, p)$ belongs to $\mathbf{H}^2(\Omega_\delta \setminus \overline{\omega}) \times \mathbf{H}^1(\Omega_\delta \setminus \overline{\omega})$ and $\Omega_\delta \setminus \overline{\omega}$ is bounded.

Thus, using Hadamard's formula ([72, Corollary 5.2.5]), we decompose the derivative I into $I = J_1 + J_2$ with

$$\begin{aligned} J_1 &:= \int_{\Omega \setminus \overline{\Omega_\delta}} \{ \nu \nabla \mathbf{u}' : \nabla \boldsymbol{\varphi} - p' \operatorname{div} \boldsymbol{\varphi} \}, \\ J_2 &:= \int_{\Omega_\delta \setminus \overline{\omega}} \{ \nu \nabla \mathbf{u}' : \nabla \boldsymbol{\varphi} - p' \operatorname{div} \boldsymbol{\varphi} \} + \int_{\partial(\Omega_\delta \setminus \overline{\omega})} (\nu \nabla \mathbf{u} : \nabla \boldsymbol{\varphi} - p \operatorname{div} \boldsymbol{\varphi}) V_n. \end{aligned}$$

Since $\mathbf{V} = 0$ on $\partial\Omega_\delta$, we have:

$$J_2 = \int_{\Omega_\delta \setminus \overline{\omega}} \{ \nu \nabla \mathbf{u}' : \nabla \boldsymbol{\varphi} - p' \operatorname{div} \boldsymbol{\varphi} \} + \int_{\partial\omega} (\nu \nabla \mathbf{u} : \nabla \boldsymbol{\varphi} - p \operatorname{div} \boldsymbol{\varphi}) V_n.$$

Therefore

$$I = \int_{\Omega \setminus \overline{\omega}} \{ \nu \nabla \mathbf{u}' : \nabla \boldsymbol{\varphi} - p' \operatorname{div} \boldsymbol{\varphi} \} + \int_{\partial\omega} (\nu \nabla \mathbf{u} : \nabla \boldsymbol{\varphi} - p \operatorname{div} \boldsymbol{\varphi}) V_n. \quad (\text{II.1.33})$$

By the same technics, we differentiate the right hand side of (II.1.32):

$$\frac{d}{dt} \left(\int_{\Omega \setminus \overline{\omega_t}} \mathbf{f} \cdot \boldsymbol{\varphi} \right) \Big|_{t=0} = \int_{\Omega \setminus \overline{\omega}} \operatorname{div} (\mathbf{f} \cdot \boldsymbol{\varphi}) V = \int_{\partial\omega} (\mathbf{f} \cdot \boldsymbol{\varphi}) V_n. \quad (\text{II.1.34})$$

Gathering (II.1.32), (II.1.33) and (II.1.34), we get

$$\int_{\Omega \setminus \overline{\omega}} \{ \nu \nabla \mathbf{u}' : \nabla \boldsymbol{\varphi} - p' \operatorname{div} \boldsymbol{\varphi} \} + \int_{\partial\omega} (\nu \nabla \mathbf{u} : \nabla \boldsymbol{\varphi} - p \operatorname{div} \boldsymbol{\varphi}) V_n = \int_{\partial\omega} (\mathbf{f} \cdot \boldsymbol{\varphi}) V_n. \quad (\text{II.1.35})$$

We use the definition of tangential gradient and divergence and the formulae

$$(\partial_n \mathbf{u} \otimes \mathbf{n}) : (\partial_n \boldsymbol{\varphi} \otimes \mathbf{n}) = \partial_n \mathbf{u} \cdot \partial_n \boldsymbol{\varphi} \quad \text{and} \quad \nabla \mathbf{u} : (\partial_n \boldsymbol{\varphi} \otimes \mathbf{n}) = \partial_n \mathbf{u} \cdot \partial_n \boldsymbol{\varphi} = \nabla \boldsymbol{\varphi} : (\partial_n \mathbf{u} \otimes \mathbf{n})$$

to get

$$\nabla \mathbf{u} : \nabla \boldsymbol{\varphi} = \nabla_\Gamma \mathbf{u} : \nabla_\Gamma \boldsymbol{\varphi} + \partial_n \mathbf{u} \cdot \partial_n \boldsymbol{\varphi}.$$

Plugging this relation in (II.1.35), we obtain:

$$\begin{aligned} \int_{\Omega \setminus \bar{\omega}} \{ \nu \nabla \mathbf{u}' : \nabla \boldsymbol{\varphi} - p' \operatorname{div} \boldsymbol{\varphi} \} + \int_{\partial \omega} (\nu \nabla_{\Gamma} \mathbf{u} : \nabla_{\Gamma} \boldsymbol{\varphi} - p \operatorname{div}_{\Gamma} \boldsymbol{\varphi}) V_n \\ = \int_{\partial \omega} \{ (\mathbf{f} \cdot \boldsymbol{\varphi}) V_n - \nu \partial_{\mathbf{n}} \mathbf{u} \cdot \partial_{\mathbf{n}} \boldsymbol{\varphi} V_n + \partial_{\mathbf{n}} \boldsymbol{\varphi} \cdot (p \mathbf{n}) V_n \}. \end{aligned}$$

We use the boundary condition $-\nu \partial_{\mathbf{n}} \mathbf{u} + p \mathbf{n} = \mathbf{0}$ on $\partial \omega$ to simplify this relation in:

$$\int_{\Omega \setminus \bar{\omega}} \{ \nu \nabla \mathbf{u}' : \nabla \boldsymbol{\varphi} - p' \operatorname{div} \boldsymbol{\varphi} \} + \int_{\partial \omega} (\nu \nabla_{\Gamma} \mathbf{u} : \nabla_{\Gamma} \boldsymbol{\varphi} - p \operatorname{div}_{\Gamma} \boldsymbol{\varphi}) V_n = \int_{\partial \omega} (\mathbf{f} \cdot \boldsymbol{\varphi}) V_n. \quad (\text{II.1.36})$$

Fifth step: derivative of the boundary values condition. We differentiate with respect to t the relation $-\nu \partial_{\mathbf{n}} \mathbf{u}_t + p_t \mathbf{n} = \mathbf{0}$ on $\partial \Omega$. At $t = 0$, we obtain on $\partial \Omega$ that $-\nu \partial_{\mathbf{n}} \mathbf{u}' + p' \mathbf{n} = \mathbf{0}$.

Let us set $\mathbf{v}_t := \mathbf{u}_t \circ \phi(t)$. By differentiation with respect to t (see [72, Corollary 5.2.5]), we obtain $\mathbf{u}' = \mathbf{v}' - \nabla \mathbf{u} \mathbf{V}$ on $\Omega_{\delta} \setminus \bar{\omega}$, where $\mathbf{v}'$ is the material derivative of $\mathbf{v}_t$ at $t = 0$. Thus, $\mathbf{u}'$ belongs to $\mathbf{H}^2(\Omega_{\delta} \setminus \bar{\omega})$ as a sum of two terms in this space. Indeed, the derivative $\mathbf{v}'$ is the limit of the differential quotient $(\mathbf{v}_t - \mathbf{v})/t$ that belongs to $\mathbf{H}^2(\Omega_{\delta} \setminus \bar{\omega})$ since $\mathbf{v}_t \in \mathbf{H}^2(\Omega_{\delta} \setminus \bar{\omega})$ for all $t \in [0, T]$. The second term $\nabla \mathbf{u} \mathbf{V}$ is also in $\mathbf{H}^2(\Omega_{\delta} \setminus \bar{\omega})$ by the local regularity result A.1.4. The crucial point here is the $C^{2,1}$ regularity of $\partial \omega$. Therefore, $\Delta \mathbf{u}' \in \mathbf{L}^2(\Omega_{\delta} \setminus \bar{\omega})$ and, using (II.1.31), $\nabla p' \in \mathbf{L}^2(\Omega_{\delta} \setminus \bar{\omega})$.

Now, consider $\boldsymbol{\varphi} \in \mathbf{H}^2(\Omega_{\delta} \setminus \bar{\omega})$ with compact support in Ω_{δ} and compute

$$\begin{aligned} \int_{\Omega \setminus \bar{\omega}} \{ \nu \nabla \mathbf{u}' : \nabla \boldsymbol{\varphi} - p' \operatorname{div} \boldsymbol{\varphi} \} &= \int_{\Omega_{\delta} \setminus \bar{\omega}} \{ \nu \nabla \mathbf{u}' : \nabla \boldsymbol{\varphi} - p' \operatorname{div} \boldsymbol{\varphi} \} \\ &= \int_{\Omega_{\delta} \setminus \bar{\omega}} (-\nu \Delta \mathbf{u}' + \nabla p') \cdot \boldsymbol{\varphi} + \int_{\partial(\Omega_{\delta} \setminus \bar{\omega})} (\nu \partial_{\mathbf{n}} \mathbf{u}' - p' \mathbf{n}) \cdot \boldsymbol{\varphi} \\ &= \int_{\partial(\Omega_{\delta} \setminus \bar{\omega})} (\nu \partial_{\mathbf{n}} \mathbf{u}' - p' \mathbf{n}) \cdot \boldsymbol{\varphi} \quad \text{by (II.1.31)}. \end{aligned}$$

Since $\boldsymbol{\varphi} = \mathbf{0}$ on $\partial \Omega_{\delta}$, the boundary term on $\partial \Omega_{\delta}$ cancels. Inserting (II.1.36), we obtain

$$\int_{\partial \omega} (\nu \partial_{\mathbf{n}} \mathbf{u}' - p' \mathbf{n}) \cdot \boldsymbol{\varphi} + \int_{\partial \omega} (\nu \nabla_{\Gamma} \mathbf{u} : \nabla_{\Gamma} \boldsymbol{\varphi} - p \operatorname{div}_{\Gamma} \boldsymbol{\varphi}) V_n = \int_{\partial \omega} \mathbf{f} \cdot \boldsymbol{\varphi} V_n. \quad (\text{II.1.37})$$

We integrate by parts on the boundary $\partial \omega$ (which has no boundary)(see [72, Proposition 5.4.9]) to get

$$\begin{aligned} \int_{\partial \omega} p \operatorname{div}_{\Gamma} \boldsymbol{\varphi} V_n &= \int_{\partial \omega} \{ H p V_n \boldsymbol{\varphi} \cdot \mathbf{n} - \boldsymbol{\varphi} \cdot \nabla_{\Gamma} (p V_n) \}, \\ \int_{\partial \omega} -\nu \nabla_{\Gamma} \mathbf{u} : \nabla_{\Gamma} \boldsymbol{\varphi} V_n &= \int_{\partial \omega} \{ -\boldsymbol{\varphi} \cdot \operatorname{div}_{\Gamma} (-\nu \nabla_{\Gamma} \mathbf{u} V_n) + H \boldsymbol{\varphi} \cdot (-\nu \nabla_{\Gamma} \mathbf{u} V_n \mathbf{n}) \}, \end{aligned}$$

where H is the mean curvature of $\partial \omega$. Inserting these results in (II.1.37), we obtain

$$\begin{aligned} -\nu \partial_{\mathbf{n}} \mathbf{u}' + p' \mathbf{n} &= -\mathbf{f}(\mathbf{V} \cdot \mathbf{n}) - \operatorname{div}_{\Gamma} (\nu \nabla_{\Gamma} \mathbf{u}(\mathbf{V} \cdot \mathbf{n})) + \nu H \nabla_{\Gamma} \mathbf{u}(\mathbf{V} \cdot \mathbf{n}) \mathbf{n} \\ &\quad - H p(\mathbf{V} \cdot \mathbf{n}) \mathbf{n} + (\mathbf{V} \cdot \mathbf{n}) \nabla_{\Gamma} p + p \nabla_{\Gamma}(\mathbf{V} \cdot \mathbf{n}). \end{aligned} \quad (\text{II.1.38})$$

We used the fact that $\nabla_{\Gamma}(p V_n) = V_n \nabla_{\Gamma} p + p \nabla_{\Gamma}(V_n)$ and the density in $\mathbf{L}^2(\partial \omega)$ of the traces on $\partial \omega$ of functions $\mathbf{H}^2(\Omega_{\delta} \setminus \bar{\omega})$ with compact support in Ω_{δ} .

Let us simplify this expression. According to the definition of the tangential gradient (see the introduction of this Chapter), $\nabla_{\Gamma} \mathbf{u} \nabla_{\Gamma}(V_n) = \nabla \mathbf{u} \nabla_{\Gamma}(V_n)$. We expand the tangential divergence

$$\operatorname{div}_{\Gamma} (\nu \nabla_{\Gamma} \mathbf{u} V_n) = \nu \operatorname{div}_{\Gamma} (\nabla_{\Gamma} \mathbf{u}) V_n + \nu \nabla_{\Gamma} \mathbf{u} \nabla_{\Gamma}(V_n)$$

and obtain the decomposition

$$\operatorname{div}_\Gamma (\nu \nabla_\Gamma \mathbf{u} V_n) = \nu \Delta_\Gamma \mathbf{u} V_n + \nu \nabla \mathbf{u} \nabla_\Gamma (V_n). \quad (\text{II.1.39})$$

Using Stokes equations $-\nu \Delta \mathbf{u} + \nabla p = \mathbf{f}$ in $\Omega \setminus \bar{\omega}$ and the decomposition of the Laplace-Beltrami operator (see [72, Proposition 5.4.12]), we compute

$$\mathbf{f} = -\nu (\Delta_\Gamma \mathbf{u} + \mathbf{H} \partial_n \mathbf{u} + \partial_{nn}^2 \mathbf{u}) + \nabla p \quad \text{on } \partial\omega. \quad (\text{II.1.40})$$

Inserting (II.1.39) and (II.1.40) in (II.1.38), the following equality holds on $\partial\omega$

$$-\nu \partial_n \mathbf{u}' + p' \mathbf{n} = \nu \partial_{nn}^2 \mathbf{u} V_n - \nabla p V_n - \nu \nabla \mathbf{u} \nabla_\Gamma (V_n) + \nu \mathbf{H} \nabla_\Gamma \mathbf{u} V_n \mathbf{n} + V_n \nabla_\Gamma p + p \nabla_\Gamma (V_n),$$

since $(\mathbf{u}, p)$ satisfies the boundary conditions $-\nu \partial_n \mathbf{u} + p \mathbf{n} = \mathbf{0}$ on $\partial\omega$. We conclude the proof by noting that $\nabla_\Gamma \mathbf{u} \mathbf{n} = \mathbf{0}$. $\square$

First order shape derivatives of the functional. Define, for all $t \in [0, T)$,

$$J(\omega_t) := j(t) := \int_O (\mathbf{u}_t - \mathbf{f}_b)^2.$$

Proof of Proposition II.1.13. First step: derivative of j and adjoint problem. By Proposition II.1.11, the solution $(\mathbf{u}, p)$ is differentiable with respect to the shape. We denote the shape derivative by $(\mathbf{u}', p')$. Differentiating j with respect to t at 0, we obtain

$$j'(0) = \int_O 2\mathbf{u}'(\mathbf{u} - \mathbf{f}_b).$$

Then, we consider the adjoint problem (II.1.18). According to Theorem A.1.2, it admits a unique solution $(\mathbf{w}, q) \in \mathbf{H}^1(\Omega \setminus \bar{\omega}) \times L^2(\Omega \setminus \bar{\omega})$ with $\int_O \mathbf{w} = \mathbf{0}$. Indeed, the compatibility condition for the adjoint is true as consequence of our choice of compatibility condition for the state.

Second step: writing of $j'(0)$ as an integral on $\partial\omega$. We proceed by successive integrations by parts. We multiply the first equation of the adjoint problem (II.1.18) by $\mathbf{u}'$ to get:

$$\int_{\Omega \setminus \bar{\omega}} \nu \nabla \mathbf{w} : \nabla \mathbf{u}' = - \langle -\nu \partial_n \mathbf{w} + q \mathbf{n}, \mathbf{u}' \rangle_{\partial(\Omega \setminus \bar{\omega})}, \quad (\text{II.1.41})$$

since $\operatorname{div} \mathbf{u}' = 0$ in $\Omega \setminus \bar{\omega}$ (see Proposition II.1.11). Then, we multiply the first equation of Problem (II.1.17) by $\mathbf{w}$ to obtain

$$\int_{\Omega \setminus \bar{\omega}} \nu \nabla \mathbf{u}' : \nabla \mathbf{w} = - \int_{\partial(\Omega \setminus \bar{\omega})} (-\nu \partial_n \mathbf{u}' + p' \mathbf{n}) \cdot \mathbf{w}, \quad (\text{II.1.42})$$

since $\operatorname{div} \mathbf{w} = 0$ in $\Omega \setminus \bar{\omega}$. Therefore, gathering (II.1.41) and (II.1.42) and using the boundary conditions of $(\mathbf{u}', p')$ and $(\mathbf{w}, q)$ (see Problems (II.1.17) and (II.1.18)), we obtain

$$\begin{aligned} \int_O 2(\mathbf{u} - \mathbf{f}_b) \mathbf{u}' &= \int_{\partial\omega} [(\nu \partial_{nn}^2 \mathbf{u} - \partial_n p \mathbf{n}) V_n + p \nabla_\Gamma (V_n) - \nu \nabla \mathbf{u} \nabla_\Gamma (V_n)] \cdot \mathbf{w} \\ &= \int_{\partial\omega} \{ (\nu \partial_{nn}^2 \mathbf{u} - \partial_n p \mathbf{n}) \cdot \mathbf{w} V_n + [{}^t(p \mathbf{I} - \nu \nabla \mathbf{u}) \mathbf{w}] \cdot \nabla_\Gamma (V_n) \}. \end{aligned}$$

Finally, we integrate by parts on the boundary $\partial\omega$ to obtain

$$\int_O 2(\mathbf{u} - \mathbf{f}_b) \mathbf{u}' = \int_{\partial\omega} [(\nu \partial_{nn}^2 \mathbf{u} - \partial_n p \mathbf{n}) \cdot \mathbf{w} - \operatorname{div}_\Gamma ((p \mathbf{I} - \nu \nabla \mathbf{u}) \mathbf{w})] V_n.$$

$\square$

Characterization of the shape Hessian at a critical point. We consider $\omega^* \in \mathcal{O}_\delta$ (with $\mathcal{O}_\delta$ defined by (II.0.1)) a critical shape of the functional J . Thanks to the local regularity result A.1.4,

$$(\mathbf{u}, p) \in \mathbf{H}^3(\Omega_\delta \setminus \overline{\omega^*}) \times \mathbf{H}^2(\Omega_\delta \setminus \overline{\omega^*}) \quad \text{and} \quad (\mathbf{u}', p') \in \mathbf{H}^2(\Omega_\delta \setminus \overline{\omega^*}) \times \mathbf{H}^1(\Omega_\delta \setminus \overline{\omega^*}).$$

Proof of Proposition II.1.16. First step : second order shape differentiability. By Lemma I.1.17, $(\mathbf{u}, p)$ is twice differentiable with respect to the domain. We denote by $(\mathbf{u}'', p'')$ the second order shape derivatives.

Second step: second derivative of j and derivative of the adjoint problem. Consider $\mathbf{V} \in \mathbf{U}$. We differentiate the function j twice with respect to t . At $t = 0$, it holds

$$j''(0) = \mathbf{D}^2 J(\omega) \cdot \mathbf{V} \cdot \mathbf{V} = \int_O \{2(\mathbf{u}')^2 + 2\mathbf{u}''(\mathbf{u} - \mathbf{f}_b)\}.$$

Since ω^* solves the inverse problem, $\mathbf{u} = \mathbf{f}_b$ on O . Therefore

$$\mathbf{D}^2 J(\omega^*) \cdot \mathbf{V} \cdot \mathbf{V} = \int_O 2(\mathbf{u}')^2.$$

We introduce $(\mathbf{w}, q) \in \mathbf{H}^1(\Omega \setminus \overline{\omega}) \times \mathbf{L}^2(\Omega \setminus \overline{\omega})$ with $\int_O \mathbf{w} = \mathbf{0}$ the solution of the adjoint system (II.1.18). As we characterized $\mathbf{u}'$ and p' (see Proposition II.1.11), we characterize $\mathbf{w}'$ and q' , the shape derivatives of $\mathbf{w}$ and q to get

$$-\nu \partial_{\mathbf{n}} \mathbf{w}' + q' \mathbf{n} = (\nu \partial_{\mathbf{nn}}^2 \mathbf{w} - \partial_{\mathbf{n}} q \mathbf{n}) V_{\mathbf{n}} + q \nabla_{\Gamma}(V_{\mathbf{n}}) - \nu \nabla \mathbf{w} \nabla_{\Gamma}(V_{\mathbf{n}}) \quad \text{on } \partial \omega.$$

In particular, for $\omega = \omega^*$, $\mathbf{u} = \mathbf{f}_b$ on O . Uniqueness of the solution of Stokes problem then enforces that $(\mathbf{w}, q) = (\mathbf{c}, 0)$ where $\mathbf{c}$ is constant. Using the fact that $\mathbf{w} = \mathbf{c}$ and $q = 0$, we obtain System (II.1.19).

Third step: writing of $j''(0)$ as an integral on $\partial \omega$. We multiply the first equation of Problem (II.1.19) by $\mathbf{u}'$ to get

$$\int_{\Omega \setminus \overline{\omega^*}} \nu \nabla \mathbf{w}' : \nabla \mathbf{u}' = - \langle -\nu \partial_{\mathbf{n}} \mathbf{w}' + q' \mathbf{n}, \mathbf{u}' \rangle_{\partial \Omega}. \quad (\text{II.1.43})$$

We multiply the first equation of Problem (II.1.17) by $\mathbf{w}'$ to get

$$\int_{\Omega \setminus \overline{\omega^*}} \nu \nabla \mathbf{u}' : \nabla \mathbf{w}' = - \langle -\nu \partial_{\mathbf{n}} \mathbf{u}' + p' \mathbf{n}, \mathbf{w}' \rangle_{\partial \omega^*}. \quad (\text{II.1.44})$$

Therefore, gathering (II.1.43) and (II.1.44), we obtain

$$\int_O 2(\mathbf{u}')^2 = \int_{\partial \omega^*} [(\nu \partial_{\mathbf{nn}}^2 \mathbf{u} - \partial_{\mathbf{n}} p \mathbf{n}) V_{\mathbf{n}} + p \nabla_{\Gamma}(V_{\mathbf{n}}) - \nu \nabla \mathbf{u} \nabla_{\Gamma}(V_{\mathbf{n}})] \cdot \mathbf{w}'.$$

□

Justifying the ill-posedness of the problem. To prove Proposition II.1.17, we first decompose the shape Hessian at a critical point: for $\mathbf{V} \in \mathbf{U}$,

$$\mathbf{D}^2 J(\omega^*) \cdot \mathbf{V} \cdot \mathbf{V} = \langle T_{\mathbf{u},p}(\mathbf{V}), M_2 \circ M_1(\mathbf{V}) \rangle_{\partial \omega^*}.$$

Here, $T_{\mathbf{u},p} : \mathbf{H}^{1/2}(\partial \omega^*) \rightarrow \mathbf{H}^{-1/2}(\partial \omega^*)$ is defined by

$$T_{\mathbf{u},p}(\mathbf{V}) = (\nu \partial_{\mathbf{nn}}^2 \mathbf{u} - \partial_{\mathbf{n}} p \mathbf{n}) V_{\mathbf{n}} + p \nabla_{\Gamma}(V_{\mathbf{n}}) - \nu \nabla \mathbf{u} \nabla_{\Gamma}(V_{\mathbf{n}}),$$

and where $M_1 : \mathbf{H}^{1/2}(\partial\omega^*) \rightarrow \mathbf{H}^{1/2}(\partial\Omega)$ is defined by $M_1(\mathbf{V}) = \mathbf{u}'$, the solution of (II.1.17). Finally, $M_2 : \mathbf{H}^{1/2}(\partial\Omega) \rightarrow \mathbf{H}^{1/2}(\partial\omega^*)$ satisfies $M_2(\varphi) = \psi$, where ψ is the trace on $\partial\omega^*$ of the solution $\mathbf{v}$ of the following Stokes problem

$$\begin{cases} -\nu\Delta\mathbf{v} + \nabla\chi &= \mathbf{0} & \text{in } \Omega \setminus \overline{\omega^*} \\ \operatorname{div} \mathbf{v} &= 0 & \text{in } \Omega \setminus \overline{\omega^*} \\ -\nu\partial_{\mathbf{n}}\mathbf{v} + \chi \mathbf{n} &= 2\varphi \mathbf{1}_O & \text{on } \partial\Omega \\ -\nu\partial_{\mathbf{n}}\mathbf{v} + \chi \mathbf{n} &= \mathbf{0} & \text{on } \partial\omega^*. \end{cases}$$

We study each operator: Lemma II.1.23 states that the operators $T_{\mathbf{u},p}$ and M_1 are linear continuous and Lemma II.1.24 claims that M_2 is compact. Hence, compactness is obtained by composition of linear continuous operator with a compact one.

Lemma II.1.23. *The operators $T_{\mathbf{u},p}$ and M_1 are linear continuous.*

Proof. Since $\mathbf{u}$ and p don't depend on $\mathbf{V}$ and using the continuity of the tangential gradient operator ∇_Γ , the operator $T_{\mathbf{u},p}$ is linear continuous as multiplier by a smooth function (see [82]).

Let $\mathbf{V} \in \mathbf{H}^{1/2}(\partial\omega^*)$. We define $M_1(\mathbf{V}) =: \mathbf{u}'$ and

$$\mathbf{g}_b(\mathbf{V}) := (\nu\partial_{\mathbf{nn}}^2\mathbf{u} - \partial_{\mathbf{n}}p\mathbf{n}) V_{\mathbf{n}} + p\nabla_\Gamma(V_{\mathbf{n}}) - \nu\nabla\mathbf{u}\nabla_\Gamma(V_{\mathbf{n}}).$$

Since ∇_Γ is linear continuous from $\mathbf{H}^{1/2}(\partial\omega^*)$ into $\mathbf{H}^{-1/2}(\partial\omega^*)$,

$$\varphi_1 : \mathbf{V} \in \mathbf{H}^{1/2}(\partial\omega^*) \mapsto \mathbf{g}_b(\mathbf{V}) \in \mathbf{H}^{-1/2}(\partial\omega^*)$$

is linear continuous. Moreover the following operator is linear continuous:

$$\varphi_2 : \mathbf{B} \in \mathbf{H}^{-1/2}(\partial\omega^*) \mapsto \psi \in \mathbf{H}^{1/2}(\partial\Omega),$$

where ψ is the trace on $\partial\Omega$ of the solution $\mathbf{v}$ of the problem

$$\begin{cases} -\nu\Delta\mathbf{v} + \nabla\pi &= \mathbf{0} & \text{in } \Omega \setminus \overline{\omega^*} \\ \operatorname{div} \mathbf{v} &= 0 & \text{in } \Omega \setminus \overline{\omega^*} \\ -\nu\partial_{\mathbf{n}}\mathbf{v} + \pi \mathbf{n} &= \mathbf{0} & \text{on } \partial\Omega \\ -\nu\partial_{\mathbf{n}}\mathbf{v} + \pi \mathbf{n} &= \mathbf{B} & \text{on } \partial\omega^*, \end{cases}$$

with $\int_O \mathbf{v} = \mathbf{0}$. Finally, by composition, $M_1 = \varphi_2 \circ \varphi_1$ is linear continuous from $\mathbf{H}^{1/2}(\partial\omega^*)$ in $\mathbf{H}^{1/2}(\partial\Omega)$. $\square$

Lemma II.1.24. *The operator $M_2 : \mathbf{H}^{1/2}(\partial\Omega) \rightarrow \mathbf{H}^{1/2}(\partial\omega^*)$ is compact.*

Proof. According to the local regularity result A.1.4, the operator

$$\varphi \in \mathbf{H}^{1/2}(\partial\Omega) \mapsto \mathbf{v} \in \mathbf{H}^2(\Omega_\delta \setminus \overline{\omega^*})$$

is linear continuous. Moreover, the domain $\Omega_\delta \setminus \overline{\omega^*}$ is bounded and the embedding of $\mathbf{H}^2(\Omega_\delta \setminus \overline{\omega^*})$ in $\mathbf{H}^1(\Omega_\delta \setminus \overline{\omega^*})$ is then compact. Thus, since the trace operator of $\partial\omega^*$ from $\mathbf{H}^1(\Omega_\delta \setminus \overline{\omega^*})$ to $\mathbf{H}^{1/2}(\partial\omega^*)$ is linear continuous, we get, by composition, that M_2 is compact. $\square$

II.2 A Kohn-Vogelius approach for the Stokes case

II.2.1 The results

The problem setting. Let $\mathbf{f} \in \mathbf{H}^{1/2}(\partial\Omega)$ such that $\mathbf{f} \neq \mathbf{0}$ satisfying the compatibility condition

$$\int_{\partial\Omega} \mathbf{f} \cdot \mathbf{n} = 0. \quad (\text{II.2.1})$$

Let O be a non-empty subset of $\partial\Omega$ and $\mathbf{g} \in \mathbf{H}^{-1/2}(O)$ be an admissible boundary measurement. For $\omega \in \mathcal{O}_\delta$ (with $\mathcal{O}_\delta$ defined by (II.0.1)), let us consider the following overdetermined Stokes boundary value problem

$$\begin{cases} -\nu\Delta \mathbf{u} + \nabla p = \mathbf{0} & \text{in } \Omega \setminus \bar{\omega} \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega \setminus \bar{\omega} \\ \mathbf{u} = \mathbf{f} & \text{on } \partial\Omega \\ \mathbf{u} = \mathbf{0} & \text{on } \partial\omega \\ -\nu\partial_{\mathbf{n}} \mathbf{u} + p\mathbf{n} = \mathbf{g} & \text{on } O. \end{cases} \quad (\text{II.2.2})$$

We assume here that there exists $\omega^* \in \mathcal{O}_\delta$ such that (II.2.2) has a solution. Thus, we consider the following geometric inverse problem:

$$\text{find } \omega \in \mathcal{O}_\delta \text{ and a pair } (\mathbf{u}, p) \text{ which satisfy the overdetermined system (II.2.2).} \quad (\text{II.2.3})$$

We will tackle the inverse problem of reconstructing ω thanks to the tools of shape optimization. In order to recover the shape of the inclusion ω , a usual strategy is then to minimize a cost functional. In this work, we consider the following Kohn-Vogelius cost functional

$$J_{KV}(\omega) := \frac{1}{2} \int_{\Omega \setminus \bar{\omega}} \nu |\nabla(\mathbf{u}_D - \mathbf{u}_N)|^2,$$

where $(\mathbf{u}_D, p_D) \in \mathbf{H}^1(\Omega \setminus \bar{\omega}) \times L^2_0(\Omega \setminus \bar{\omega})$ is the unique solution of the Stokes problem with Dirichlet boundary conditions

$$\begin{cases} -\nu\Delta \mathbf{u}_D + \nabla p_D = \mathbf{0} & \text{in } \Omega \setminus \bar{\omega} \\ \operatorname{div} \mathbf{u}_D = 0 & \text{in } \Omega \setminus \bar{\omega} \\ \mathbf{u}_D = \mathbf{f} & \text{on } \partial\Omega \\ \mathbf{u}_D = \mathbf{0} & \text{on } \partial\omega \end{cases} \quad (\text{II.2.4})$$

and $(\mathbf{u}_N, p_N) \in \mathbf{H}^1(\Omega \setminus \bar{\omega}) \times L^2(\Omega \setminus \bar{\omega})$ is the unique solution of the Stokes problem with mixed boundary conditions

$$\begin{cases} -\nu\Delta \mathbf{u}_N + \nabla p_N = \mathbf{0} & \text{in } \Omega \setminus \bar{\omega} \\ \operatorname{div} \mathbf{u}_N = 0 & \text{in } \Omega \setminus \bar{\omega} \\ -\nu\partial_{\mathbf{n}} \mathbf{u}_N + p_N \mathbf{n} = \mathbf{g} & \text{on } O \\ \mathbf{u}_N = \mathbf{f} & \text{on } \partial\Omega \setminus \bar{O} \\ \mathbf{u}_N = \mathbf{0} & \text{on } \partial\omega. \end{cases} \quad (\text{II.2.5})$$

We refer, for example, to [13, 31, 44, 56, 61, 97], for the results of existence, uniqueness and regularity of the solution of the Stokes system (II.2.4) with Dirichlet boundary conditions. In particular, a local regularity result can be found in [56, Theorem IV.5.1]. Notice that we assume the compatibility condition (II.2.1) associated to Problem (II.2.4) is satisfied. Hence, the solution $(\mathbf{u}_D, p_D)$ exists and is unique. Moreover, the existence and the uniqueness of the couple $(\mathbf{u}_N, p_N)$ is guaranteed by Theorem A.1.5 and other results on the mixed boundary value problem (II.2.5) are recalled in Appendix A.1.2.2.

Then, we try to minimize the Kohn-Vogelius cost functional J_{KV} :

$$\omega^* = \operatorname{argmin}_{\omega \in \mathcal{O}_\delta} J_{KV}(\omega). \quad (\text{II.2.6})$$

Indeed, if ω^* is solution of the inverse problem (II.2.3), then $J_{KV}(\omega^*) = 0$ and (II.2.6) holds. Conversely, if ω^* solves (II.2.6) with $J_{KV}(\omega^*) = 0$, then this domain ω^* is a solution of the inverse problem.

Identifiability result. The identifiability result II.1.2 of Alvarez *et al.* can be adapted to our problem, *i.e.* with $-\nu\partial_{\mathbf{n}}\mathbf{u} + p\mathbf{n}$ instead of $\sigma(\mathbf{u}, p)\mathbf{n}$, with $\varepsilon_* = 0$ and with Ω Lipschitz (see the proof of the identifiability result II.1.10 in the Neumann case). Hence this result states that given a fixed $\mathbf{f}$, two different geometries ω_0 and ω_1 in $\mathcal{O}_\delta$ yield two different measures $\mathbf{g}_1$ and $\mathbf{g}_2$. Thus Problem (II.2.3) admits a unique solution.

Sensitivity with respect to the domain. The following result is based on the corresponding Proposition in the least squares approach (see Proposition II.1.3) and ensures that the solutions $(\mathbf{u}_D, p_D)$ and $(\mathbf{u}_N, p_N)$ are differentiable with respect to the domain. Moreover, we characterize the shape derivatives of these solutions.

Proposition II.2.1 (First order shape derivatives of the states). *Let $\mathbf{V} \in \mathbf{U}$ be an admissible deformation. The solutions $(\mathbf{u}_D, p_D)$ and $(\mathbf{u}_N, p_N)$ are differentiable with respect to the domain and the shape derivatives $(\mathbf{u}'_D, p'_D)$ and $(\mathbf{u}'_N, p'_N)$ belong to $\mathbf{H}^2(\Omega_{d_0} \setminus \bar{\omega}) \times \mathbf{H}^1(\Omega_{d_0} \setminus \bar{\omega})$. The couples $(\mathbf{u}'_D, p'_D) \in \mathbf{H}^1(\Omega \setminus \bar{\omega}) \times L^2_0(\Omega \setminus \bar{\omega})$ and $(\mathbf{u}'_N, p'_N) \in \mathbf{H}^1(\Omega \setminus \bar{\omega}) \times L^2(\Omega \setminus \bar{\omega})$ are respectively the only solutions of the following boundary value problems*

$$\left\{ \begin{array}{ll} -\nu\Delta\mathbf{u}'_D + \nabla p'_D = \mathbf{0} & \text{in } \Omega \setminus \bar{\omega} \\ \operatorname{div} \mathbf{u}'_D = 0 & \text{in } \Omega \setminus \bar{\omega} \\ \mathbf{u}'_D = \mathbf{0} & \text{on } \partial\Omega \\ \mathbf{u}'_D = -V_n \partial_{\mathbf{n}} \mathbf{u}_D & \text{on } \partial\omega \end{array} \right. \quad (\text{II.2.7})$$

and

$$\left\{ \begin{array}{ll} -\nu\Delta\mathbf{u}'_N + \nabla p'_N = \mathbf{0} & \text{in } \Omega \setminus \bar{\omega} \\ \operatorname{div} \mathbf{u}'_N = 0 & \text{in } \Omega \setminus \bar{\omega} \\ -\nu\partial_{\mathbf{n}} \mathbf{u}'_N + p'_N \mathbf{n} = \mathbf{0} & \text{on } O \\ \mathbf{u}'_N = \mathbf{0} & \text{on } \partial\Omega \setminus \bar{O} \\ \mathbf{u}'_N = -V_n \partial_{\mathbf{n}} \mathbf{u}_N & \text{on } \partial\omega. \end{array} \right. \quad (\text{II.2.8})$$

In order to simplify the expressions, we introduce the following notations:

$$\mathbf{w} := \mathbf{u}_D - \mathbf{u}_N \quad \text{and} \quad q := p_D - p_N, \quad (\text{II.2.9})$$

where $(\mathbf{u}_D, p_D)$ solves (II.2.4) and $(\mathbf{u}_N, p_N)$ solves (II.2.5).

Proposition II.2.2 (First order shape derivative of the functional). *For $\mathbf{V} \in \mathbf{U}$, the Kohn-Vogelius cost functional J_{KV} is differentiable at ω in the direction $\mathbf{V}$ with*

$$DJ_{KV}(\omega) \cdot \mathbf{V} = - \int_{\partial\omega} (\nu\partial_{\mathbf{n}}\mathbf{w} - q\mathbf{n}) \cdot \partial_{\mathbf{n}}\mathbf{u}_D V_n + \frac{1}{2}\nu \int_{\partial\omega} |\nabla\mathbf{w}|^2 V_n, \quad (\text{II.2.10})$$

where $(\mathbf{w}, q)$ is defined by (II.2.9).

Second order analysis: justification of the instability. We study the stability of the problem. First, we give an explicit formula for the shape Hessian when $\omega \in \mathcal{O}_\delta$ (with $\mathcal{O}_\delta$ defined by (II.0.1)) in the following proposition. In order to simplify the expressions, we will use the following notations:

$$\mathbf{w}' := \mathbf{u}'_D - \mathbf{u}'_N \quad \text{and} \quad q' := p'_D - p'_N, \quad (\text{II.2.11})$$

where $(\mathbf{u}'_D, p'_D)$ solves (II.2.7) and $(\mathbf{u}'_N, p'_N)$ solves (II.2.8).

Proposition II.2.3 (Second order shape derivative of the functional). *The solution $(\mathbf{u}, p)$ is twice differentiable with respect to the domain. Moreover, for $\mathbf{V} \in \mathcal{U}$, we have*

$$\begin{aligned} D^2 J_{KV}(\omega) \cdot \mathbf{V} \cdot \mathbf{V} = & \int_{\partial\omega} \left[\nu \nabla \mathbf{w}' : \nabla \mathbf{w} - \left(\nu \partial_{\mathbf{n}} \mathbf{w}' + \nu \nabla \mathbf{w} \mathbf{n}' - q' \mathbf{n} - q \mathbf{n}' \right) \cdot \partial_{\mathbf{n}} \mathbf{u}_D \right. \\ & \left. - \left(\nu \partial_{\mathbf{n}} \mathbf{w} - q \mathbf{n} \right) \cdot \left(\partial_{\mathbf{n}} \mathbf{u}'_D + \nabla \mathbf{u}_D \mathbf{n}' \right) \right] V_n + \int_{\partial\omega} \operatorname{div} \left[\left(\frac{1}{2} \nu |\nabla \mathbf{w}|^2 - (\nu \partial_{\mathbf{n}} \mathbf{w} - q \mathbf{n}) \cdot \partial_{\mathbf{n}} \mathbf{u}_D \right) \mathbf{V} \right] V_n, \end{aligned} \quad (\text{II.2.12})$$

where $(\mathbf{w}, q)$ and $(\mathbf{w}', q')$ are defined respectively by (II.2.9) and (II.2.11).

Let $\omega^* \in \mathcal{O}_\delta$ (with $\mathcal{O}_\delta$ defined by (II.0.1)) be a critical point of the Kohn-Vogelius functional. The following proposition shows that the optimization problem (II.2.6) is unstable.

Proposition II.2.4 (Compactness at a critical point). *Let $\omega^* \in \mathcal{O}_\delta$ be a solution of the inverse problem (II.2.3). We have*

$$D^2 J_{KV}(\omega^*) \cdot \mathbf{V} \cdot \mathbf{V} = \int_{\partial\omega^*} -(\nu \partial_{\mathbf{n}} \mathbf{w}' - q' \mathbf{n}) \cdot \partial_{\mathbf{n}} \mathbf{u}_D V_n,$$

where $(\mathbf{w}', q')$ is defined by (II.2.11). Moreover, the Riesz operator corresponding to $D^2 J_{KV}(\omega^*)$ defined from $\mathbf{H}^{1/2}(\partial\omega^*)$ to $\mathbf{H}^{-1/2}(\partial\omega^*)$ is compact.

II.2.2 The proofs

First order shape derivative. The proof of Proposition II.2.1 is directly adapted of the proof of corresponding Proposition II.1.3 (see Lemma I.1.4 for the existence of the shape derivatives) in the least squares approach and we refer to it for the details. The main difficulty is to prove the existence of the shape derivatives: it is obtained through the classical implicit functions theorem. We can notice that there is no compatibility condition in the case of mixed boundary conditions. The characterization of $(\mathbf{u}'_D, p'_D)$ and $(\mathbf{u}'_N, p'_N)$ is obtained using classical results of shape derivatives calculus (see [72, Chapter 5]).

Proof of Proposition II.2.2. The Hadamard's formula (see [72, Theorem 5.2.2]) leads:

$$\begin{aligned} DJ_{KV}(\omega) \cdot \mathbf{V} &= \int_{\Omega \setminus \bar{\omega}} \left(\nu \nabla \mathbf{w} : \nabla \mathbf{w}' + \frac{1}{2} \nu \operatorname{div} (|\nabla \mathbf{w}|^2 \mathbf{V}) \right) \\ &= \int_{\Omega \setminus \bar{\omega}} \nu \nabla \mathbf{w} : \nabla (\mathbf{u}'_D - \mathbf{u}'_N) + \frac{1}{2} \int_{\partial\omega} \nu |\nabla \mathbf{w}|^2 V_n \end{aligned} \quad (\text{II.2.13})$$

using the fact that $\mathbf{V} = \mathbf{0}$ on $\partial\Omega$. Let us prove that

$$\int_{\Omega \setminus \bar{\omega}} \nu \nabla \mathbf{w} : \nabla \mathbf{u}'_N = 0. \quad (\text{II.2.14})$$

Applying Green's Formula and using System (II.2.8) solved by $(\mathbf{u}'_N, p'_N)$, we get

$$\begin{aligned} \int_{\Omega \setminus \bar{\omega}} \nu \nabla \mathbf{w} : \nabla \mathbf{u}'_N &= - \int_{\Omega \setminus \bar{\omega}} \nu \Delta \mathbf{u}'_N \cdot \mathbf{w} + \int_{\partial(\Omega \setminus \bar{\omega})} \nu \partial_{\mathbf{n}} \mathbf{u}'_N \cdot \mathbf{w} \\ &= - \int_{\Omega \setminus \bar{\omega}} \nabla p'_N \cdot \mathbf{w} + \int_{\partial(\Omega \setminus \bar{\omega})} \nu \partial_{\mathbf{n}} \mathbf{u}'_N \cdot \mathbf{w} \\ &= \int_{\Omega \setminus \bar{\omega}} p'_N \operatorname{div} \mathbf{w} - \int_{\partial(\Omega \setminus \bar{\omega})} p'_N \mathbf{w} \cdot \mathbf{n} + \int_{\partial(\Omega \setminus \bar{\omega})} \nu \partial_{\mathbf{n}} \mathbf{u}'_N \cdot \mathbf{w} \\ &= 0, \end{aligned}$$

since $\operatorname{div} \mathbf{w} = 0$ in $\Omega \setminus \bar{\omega}$, $-\nu \partial_{\mathbf{n}} \mathbf{u}'_N + p'_N \mathbf{n} = \mathbf{0}$ on O and $\mathbf{w} = \mathbf{0}$ on $\partial\omega$ and on $\partial\Omega \setminus \bar{O}$. Notice that $\mathbf{w} \in \mathbf{H}_{00}^{1/2}(O)$ since it belongs to $\mathbf{H}^{1/2}(\partial\Omega)$ and vanishes on $\partial\Omega \setminus \bar{O}$.

Then, let us write $\int_{\Omega \setminus \bar{\omega}} \nu \nabla \mathbf{w} : \nabla \mathbf{u}'_D$ as an integral on $\partial\omega$. Proceeding as above, we apply Green's Formula on this term and use System (II.2.7) solved by $(\mathbf{u}'_D, p'_D)$ to obtain

$$\begin{aligned} \int_{\Omega \setminus \bar{\omega}} \nu \nabla \mathbf{w} : \nabla \mathbf{u}'_D &= - \int_{\Omega \setminus \bar{\omega}} \nu \Delta \mathbf{w} \cdot \mathbf{u}'_D + \int_{\partial(\Omega \setminus \bar{\omega})} \nu \partial_{\mathbf{n}} \mathbf{w} \cdot \mathbf{u}'_D \\ &= - \int_{\Omega \setminus \bar{\omega}} \nabla q \cdot \mathbf{u}'_D + \int_{\partial(\Omega \setminus \bar{\omega})} \nu \partial_{\mathbf{n}} \mathbf{w} \cdot \mathbf{u}'_D \\ &= \int_{\Omega \setminus \bar{\omega}} q \operatorname{div} \mathbf{u}'_D - \int_{\partial(\Omega \setminus \bar{\omega})} q \mathbf{u}'_D \cdot \mathbf{n} + \int_{\partial(\Omega \setminus \bar{\omega})} \nu \partial_{\mathbf{n}} \mathbf{w} \cdot \mathbf{u}'_D \\ &= \int_{\partial(\Omega \setminus \bar{\omega})} (\nu \partial_{\mathbf{n}} \mathbf{w} - q \mathbf{n}) \cdot \mathbf{u}'_D. \end{aligned}$$

Since $\mathbf{u}'_D = \mathbf{0}$ on $\partial\Omega$ and $\mathbf{u}'_D = -\partial_{\mathbf{n}} \mathbf{u}_D V_n$ on $\partial\omega$, we obtain

$$\int_{\Omega \setminus \bar{\omega}} \nu \nabla \mathbf{w} : \nabla \mathbf{u}'_D = - \int_{\partial\omega} (\nu \partial_{\mathbf{n}} \mathbf{w} - q \mathbf{n}) \cdot \partial_{\mathbf{n}} \mathbf{u}_D V_n. \quad (\text{II.2.15})$$

Gathering Equations (II.2.13), (II.2.14) and (II.2.15), we check that

$$\operatorname{D}J_{KV}(\omega) \cdot \mathbf{V} = - \int_{\partial\omega} (\nu \partial_{\mathbf{n}} \mathbf{w} - q \mathbf{n}) \cdot \partial_{\mathbf{n}} \mathbf{u}_D V_n + \frac{1}{2} \nu \int_{\partial\omega} |\nabla \mathbf{w}|^2 V_n.$$

□

Characterization of the shape Hessian.

Proof of Proposition II.2.3. The proof of the existence of the second order shape differentiability of the states $(\mathbf{u}_D, p_D)$ and $(\mathbf{u}_N, p_N)$ is given by Lemma I.1.8. We can just notice that there is no compatibility condition in the case of mixed boundary conditions.

From Proposition II.2.2, we have

$$\begin{aligned} \operatorname{D}J_{KV}(\omega) \cdot \mathbf{V} &= \int_{\partial\omega} \left[\frac{1}{2} \nu |\nabla \mathbf{w}|^2 - (\nu \partial_{\mathbf{n}} \mathbf{w} - q \mathbf{n}) \cdot \partial_{\mathbf{n}} \mathbf{u}_D \right] V_n \\ &= \int_{\Omega \setminus \bar{\omega}} \operatorname{div} \left[\left(\frac{1}{2} \nu |\nabla \mathbf{w}|^2 - (\nu \partial_{\mathbf{n}} \mathbf{w} - q \mathbf{n}) \cdot \partial_{\mathbf{n}} \mathbf{u}_D \right) \mathbf{V} \right], \end{aligned}$$

since $\mathbf{V} \equiv \mathbf{0}$ on $\partial\Omega$. From Hadamard's formula (see [72, Theorem 5.2.2]), this leads

$$\begin{aligned} \operatorname{D}^2 J_{KV}(\omega) \cdot \mathbf{V} \cdot \mathbf{V} &= \int_{\Omega \setminus \bar{\omega}} \operatorname{div} \left(\left[\nu \nabla \mathbf{w}' : \nabla \mathbf{w} - (\nu \partial_{\mathbf{n}} \mathbf{w}' + \nu \nabla \mathbf{w} \mathbf{n}' - q' \mathbf{n} - q \mathbf{n}') \cdot \partial_{\mathbf{n}} \mathbf{u}_D \right. \right. \\ &\quad \left. \left. - (\nu \partial_{\mathbf{n}} \mathbf{w} - q \mathbf{n}) \cdot (\partial_{\mathbf{n}} \mathbf{u}'_D + \nabla \mathbf{u}_D \mathbf{n}') \right] \mathbf{V} \right) \\ &\quad + \int_{\partial\omega} \operatorname{div} \left(\left[\frac{1}{2} \nu |\nabla \mathbf{w}|^2 - (\nu \partial_{\mathbf{n}} \mathbf{w} - q \mathbf{n}) \cdot \partial_{\mathbf{n}} \mathbf{u}_D \right] \mathbf{V} \right) V_n. \end{aligned}$$

Since $\mathbf{V} = \mathbf{0}$ on $\partial\Omega$, we then obtain

$$\begin{aligned} \operatorname{D}^2 J_{KV}(\omega) \cdot \mathbf{V} \cdot \mathbf{V} &= \int_{\partial\omega} \left[\nu \nabla \mathbf{w}' : \nabla \mathbf{w} - (\nu \partial_{\mathbf{n}} \mathbf{w}' + \nu \nabla \mathbf{w} \mathbf{n}' - q' \mathbf{n} - q \mathbf{n}') \cdot \partial_{\mathbf{n}} \mathbf{u}_D \right. \\ &\quad \left. - (\nu \partial_{\mathbf{n}} \mathbf{w} - q \mathbf{n}) \cdot (\partial_{\mathbf{n}} \mathbf{u}'_D + \nabla \mathbf{u}_D \mathbf{n}') \right] V_n \\ &\quad + \int_{\partial\omega} \operatorname{div} \left(\left[\frac{1}{2} \nu |\nabla \mathbf{w}|^2 - (\nu \partial_{\mathbf{n}} \mathbf{w} - q \mathbf{n}) \cdot \partial_{\mathbf{n}} \mathbf{u}_D \right] \mathbf{V} \right) V_n. \end{aligned}$$

□

Justifying the ill-posedness of the problem. In order to prove Proposition II.2.4, we investigate the properties of stability of the cost functional J_{KV} . Then we consider an admissible inclusion $\omega^* \in \mathcal{O}_\delta$ (with $\mathcal{O}_\delta$ defined by (II.0.1)) which is solution of Problem (II.2.3). Then, ω^* realizes the absolute minimum of the criterion J_{KV} , $J_{KV}(\omega^*) = 0$ and then $\mathbf{u}_D = \mathbf{u}_N$ and $p_D = p_N$ in $\Omega \setminus \overline{\omega^*}$. Therefore

$$D^2 J_{KV}(\omega^*) \cdot \mathbf{V} \cdot \mathbf{V} = - \int_{\partial \omega^*} (\nu \partial_{\mathbf{n}} \mathbf{w}' - q' \mathbf{n}) \cdot \partial_{\mathbf{n}} \mathbf{u}_D V_n.$$

Proof of Proposition II.2.4. We first decompose $D^2 J_{KV}(\omega^*)$ as a composition of two operators: for $\mathbf{V} \in \mathbf{U}$,

$$D^2 J_{KV}(\omega^*) \cdot \mathbf{V} \cdot \mathbf{V} = \langle M(\mathbf{V}), T(\mathbf{V}) \rangle_{\mathbf{H}^{-1/2}(\partial \omega^*), \mathbf{H}^{1/2}(\partial \omega^*)}.$$

Here, the operator $T : \mathbf{H}^{1/2}(\partial \omega^*) \rightarrow \mathbf{H}^{1/2}(\partial \omega^*)$ is defined by

$$T(\mathbf{V}) := -\partial_{\mathbf{n}} \mathbf{u}_D V_n$$

and the operator $M : \mathbf{H}^{1/2}(\partial \omega^*) \rightarrow \mathbf{H}^{-1/2}(\partial \omega^*)$ is defined by

$$M(\mathbf{V}) := \nu \partial_{\mathbf{n}} \mathbf{w}' - q' \mathbf{n}.$$

From Systems (II.2.7) and (II.2.8) and since $\mathbf{u}_D = \mathbf{u}_N$ in $\Omega \setminus \overline{\omega^*}$ (and then $\partial_{\mathbf{n}} \mathbf{u}_D = \partial_{\mathbf{n}} \mathbf{u}_N$ on $\partial \omega^*$), we obtain the following system:

$$\begin{cases} -\nu \Delta \mathbf{w}' + \nabla q' &= \mathbf{0} & \text{in } \Omega \setminus \overline{\omega^*} \\ \operatorname{div} \mathbf{w}' &= 0 & \text{in } \Omega \setminus \overline{\omega^*} \\ \mathbf{w}' &= -\mathbf{u}'_N & \text{on } \partial \Omega \\ \mathbf{w}' &= \mathbf{0} & \text{on } \partial \omega^*. \end{cases}$$

Since $\partial_{\mathbf{n}} \mathbf{u}_D$ does not depend on $\mathbf{V}$, notice that the operator T is linear continuous as multiplier by a smooth function (see [82]). Now, let us prove the following lemma which states that the operator M is compact. Hence, the Riesz operator corresponding to the shape Hessian is compact as composition of linear continuous operator with a compact one. □

Lemma II.2.5. *The operator M is compact.*

Proof. We decompose the operator M as follows:

$$M := M_2 \circ M_1$$

with

$$M_1 : \mathbf{V} \in \mathbf{H}^{1/2}(\partial \omega^*) \mapsto M_1(\mathbf{V}) := -\mathbf{u}'_N \in \mathbf{H}^{1/2}(\partial \Omega),$$

and

$$M_2 : \Psi \in \mathbf{H}^{1/2}(\partial \Omega) \mapsto M_2(\Psi) := \nu \partial_{\mathbf{n}} \Phi - \chi \mathbf{n} \in \mathbf{H}^{-1/2}(\partial \omega^*),$$

where $(\mathbf{u}'_N, p'_N)$ is solution of (II.2.8) and where $(\Phi, \chi) \in \mathbf{H}^1(\Omega \setminus \overline{\omega^*}) \times L_0^2(\Omega \setminus \overline{\omega^*})$ solves

$$\begin{cases} -\nu \Delta \Phi + \nabla \chi &= \mathbf{0} & \text{in } \Omega \setminus \overline{\omega^*} \\ \operatorname{div} \Phi &= 0 & \text{in } \Omega \setminus \overline{\omega^*} \\ \Phi &= \Psi & \text{on } \partial \Omega \\ \Phi &= \mathbf{0} & \text{on } \partial \omega^*. \end{cases} \quad (\text{II.2.16})$$

Let us prove that M_1 is linear continuous and M_2 is compact. We check that

$$M_1 = M_{1,2} \circ M_{1,1}$$

with

$$M_{1,1} : \mathbf{V} \in \mathbf{H}^{1/2}(\partial\omega^*) \mapsto M_{1,1}(\mathbf{V}) := -\partial_{\mathbf{n}} \mathbf{u}_{\mathbf{N}} V_n \in \mathbf{H}^{1/2}(\partial\omega^*)$$

and with

$$M_{1,2} : \boldsymbol{\eta} \in \mathbf{H}^{1/2}(\partial\omega^*) \mapsto M_{1,2}(\mathbf{V}) := -\mathbf{z} \in \mathbf{H}^{1/2}(\partial\Omega),$$

where the couple $(\mathbf{z}, q) \in \mathbf{H}^1(\Omega \setminus \overline{\omega^*}) \times L^2(\Omega \setminus \overline{\omega^*})$ is solution of

$$\begin{cases} -\nu \Delta \mathbf{z} + \nabla q &= \mathbf{0} & \text{in } \Omega \setminus \overline{\omega^*} \\ \operatorname{div} \mathbf{z} &= 0 & \text{in } \Omega \setminus \overline{\omega^*} \\ -\nu \partial_{\mathbf{n}} \mathbf{z} + q \mathbf{n} &= \mathbf{0} & \text{on } \partial\Omega \\ \mathbf{z} &= \boldsymbol{\eta} & \text{on } \partial\omega^*. \end{cases}$$

Since $\partial_{\mathbf{n}} \mathbf{u}_{\mathbf{N}}$ does not depend on $\mathbf{V}$, the operator $M_{1,1}$ is linear continuous as multiplier by a smooth function (see [82]) and the operator $M_{1,2}$ is clearly linear continuous. Thus M_1 is linear continuous.

Finally, we decompose M_2 as follows:

$$M_2 = M_{2,3} \circ M_{2,2} \circ M_{2,1}$$

where

$$M_{2,1} : \boldsymbol{\Psi} \in \mathbf{H}^{1/2}(\partial\Omega) \mapsto (\boldsymbol{\Phi}, \chi) \in \mathbf{H}^3(\Omega_{d_0} \setminus \overline{\omega^*}) \times \mathbf{H}^2(\Omega_{d_0} \setminus \overline{\omega^*})$$

with $(\boldsymbol{\Phi}, \chi)$ solution of (II.2.16),

$$M_{2,2} : (\mathbf{v}, \xi) \in \mathbf{H}^3(\Omega_{d_0} \setminus \overline{\omega^*}) \times \mathbf{H}^2(\Omega_{d_0} \setminus \overline{\omega^*}) \mapsto \nu \partial_{\mathbf{n}} \mathbf{v} - \xi \mathbf{n} \in \mathbf{H}^{3/2}(\partial\omega^*)$$

and $M_{2,3}$ is the compact imbedding of $\mathbf{H}^{3/2}(\partial\omega^*)$ into $\mathbf{H}^{-1/2}(\partial\omega^*)$. From the local regularity Theorem A.1.6, the operator $M_{2,1}$ is linear continuous and the operator $M_{2,2}$ is clearly linear continuous. Then, by composition M_2 is compact, which concludes the proof. $\square$

II.2.3 Explicit computations of the shape Hessian

We illustrate the compact behavior of the shape Hessian in the case $O = \partial\Omega$. We study it in the bi-dimensional situation for some particular geometries. In order to simplify the expressions, we assume $\nu = 1$. In the following, $(\vec{e}_r, \vec{e}_\theta) := \left(\begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix} \right)$ represents the polar coordinates system.

II.2.3.1 Main result and examples

Proposition II.2.6 (Explicit calculus of the shape Hessian in a concentric annulus). *Let us assume that the optimum ω^* corresponds to the concentric annulus case*

$$\Omega_\rho = \{x \in \mathbb{R}^2, \rho < |x| < 1\}.$$

Set $\mathbf{f} = \cos(n\theta) \vec{e}_r$, $n > 1$ and the associated Neumann boundary condition $\mathbf{g}$ (see (II.2.20)). Let $p > 1$ and let us define

$$\begin{aligned} \xi_n &:= \rho^2 - n^2 \rho^{2n} + 2(-1 + n^2) \rho^{2+2n} - n^2 \rho^{4+2n} + \rho^{2+4n} \\ \kappa_n &:= 2(-1 + \rho^{2n}) + n^2(-1 + \rho^2)(-1 + \rho^{2n}) - n(-1 + \rho^2)(1 + \rho^{2n}) \\ A_p &:= 2(-1 + \rho^{2p})^2 + p^3(-1 + \rho^2)^2(-1 + \rho^{4p}) + 2p(1 + \rho^2)(-1 + \rho^{4p}) \\ &\quad - p^2(-1 + \rho^2)(1 + \rho^2 + 4\rho^{2p} + \rho^{4p} + \rho^{2+4p}) \\ B_p &:= (\rho^2 - p^2 \rho^{2p} + 2(-1 + p^2) \rho^{2+2p} - p^2 \rho^{4+2p} + \rho^{2+4p}) \\ &\quad (3\rho^2 + (8 + p^2) \rho^{2p} - 2(-1 + p^2) \rho^{2+2p} + p^2 \rho^{4+2p} + 3\rho^{2+4p}). \end{aligned}$$

II.2 Une approche de Kohn-Vogelius pour le cas des équations de Stokes

We have

$$D^2 J_{KV}(\omega^*) [(\cos k\theta)\vec{e}_r, (\sin l\theta)\vec{e}_r] = 0$$

and

$$\begin{aligned} D^2 J_{KV}(\omega^*) [(\cos k\theta)\vec{e}_r, (\cos l\theta)\vec{e}_r] &= D^2 J_{KV}(\omega^*) [(\sin k\theta)\vec{e}_r, (\sin l\theta)\vec{e}_r] \\ &= \begin{cases} -\frac{1}{2} (R_{n,|n-k|} + R_{n,|n+l|}) & \text{if } k-l=2n \\ -\frac{1}{2} (R_{n,|n+k|} + R_{n,|n-l|}) & \text{if } l-k=2n \\ R_{n,|n+k|} + R_{n,|n-l|} & \text{if } k=l, \end{cases} \end{aligned}$$

where

$$\begin{aligned} R_{n,p} &:= \frac{-8\pi\rho^{2+2n}}{\xi_n^2} \kappa_n^2 \frac{\rho^{2p} A_p}{B_p} \quad \text{if } p > 1 \\ R_{n,1} &:= -\frac{\pi\rho^{2n}}{\xi_n^2} \kappa_n^2 \frac{-3 + 12\rho^2 - 6\rho^4 - 4\rho^6 + \rho^8 + 16\rho^4 \ln(\rho)}{(-3 + 3\rho^2 - \rho^4 + \rho^6)(1 - \rho^2 + (1 + \rho^2) \ln(\rho))} \\ R_{n,0} &:= 0. \end{aligned}$$

Remark II.2.7. In particular, if $|n-k| \notin \{0,1\}$, notice that we can write the diagonal elements of the shape Hessian under the form

$$\begin{aligned} D^2 J_{KV}(\omega^*) [(\cos k\theta)\vec{e}_r, (\cos k\theta)\vec{e}_r] &= D^2 J_{KV}(\omega^*) [(\sin k\theta)\vec{e}_r, (\sin k\theta)\vec{e}_r] \\ &= \frac{-8\pi\rho^{2+2n}}{\xi_n^2} \kappa_n^2 \left(\frac{\rho^{2|n+k|} A_{|n+k|}}{B_{|n+k|}} + \frac{\rho^{2|n-k|} A_{|n-k|}}{B_{|n-k|}} \right) \end{aligned}$$

Let us compare our Stokes case with the electrical impedance tomography (E.I.T.) situation (see [3, Proposition 3]). We make the computations of the E.I.T. case with Dirichlet boundary conditions (in [3], they impose Neumann boundary conditions). Proceeding as in [3] and as previously, we prove that, in the E.I.T. situation with Dirichlet boundary conditions (and with $\mathbf{f} = \cos(n\theta)\vec{e}_r$, $n > 1$), the Hessian matrix is defined similarly as in Proposition II.2.6 with

$$\begin{aligned} R_{n,p} &:= \frac{-4\pi n^2 \rho^{-2+2n}}{(1 - \rho^{2n})^2} \frac{p\rho^{2p}}{-1 + \rho^{4p}} \quad \text{if } p \geq 1 \\ R_{n,0} &:= 0. \end{aligned}$$

Hence our shape Hessian has the same aspect than the Hessian in the E.I.T. case.

Exemple II.2.8. In order to be complete, we give some examples of the influence of the different parameters on the value of the smallest eigenvalue λ_1 of the shape Hessian $D^2 J_{KV}(\omega^*)$ and hence on the reconstruction of the object.

In Table II.1, we study the influence of the size of the object ω^* . We here assume that $\mathbf{f} = \cos(2\theta)\vec{e}_r$ (i.e. $n = 2$) and $k, l = 1, \dots, 7$. The regularizing behavior is emphasized: the more the object is far of

Tableau II.1 – Influence of the size of the object ρ on the smallest eigenvalue λ_1 of the shape Hessian $D^2 J_{KV}(\omega^*)$.

ρ	0.9	0.7	0.5	0.3	0.1	0.05
λ_1	8.9401e+05	1.0921e+03	3.9992	0.0019	3.1851e-10	1.9032e-14

the exterior boundary $\partial\Omega$, the more the functional is degenerated (and the more it is difficult to detect it).

Tableau II.2 – Influence of the *high frequencies* on the smallest eigenvalue λ_1 of the shape Hessian $D^2 J_{KV}(\omega^*)$.

k, l	1, ..., 5	1, ..., 10	1, ..., 15	1, ..., 30	1, ..., 40	1, ..., 60
λ_1	6.4483e+03	137.7707	14.5852	6.9862e-04	3.6934e-06	4.7546e-12

In Table II.2, we study the influence of the deformation directions. Here again we assume $n = 2$ and we fix $\rho = 0.7$. As we expected taking into account of our compactness result II.2.4, we see that the problem is severely ill-posed for the high frequencies.

In order to highlight this degeneracy and to compare our Stokes case with the E.I.T. case (with Dirichlet boundary conditions), we present in Figure II.5 the spectrum of the shape Hessian matrix for 50 Fourier modes. Here again we fix $n = 2$ and $\rho = 0.7$. The results are presented in decimal

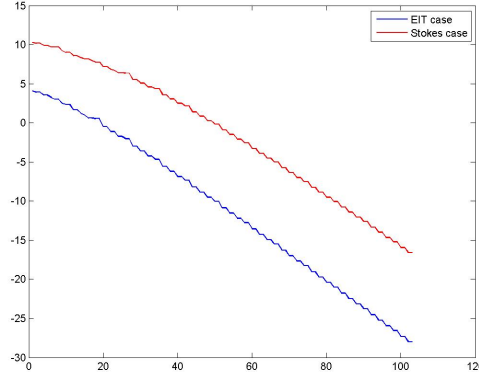


Figure II.5 – The typical spectrum of the shape Hessian in decimal logarithm scale

logarithm scale to emphasize the behavior of the spectrum. As a clear consequence, the corresponding mode is poorly distinguished by the functional: the functional is almost flat in the direction of high frequencies. We will have to take into account of this point for the numerical simulations. Figure II.5 also suggests that the E.I.T. case is as ill-posed as the Stokes case.

II.2.3.2 Resolution of the Stokes equations in a concentric annulus: using the PDE

In this subsection, we solve explicitly some Stokes boundary value problems set in the concentric annulus

$$\Omega_\rho = \{x \in \mathbb{R}^2, \rho < |x| < 1\}.$$

We recall that the pair $(\vec{e}_r, \vec{e}_\theta) := \left(\begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix} \right)$ represents the polar coordinates system.

In order to simplify the expressions, we recall that we assumed $\nu = 1$ and $O = \partial\Omega$. We first use the partial differential equations to derive the special form of the solution, then reduce the resolution of the boundary value problem to some linear system.

Noticing that the incompressibility condition $\operatorname{div} \mathbf{u} = 0$ implies that the pressure p is harmonic, we pass in polar coordinates (r, θ) in which p can be expanded in Laurent's series. Hence we seek the

velocity $\mathbf{u}$ and the pressure under the form:

$$\mathbf{u}(r, \theta) = \begin{pmatrix} u_r(r, \theta) \\ u_\theta(r, \theta) \end{pmatrix} = \begin{pmatrix} \sum_{n \in \mathbb{Z}} u_{n,r}(r) e^{in\theta} \\ \sum_{n \in \mathbb{Z}} u_{n,\theta}(r) e^{in\theta} \end{pmatrix} \quad \text{and} \quad p(r, \theta) = \sum_{n \in \mathbb{Z}} p_n(r) e^{in\theta},$$

with

$$p_0(r) = \alpha_0 + \beta_0 \ln r \quad \text{and} \quad p_n(r) = \alpha_n r^{|n|} + \beta_n r^{-|n|}, \quad n \in \mathbb{Z}^*.$$

We distinguish the case $n = 0$ of the other Fourier modes.

Case $n = 0$. We are led to solve:

$$u''_{0,r} + \frac{1}{r} u'_{0,r} - \frac{1}{r^2} u_{0,r} = \frac{\beta_0}{r} \quad \text{and} \quad u''_{0,\theta} + \frac{1}{r} u'_{0,\theta} - \frac{1}{r^2} u_{0,\theta} = 0,$$

whose solutions are

$$u_{0,r}(r) = A_{0,r} r + \frac{B_{0,r}}{r} + \frac{\beta_0}{2} r \ln r \quad \text{and} \quad u_{0,\theta}(r) = A_{0,\theta} r + \frac{B_{0,\theta}}{r}.$$

The incompressibility condition imposes:

$$u_{0,r}(r) = \frac{B_{0,r}}{r} \quad \text{and} \quad u_{0,\theta}(r) = A_{0,\theta} r + \frac{B_{0,\theta}}{r}.$$

Case $n \neq 0$. We are led to solve the system:

$$\begin{cases} u''_{n,r} + \frac{3}{r} u'_{n,r} - \frac{n^2-1}{r^2} u_{n,r} = |n| [\alpha_n r^{|n|-1} - \beta_n r^{-|n|-1}] \\ u''_{n,\theta} + \frac{1}{r} u'_{n,\theta} - \frac{n^2+1}{r^2} u_{n,\theta} = in [\alpha_n r^{|n|-1} + \beta_n r^{-|n|-1} - 2r^{-2} u_{n,r}] \end{cases}.$$

As we have a triangular system, we first determine the radial component $u_{n,r}$. We get after some calculations

$$\begin{cases} u_{n,r}(r) = A_{n,r} r^{n-1} + B_{n,r} r^{-n-1} + \frac{\alpha_n}{8} r^2 - \frac{\beta_n}{2} \ln r & \text{if } n = \pm 1 \\ u_{n,r}(r) = A_{n,r} r^{n-1} + B_{n,r} r^{-n-1} + \frac{|n|}{4} \left[\frac{\alpha_n}{|n|+1} r^{|n|+1} + \frac{\beta_n}{|n|-1} r^{-|n|+1} \right] & \text{else.} \end{cases}$$

Then we solve the equation in the angular component and impose the incompressibility condition to get

$$\begin{cases} u_{n,\theta}(r) = A_{n,r} i r^{n-1} - B_{n,r} i r^{-n-1} + \alpha_n i \left(\frac{sg(n)}{2} - \frac{n}{8} \right) r^2 \\ \quad - \frac{i}{2} \beta_n (sg(n) + n \ln r) & \text{if } n = \pm 1, \\ u_{n,\theta}(r) = A_{n,r} i r^{n-1} - B_{n,r} i r^{-n-1} + \alpha_n i \left(\frac{sg(n)}{2} - \frac{n}{4(|n|+1)} \right) r^{|n|+1} \\ \quad + \beta_n i \left(-\frac{sg(n)}{2} + \frac{n}{4(|n|-1)} \right) r^{-|n|+1} & \text{else,} \end{cases}$$

where $sg(n)$ denotes the sign of n . Since we deal with states taking real values, we have obviously

$$A_{-n,r} = \overline{B_{n,r}}, \quad \alpha_{-n} = \overline{\alpha_n}, \quad \beta_{-n} = \overline{\beta_n}.$$

The solution $\mathbf{u}$. Noticing $\mathbf{u}_n = \begin{pmatrix} u_{n,r} \\ u_{n,\theta} \end{pmatrix}$, we have for $n > 1$

$$\mathbf{u}_n(r) = A_{n,r} r^{n-1} \begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{B_{n,r}}{r^{n+1}} \begin{pmatrix} 1 \\ -i \end{pmatrix} + \alpha_n r^{n+1} \begin{pmatrix} \frac{n}{4(1+n)} \\ \frac{n+2}{4(1+n)} i \end{pmatrix} + \beta_n r^{1-n} \begin{pmatrix} \frac{n}{4(n-1)} \\ \frac{2-n}{4(n-1)} i \end{pmatrix}$$

with $p = \sum_{n \in \mathbb{Z}^*} (\alpha_n r^{|n|} + \beta_n r^{-|n|}) + \alpha_0$. We characterize in the same way the solution for $n \leq 1$.

The normal derivative $\partial_{\mathbf{n}} \mathbf{u}$. We easily obtain for $n > 1$

$$\begin{aligned} \partial_{\mathbf{n}} \mathbf{u}_n(r) = (n-1) A_{n,r} r^{n-2} \begin{pmatrix} 1 \\ i \end{pmatrix} + (n+1) \frac{B_{n,r}}{r^{n+2}} \begin{pmatrix} -1 \\ i \end{pmatrix} \\ + \alpha_n r^n \begin{pmatrix} \frac{n}{4} \\ \frac{n+2}{4} i \end{pmatrix} + \beta_n r^{-n} \begin{pmatrix} -\frac{n}{4} \\ \frac{n-2}{4} i \end{pmatrix} \end{aligned} \quad (\text{II.2.17})$$

and we characterize $\partial_{\mathbf{n}} \mathbf{u}_n$ in the same way for $n \leq 1$.

The Neumann boundary condition $-\partial_{\mathbf{n}} \mathbf{u} + p \mathbf{n}$. We easily obtain for $n > 1$

$$\begin{aligned} -\partial_{\mathbf{n}} \mathbf{u}_n(r) + p_n \mathbf{n} = A_{n,r} r^{n-2} \begin{pmatrix} 1-n \\ i(1-n) \end{pmatrix} + \frac{B_{n,r}}{r^{n+2}} \begin{pmatrix} n+1 \\ -i(n+1) \end{pmatrix} \\ + \alpha_n r^n \begin{pmatrix} 1-\frac{n}{4} \\ -\frac{n+2}{4} i \end{pmatrix} + \beta_n r^{-n} \begin{pmatrix} 1+\frac{n}{4} \\ \frac{2-n}{4} i \end{pmatrix} \end{aligned} \quad (\text{II.2.18})$$

and we characterize $-\partial_{\mathbf{n}} \mathbf{u}_n + p_n \mathbf{n}$ in the same way for $n \leq 1$.

II.2.3.3 Resolution of the Stokes equations in a concentric annulus: using the boundary conditons

In order to compute the shape hessian at Ω_ρ , we have to solve two types of boundary value problem: first a Dirichlet's one then a mixed one with Dirichlet boundary condition on the inner boundary and a Neumann boundary condition on the outer one. We recall that we assume for these computations that $O = \partial\Omega$. Thus Problem (II.2.5) is now

$$\begin{cases} -\nu \Delta \mathbf{u}_N + \nabla p_N = \mathbf{0} & \text{in } \Omega \setminus \bar{\omega} \\ \operatorname{div} \mathbf{u}_N = 0 & \text{in } \Omega \setminus \bar{\omega} \\ -\nu \partial_{\mathbf{n}} \mathbf{u}_N + p_N \mathbf{n} = \mathbf{g} & \text{on } \partial\Omega \\ \mathbf{u}_N = \mathbf{0} & \text{on } \partial\omega. \end{cases}$$

From now, we only focus on the cases $n > 1$. We point out that we will choose boundary data such that all the coefficient will be real valued. Thus

$$\begin{aligned} A_{-n,r} &= \overline{B_{n,r}} = B_{n,r} \\ \alpha_{-n} &= \overline{\alpha_n} = \alpha_n \\ \beta_{-n} &= \overline{\beta_n} = \beta_n. \end{aligned}$$

The solution $\mathbf{u}_D$ of the Dirichlet Stokes system. Let $\mathbf{f}^{(ext)} \in \mathbf{L}^2(S^1)$ and $\mathbf{f}^{(int)} \in \mathbf{L}^2(\rho S^1)$. One seeks $\mathbf{u}$ and p such that

$$\begin{aligned} \mathbf{u} &= \mathbf{f}^{(ext)}(\theta) = \begin{pmatrix} f_r^{(ext)}(\theta) \\ f_\theta^{(ext)}(\theta) \end{pmatrix} & \text{on } \partial\Omega = S^1 \\ \mathbf{u} &= \mathbf{f}^{(int)}(\theta) = \begin{pmatrix} f_r^{(int)}(\theta) \\ f_\theta^{(int)}(\theta) \end{pmatrix} & \text{on } \partial\omega = \rho S^1. \end{aligned}$$

The boundary conditions are expanded in Fourier series:

$$\begin{aligned} f_r^{(ext)}(\theta) &= \sum_{n \in \mathbb{Z}} f_{n,r}^{(ext)} e^{in\theta} \quad \text{and} \quad f_\theta^{(ext)}(\theta) = \sum_{n \in \mathbb{Z}} f_{n,\theta}^{(ext)} e^{in\theta}, \\ f_r^{(int)}(\theta) &= \sum_{n \in \mathbb{Z}} f_{n,r}^{(int)} e^{in\theta} \quad \text{and} \quad f_\theta^{(int)}(\theta) = \sum_{n \in \mathbb{Z}} f_{n,\theta}^{(int)} e^{in\theta}. \end{aligned}$$

We have to seek the (unique) solution of the following linear system

$$\left\{ \begin{array}{l} A_{n,r}^D + B_{n,r}^D + \frac{n}{4n+4} \alpha_n^D + \frac{n}{4n-4} \beta_n^D = f_{n,r}^{(ext)} \\ A_{n,r}^D - B_{n,r}^D + \frac{n+2}{4n+4} \alpha_n^D + \frac{2-n}{4n-4} \beta_n^D = \frac{f_{n,\theta}^{(ext)}}{i} \\ \rho^{n-1} A_{n,r}^D + \rho^{-n-1} B_{n,r}^D + \frac{n}{4n+4} \rho^{n+1} \alpha_n^D + \frac{n}{4n-4} \rho^{1-n} \beta_n^D = f_{n,r}^{(int)} \\ \rho^{n-1} A_{n,r}^D - \rho^{-n-1} B_{n,r}^D + \frac{n+2}{4n+4} \rho^{n+1} \alpha_n^D + \frac{2-n}{4n-4} \rho^{1-n} \beta_n^D = \frac{f_{n,\theta}^{(int)}}{i}. \end{array} \right.$$

We point out that we will choose boundary data such that all the coefficients $A_{n,r}^D$, $B_{n,r}^D$, α_n^D and β_n^D will be real valued. We denote

$$M_n = (\rho^n - \rho^{-n})^2 - n^2(\rho - \rho^{-1})^2.$$

Using the fact that

$$\rho^2 - n^2 \rho^{2n} + 2(-1 + n^2) \rho^{2+2n} - n^2 \rho^{4+2n} + \rho^{2+4n} = \rho^2 \rho^{2n} M_n,$$

we then get the solution of the linear system:

$$\begin{aligned} A_{n,r}^D &= \frac{n^2(1 - \rho^2) + (n+2)(\rho^{-2n} - 1)}{2M_n} f_{n,r}^{(ext)} + \frac{n((1 - \rho^{-2n}) + n^2(1 - \rho^2))}{2M_n} \frac{f_{n,\theta}^{(ext)}}{i} \\ &\quad + \frac{n\rho^{-n}(\rho - \rho^{-1}) - \rho(\rho^n - \rho^{-n})}{2M_n} f_{n,r}^{(int)} + \frac{n\rho^{-n}(\rho - \rho^{-1}) + (n+2)\rho(\rho^n - \rho^{-n})}{2M_n} \frac{f_{n,\theta}^{(int)}}{i} \\ B_{n,r}^D &= \frac{n^2(1 - \rho^2) + (n-2)(1 - \rho^{2n})}{2M_n} f_{n,r}^{(ext)} + \frac{(1 - \rho^{2n}) + n(\rho^2 - 1)}{2M_n} \frac{f_{n,\theta}^{(ext)}}{i} \\ &\quad + \frac{(1-n)\rho^{-n}(\rho^{-1} - \rho) + \rho^{n-1} - \rho^{1-n}}{M_n} f_{n,r}^{(int)} + \frac{n^2\rho^n(\rho - \rho^{-1}) + n\rho(\rho^n - \rho^{-n})}{2M_n} \frac{f_{n,\theta}^{(int)}}{i} \\ \alpha_n^D &= \frac{(1 - \rho^{-2n}) + (n-2)(1 - \rho^{-2})}{M_n} f_{n,r}^{(ext)} + 2(1+n) \frac{n(1 - \rho^{-2}) - (1 - \rho^{-2n})}{M_n} \frac{f_{n,\theta}^{(ext)}}{i} \\ &\quad + \frac{(n-2)\rho^{-n}(\rho - \rho^{-1}) + \rho^{-1}(\rho^n - \rho^{n-1})}{2M_n} f_{n,r}^{(int)} \\ &\quad + \frac{(1+n)\rho^n(\rho - \rho^{-1}) + (\rho^{n+1} - \frac{1}{\rho^{n+1}})}{2M_n} \frac{f_{n,\theta}^{(int)}}{i} \\ \beta_n^D &= \frac{(n-2)(1 - \rho^{-2}) + (1 - \rho^{2n})}{M_n} f_{n,r}^{(ext)} + \frac{n(1 - \rho^{-2}) + (1 - \rho^{2n})}{M_n} \frac{f_{n,\theta}^{(ext)}}{i} \\ &\quad + 2(1-n) \frac{(1+n)\rho^n(\rho - \rho^{-1}) + (\rho^{n+1} - \frac{1}{\rho^{n+1}})}{2M_n} f_{n,r}^{(int)} \\ &\quad + 2(n-1) \frac{n\rho^n(\rho - \rho^{-1}) + \rho^{-1}(\rho^n - \rho^{-n})}{M_n} \frac{f_{n,\theta}^{(int)}}{i}. \end{aligned}$$

Remark II.2.9. We quote from Engliš and Peetre that M_n is called an Almansi determinant (see [53]). This arise in connection with the following interpolation problem: try to reconstruct a function f of the type $f(x) = P(x) + e^{\mu x}Q(x)$ where P and Q are poynomials of respective degree m , and n , given its value at $m + n$ points $x_1, x_2, \dots, x_{m+n}$.

Remark II.2.10. To compare the factorization in the case of the Laplacian and the case of the Stokes system, we recall that in the case of the Laplacian, we find a denominator equal to

$$\rho^{|n|} - \rho^{-|n|} = \rho^{-|n|}(\rho^{2|n|} - 1) = \rho^{-|n|}(\rho^{|n|} - 1)(\rho^{|n|} + 1).$$

In the case of Stokes system, we rewrite the denominator as

$$M_n = \left((\rho^{|n|} - \rho^{-|n|}) + |n|(\rho^2 - \rho^{-2}) \right) \left((\rho^{|n|} - \rho^{-|n|}) - |n|(\rho^2 - \rho^{-2}) \right).$$

Application to our case for u_D . We get

$$\begin{pmatrix} A_{n,r}^D \\ B_{n,r}^D \\ \alpha_n^D \\ \beta_n^D \end{pmatrix} = \begin{pmatrix} \frac{n^2(1-\rho^2)+(n+2)(\rho^{-2n}-1)}{2M_n} & \frac{n((1-\rho^{-2n})+n^2(1-\rho^2))}{2M_n} \\ \frac{n^2(1-\rho^2)+(n-2)(1-\rho^{2n})}{2M_n} & \frac{(1-\rho^{2n})+n(\rho^2-1)}{2M_n} \\ \frac{(1-\rho^{-2n})+(n-2)(1-\rho^{-2})}{M_n} & \frac{n(1-\rho^{-2})-(1-\rho^{-2n})}{M_n} \\ \frac{(n-2)(1-\rho^{-2})+(1-\rho^{2n})}{M_n} & 2(1-n)\frac{n(1-\rho^{-2})+(1-\rho^{2n})}{M_n} \end{pmatrix} \begin{pmatrix} f_{n,r} \\ f_{n,\theta} \\ i \end{pmatrix}.$$

We choose $\mathbf{f}^{(ext)}(\theta) = \mathbf{f}(\theta) = \cos(n\theta)\vec{e}_r$, $n > 1$ and $\mathbf{f}^{(int)} = \mathbf{0}$. Hence $f_{n,r} = \frac{1}{2}$, $f_{n,\theta} = 0$ and

$$\begin{pmatrix} A_{n,r}^D \\ B_{n,r}^D \\ \alpha_n^D \\ \beta_n^D \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \frac{n^2(1-\rho^2)+(n+2)(\rho^{-2n}-1)}{2M_n} \\ \frac{n^2(1-\rho^2)+(n-2)(1-\rho^{2n})}{2M_n} \\ \frac{(1-\rho^{-2n})+(n-2)(1-\rho^{-2})}{M_n} \\ \frac{(n-2)(1-\rho^{-2})+(1-\rho^{2n})}{M_n} \end{pmatrix}.$$

The solution u_N of the mixed Stokes system. The coefficients are denoted $A_{n,r}^N$, $B_{n,r}^N$ and α_n^N and β_n^N . To fulfill the boundary conditions

$$-\partial_n u_N + p_N \vec{e}_r = \mathbf{g}^{(ext)} \quad \text{if } |x| = 1 \quad \text{and} \quad u_N = \mathbf{g}^{(int)} \quad \text{if } |x| = \rho,$$

we have to seek the (unique) solution of the following linear system

$$\begin{cases} (1-n)A_{n,r}^N + (1+n)B_{n,r}^N + \frac{4-n}{4}\alpha_n^N + \frac{4+n}{4}\beta_n^N = g_{n,r}^{(ext)} \\ (1-n)A_{n,r}^N - (1+n)B_{n,r}^N - \frac{2+n}{4}\alpha_n^N + \frac{2-n}{4}\beta_n^N = \frac{g_{n,\theta}^{(ext)}}{i} \\ \rho^{n-1}A_{n,r}^N + \rho^{-n-1}B_{n,r}^N + \rho^{n+1}\frac{n}{4n+4}\alpha_n^N + \rho^{1-n}\frac{n}{4n-4}\beta_n^N = g_{n,r}^{(int)} \\ \rho^{n-1}A_{n,r}^N - B_{n,r}^N\rho^{-n-1} + \rho^{n+1}\frac{n+2}{4n+4}\alpha_n^N + \rho^{1-n}\frac{2-n}{4n-4}\beta_n^N = \frac{g_{n,\theta}^{(int)}}{i}. \end{cases} \quad (\text{II.2.19})$$

We denote

$$N_n = 3\rho^2 + (8+n^2)\rho^{2n} - 2(-1+n^2)\rho^{2+2n} + n^2\rho^{4+2n} + 3\rho^{2+4n}$$

which can be written under the form

$$N_n = 3(\rho^n - \rho^{-n})^2 + n^2(\rho - \rho^{-1})^2 + 8(1 + \rho^{-2}).$$

II.2 Une approche de Kohn-Vogelius pour le cas des équations de Stokes

The solution of the system corresponding to a general Neumann problem with Dirichlet conditions on the interface is obtained after expanding the inverse of the matrix corresponding to (II.2.19). We get

$$\begin{aligned}
A_{n,r}^N &= -\frac{(n+2)(\rho^{-2n}-1)+n^2(\rho^2-1)+4n}{2(n-1)N_n}g_{n,r}^{(ext)} + \frac{n((1-\rho^{-2n})+n^2(1-\rho^2))}{2N_n}\frac{g_{n,\theta}^{(ext)}}{i} \\
&\quad + \frac{3(n+2)\rho(\rho^n-\rho^{-n})+4(n+2)\rho^{-n}(\rho-\rho^{-1})+4(n+4)\rho^{-1-n}}{2N_n}g_{n,r}^{(int)} \\
&\quad + \frac{n(n+4)\rho^{-n}(\rho-\rho^{-1})+3n\rho(\rho^n-\rho^{-n})+4(n+2)\rho^{-1-n}}{2N_n}\frac{g_{n,\theta}^{(int)}}{i} \\
B_{n,r}^N &= -\frac{n^2(1-\rho^2)+(n-2)(\rho^{2n}-1)+4n}{2(1+n)N_n}g_{n,r}^{(ext)} - \frac{(4+n)(\rho^{2n}-1)+n^2(\rho^2-1)+4(n+2)}{2(1+n)N_n}\frac{g_{n,\theta}^{(ext)}}{i} \\
&\quad - \frac{(3(n-2)\rho(\rho^{-n}-\rho^n)-(n^2+8)\rho^n(\rho^{-1}-\rho)-2(n+2)\rho^1+n)}{2N_n}g_{n,r}^{(int)} \\
&\quad + \frac{n^2\rho^n(\rho-\rho^{-1})-n\rho(\rho^n-\rho^{-n})-4n(\rho^{1-n}-\rho^{n-1})-4(n+2)\rho^{n-1}}{2N_n}\frac{g_{n,\theta}^{(int)}}{i} \\
\alpha_n^N &= 2\frac{(2-n)(\rho^{-2}-1)+(\rho^{-2n}-1)+4}{N_n}g_{n,r}^{(ext)} + 2\frac{(4+n)(1-\rho^{-2})+(1-\rho^{-2n})-4}{N_n}\frac{g_{n,\theta}^{(ext)}}{i} \\
&\quad - 2(1+n)\frac{(n-1)\rho(\rho^{-n}-\rho^n)+3\rho(\rho^n-\rho^{-n})+(4-n)\rho^{1-n}}{N_n}g_{n,r}^{(int)} \\
&\quad + 2(1+n)\frac{(n-1)\rho^{-n}(\rho-\rho^{-1})+3(\rho^{n-1}-\rho^{1-n})+4\rho^{1-n}}{N_n}\frac{g_{n,\theta}^{(int)}}{i} \\
\beta_n^N &= \frac{(n+2)(\rho^{-2}-1)+(\rho^{2n}-1)+4}{N_n}g_{n,r}^{(ext)} + \frac{(4-n)(\rho^{-2}-1)+(\rho^{2n}-1)+2(n-2)}{N_n}\frac{g_{n,\theta}^{(ext)}}{i} \\
&\quad + 2(n-1)\frac{(1+n)\rho^n(\rho-\rho^{-1})+3\rho^{-1}(\rho^n-\rho^{-n})+(n+4)\rho^{n-1}}{N_n}g_{n,r}^{(int)} \\
&\quad + 2(n-1)\frac{n\rho^n(\rho-\rho^{-1})-\rho^{-1}(\rho^n-\rho^{-n})-4\rho-1-n}{N_n}\frac{g_{n,\theta}^{(int)}}{i}.
\end{aligned}$$

The Dirichlet-to-Neumann map. Here, we have to take the Neumann data

$$\mathbf{g}^{(ext)} = g_r^{(ext)}\vec{e}_r + g_\theta^{(ext)}\vec{e}_\theta$$

corresponding to the choice of the Dirichlet data $\mathbf{f}^{(ext)}$. In order to simplify the expression and to get closer to our case, we assume $\mathbf{f}^{(int)} = \mathbf{0}$. Omitting the exponent $^{(ext)}$, a straightforward calculation gives

$$\begin{aligned}
g_{n,r} &= A_{n,r}^D(1-n) + B_{n,r}^D(1+n) + \frac{4-n}{4}\alpha_n^D + \frac{4+n}{4}\beta_n^D \\
g_{n,\theta} &= i\left(A_{n,r}^D(1-n) - B_{n,r}^D(1+n) - \frac{2+n}{4}\alpha_n^D + \frac{2-n}{4}\beta_n^D\right),
\end{aligned}$$

where Fourier coefficients $A_{n,r}^D, B_{n,r}^D, \alpha_n^D$ and β_n^D are given above. The expression with respect to the Fourier coefficients $f_{n,r}$ and $f_{n,\theta}$ is then given by

$$\begin{aligned}
g_{n,r} &= -\frac{4(2-n^2)(1-\rho^{-2})+2n(\rho^{-2n}-\rho^{2n})+(\rho^n-\rho^{-n})^2+n^2(\rho-\rho^{-1})^2}{M_n}f_{n,r} \\
&\quad + \frac{4n(1-\rho^{-2})+2(\rho^{-2n}-\rho^{-2n})+n^3(\rho-\rho^{-1})^2+n(\rho^n-\rho^{-n})^2}{M_n}\frac{f_{n,\theta}}{i}
\end{aligned}$$

and

$$g_{n,\theta} = i \frac{4n(1-\rho^{-2}) + 2(\rho^{-2n} - \rho^{2n}) + n(\rho^n - \rho^{-n})^2 + n^3(\rho - \rho^{-1})^2}{M_n} f_{n,r} - \frac{4n^2(1-\rho^{-2}) + 2n(\rho^{-2n} - \rho^{2n}) + (\rho^{-n} - \rho^n)^2 + n^2(\rho^{-1} - \rho)^2}{M_n} f_{n,\theta}.$$

We illustrate the mapping $(f_{n,r}, f_{n,\theta}) \mapsto (g_{n,r}, g_{n,\theta})$ via the linear transformation

$$\begin{pmatrix} g_{n,r} \\ g_{n,\theta} \end{pmatrix} = \begin{pmatrix} \Delta_{1,1}^{(n)} & \Delta_{1,2}^{(n)} \\ \Delta_{2,1}^{(n)} & \Delta_{2,2}^{(n)} \end{pmatrix} \begin{pmatrix} f_{n,r} \\ f_{n,\theta} \end{pmatrix}$$

where

$$\begin{aligned} \Delta_{1,1}^{(n)} &= -\frac{4(2-n^2)(1-\rho^{-2}) + 2n(\rho^{-2n} - \rho^{2n}) + (\rho^n - \rho^{-n})^2 + n^2(\rho - \rho^{-1})^2}{M_n} \\ \Delta_{1,2}^{(n)} &= -i \frac{4n(1-\rho^{-2}) + 2(\rho^{-2n} - \rho^{2n}) + n(\rho^n - \rho^{-n})^2 + n^3(\rho - \rho^{-1})^2}{M_n} \\ \Delta_{2,1}^{(n)} &= \overline{\Delta_{1,2}^{(n)}} \\ \Delta_{2,2}^{(n)} &= -\frac{4n^2(1-\rho^{-2}) + 2n(\rho^{-2n} - \rho^{2n}) + (\rho^{-n} - \rho^n)^2 + n^2(\rho^{-1} - \rho)^2}{M_n}. \end{aligned}$$

We can deduce the Dirichlet-to-Neumann map from this representation: in the trigonometric basis it is represented by a block diagonal matrix; its n -th diagonal is given by the matrix $\Delta^{(n)} = (\Delta_{i,j}^{(n)})_{i,j=1,2}$.

Application to our case for u_N . In our case, with $\mathbf{f}^{(ext)}(\theta) = \mathbf{f}(\theta) = \cos(n\theta) \vec{e}_r$, $n > 1$ and $\mathbf{f}^{(int)} = \mathbf{0}$, we then obtain

$$\begin{cases} g_{n,r} &= -\frac{4(2-n^2)(1-\rho^{-2}) + 2n(\rho^{-2n} - \rho^{2n}) + (\rho^n - \rho^{-n})^2 + n^2(\rho - \rho^{-1})^2}{M_n} f_{n,r} \\ g_{n,\theta} &= i \frac{4n(1-\rho^{-2}) + 2(\rho^{-2n} - \rho^{2n}) + n(\rho^n - \rho^{-n})^2 + n^3(\rho - \rho^{-1})^2}{M_n} f_{n,r}. \end{cases} \quad (\text{II.2.20})$$

Hence

$$\begin{pmatrix} A_{n,r}^N \\ B_{n,r}^N \\ \alpha_n^N \\ \beta_n^N \end{pmatrix} = \begin{pmatrix} -\frac{(n+2)(\rho^{-2n}-1)+n^2(\rho^2-1)+4n}{2(n-1)N_n} & \frac{n((1-\rho^{-2n})+n^2(1-\rho^2))}{2N_n} \\ -\frac{n^2(1-\rho^2)+(n-2)(\rho^{2n}-1)+4n}{2(1+n)N_n} & -\frac{(4+n)(\rho^{2n}-1)+n^2(\rho^2-1)+4(n+2)}{2(1+n)N_n} \\ 2\frac{(2-n)(\rho^{-2}-1)+(\rho^{-2n}-1)+4}{N_n} & 2\frac{(4+n)(1-\rho^{-2})+(1-\rho^{-2n})-4}{N_n} \\ \frac{(n+2)(\rho^{-2}-1)+(\rho^{2n}-1)+4}{N_n} & \frac{(4-n)(\rho^{-2}-1)+(\rho^{2n}-1)+2(n-2)}{N_n} \end{pmatrix} \begin{pmatrix} g_{n,r} \\ g_{n,\theta} \\ i \end{pmatrix}.$$

II.2.3.4 Explicit formulae for the shape derivatives u'_D and u'_N in a concentric annulus

In this section, we deform the domain via the vector field $\mathbf{V} = \cos k\theta \vec{e}_r$. We then have $V_n = \cos k\theta$. Here again, we only focus on the cases $n > 1$.

The formula for the shape derivative u'_D . For $n > 1$, we set

$$\begin{aligned} K_{n,r}^D &= A_{n,r}^D(1-n)\rho^{n-2} + \rho^{-n-2}B_{n,r}^D(1+n) - \alpha_n^D \rho^n \frac{n}{4} + \beta_n^D \rho^{-n} \frac{n}{4} \\ L_{n,r}^D &= A_{n,r}^D(1-n)\rho^{n-2} - \rho^{-n-2}B_{n,r}^D(1+n) - \alpha_n^D \rho^n \frac{n+2}{4} + \beta_n^D \rho^{-n} \frac{2-n}{4} \end{aligned} \quad (\text{II.2.21})$$

II.2 Une approche de Kohn-Vogelius pour le cas des équations de Stokes

the respective Fourier coefficients of $-\partial_{\mathbf{n}} \mathbf{u}_D(\rho, \theta)$. As it has been noticed, when $n > 1$, we have

$$K_{-n,r}^D = \overline{K_{n,r}^D} \quad \text{and} \quad L_{-n,r}^D = -\overline{L_{n,r}^D}.$$

We will then focus on the coefficients $A_{n,r}^{D'}$, $B_{n,r}^{D'}$, $\alpha_n^{D'}$ and $\beta_n^{D'}$ for $n > 1$. Let us point out that the computations give $K_{n,r}^D = 0$. We have (see (II.2.17))

$$\begin{aligned} -\partial_{\mathbf{n}} \mathbf{u}_D(\rho, \theta) &= \left(K_{n,r}^D e^{in\theta} + K_{-n,r}^D e^{-in\theta} \right) \vec{e}_r + i \left(L_{n,r}^D e^{in\theta} + L_{-n,r}^D e^{-in\theta} \right) \vec{e}_\theta \\ &= i \left(L_{n,r}^D e^{in\theta} - \overline{L_{n,r}^D} e^{-in\theta} \right) \vec{e}_\theta \end{aligned}$$

and then

$$-\partial_{\mathbf{n}} \mathbf{u}_D V_n = i \frac{L_{n,r}^D e^{i(n+k)\theta} + L_{-n,r}^D e^{-i(n+k)\theta}}{2} \vec{e}_\theta + i \frac{L_{n,r}^D e^{i(n-k)\theta} + L_{-n,r}^D e^{-i(n-k)\theta}}{2} \vec{e}_\theta.$$

Hence

$$-\partial_{\mathbf{n}} \mathbf{u}_D V_n = -L_{n,r}^D \left(\sin((n+k)\theta) + \sin((n-k)\theta) \right) \vec{e}_\theta.$$

We denote

$$\mathbf{u}'_D = \sum_{p \in I_{n,k}} u_{p,r}^{D'}(r) e^{ip\theta} \vec{e}_r + \sum_{p \in I_{n,k}} u_{p,\theta}^{D'}(r) e^{ip\theta} \vec{e}_\theta,$$

where $I_{n,k} = \{n+k, n-k, -n-k, -n+k\}$. Let us assume that $-1, 0, 1 \notin I_{n,k}$ (the other cases can be treated similarly). We know that

$$\begin{cases} u_{p,r}^{D'}(r) &= A_{p,r}^{D'} r^{p-1} + \frac{B_{p,r}^{D'}}{r^{p+1}} + \alpha_p^{D'} r^{|p|+1} \frac{|p|}{4|p|+4} + \beta_p^{D'} r^{1-|p|} \frac{|p|}{4|p|-4} \\ u_{p,\theta}^{D'}(r) &= i \left(A_{p,r}^{D'} r^{p-1} - \frac{B_{p,r}^{D'}}{r^{p+1}} + \alpha_p^{D'} r^{|p|+1} \left(\frac{1}{2} \operatorname{sg}(p) - \frac{p}{4p+4} \right) + \beta_p^{D'} r^{1-|p|} \left(\frac{p}{4p-4} - \frac{1}{2} \operatorname{sg}(p) \right) \right). \end{cases}$$

We use the superposition principles: we use the fact that we have to use a linear system associated to a Dirichlet problem and then the coefficients associated to $u_{p,r}^{D'}(r)$ when $p = |n+k|$ and $p = |n-k|$ must satisfy the following linear system

$$\begin{cases} A_{p,r}^{D'} + B_{p,r}^{D'} + \frac{p}{4p+4} \alpha_p^{D'} + \frac{p}{4p-4} \beta_p^{D'} = 0 \\ A_{p,r}^{D'} - B_{p,r}^{D'} + \frac{p+2}{4p+4} \alpha_p^{D'} + \frac{2-p}{4p-4} \beta_p^{D'} = 0 \\ \rho^{p-1} A_{p,r}^{D'} + \rho^{-p-1} B_{p,r}^{D'} + \frac{p}{4p+4} \rho^{p+1} \alpha_p^{D'} + \frac{p}{4p-4} \rho^{1-p} \beta_p^{D'} = K_{n,r}^D \\ \rho^{p-1} A_{p,r}^{D'} - \rho^{-p-1} B_{p,r}^{D'} + \frac{p+2}{4p+4} \rho^{p+1} \alpha_p^{D'} + \frac{2-n}{4p-4} \rho^{1-p} \beta_p^{D'} = L_{n,r}^D. \end{cases}$$

We remark that the second side of the linear system is independent of k . Once the coefficients are computed, we use (II.2.18) to get

$$\begin{aligned} (-\partial_{\mathbf{n}} \mathbf{u}'_D + p'_D \mathbf{n})(\rho, \theta) &= \left(K_{|n+k|,r}^{D'} \cos(n+k)\theta + K_{|n-k|,r}^{D'} \cos(n-k)\theta \right) \vec{e}_r \\ &\quad - \left(L_{|n+k|,r}^{D'} \sin(n+k)\theta + L_{|n-k|,r}^{D'} \sin(n-k)\theta \right) \vec{e}_\theta, \end{aligned}$$

where

$$\begin{aligned} K_{p,r}^{D'} &= A_{p,r}^{D'} (1-p) \rho^{p-2} + \rho^{-p-2} B_{p,r}^{D'} (1+p) + \alpha_p^{D'} \rho^p \left(1 - \frac{p}{4} \right) + \beta_p^{D'} \rho^{-p} \left(1 + \frac{p}{4} \right) \\ L_{p,r}^{D'} &= A_{p,r}^{D'} (1-p) \rho^{p-2} - \rho^{-p-2} B_{p,r}^{D'} (1+p) - \alpha_p^{D'} \rho^p \frac{p+2}{4} + \beta_p^{D'} \rho^{-p} \frac{2-p}{4}. \end{aligned}$$

The formula for the shape derivative u'_N . We adopt the same strategy. We have only to write the corresponding Neumann condition on the exterior boundary (*i.e.* for $|x| = 1$). For $n > 1$, we set

$$\begin{aligned} K_{n,r}^N &= A_{n,r}^N(1-n)\rho^{n-2} + \rho^{-n-2}B_{n,r}^N(1+n) - \alpha_n^N \rho^n \frac{n}{4} + \beta_n^N \rho^{-n} \frac{n}{4} \\ L_{n,r}^N &= A_{n,r}^N(1-n)\rho^{n-2} - \rho^{-n-2}B_{n,r}^N(1+n) - \alpha_n^N \rho^n \frac{n+2}{4} + \beta_n^N \rho^{-n} \frac{2-n}{4} \end{aligned}$$

the respective Fourier coefficients of $-\partial_{\mathbf{n}} \mathbf{u}_N(\rho, \theta)$. The computations give

$$K_{n,r}^N = 0$$

and

$$\begin{aligned} L_{n,r}^N &= -2 \frac{n(n+2)\rho^{-n}(1-\rho^{-2}) + 4\rho^{-2}(\rho^{-n} - \rho^n) + 2n(n-1)\rho^n(\rho^{-2} - 1)}{M_n} f_{n,r}^{(ext)} \\ &\quad - 2 \frac{2n^2(\rho^n + \rho^{-n})(\rho^{-2} - 1) + 2n(\rho^n - \rho^{-n}) + (1 + \rho^{-2})}{M_n} \frac{f_{n,\theta}^{(ext)}}{i}. \end{aligned}$$

We solve the system

$$\left\{ \begin{array}{l} (1-p)A_{p,r}^{N'} + (1+p)\rho^{-p-2}B_{p,r}^{N'} + \frac{4-p}{4}\alpha_p^{N'} + \frac{4+p}{4}\beta_p^{N'} = 0 \\ (1-p)A_{p,r}^{N'} - (1+p)B_{p,r}^{N'} - \frac{p+2}{4}\alpha_p^{N'} + \frac{2-p}{4}\beta_p^{N'} = 0 \\ \rho^{p-1}A_{p,r}^{N'} + \rho^{-p-1}B_{p,r}^{N'} + \frac{p}{4p+4}\rho^{p+1}\alpha_p^{N'} + \frac{p}{4p-4}\rho^{1-p}\beta_p^{N'} = K_{n,r}^N \\ \rho^{p-1}A_{p,r}^{N'} - \rho^{-p-1}B_{p,r}^{N'} + \frac{p+2}{4p+4}\rho^{p+1}\alpha_p^{N'} + \frac{2-p}{4p-4}\rho^{1-p}\beta_p^{N'} = L_{n,r}^N. \end{array} \right.$$

We get

$$\begin{aligned} A_{p,r}^{N'} &= \frac{p(p+4)\rho^{-p}(\rho - \rho^{-1}) + 3p\rho(\rho^p - \rho^{-p}) + 4(p+2)\rho^{-1-p}}{2N_p} L_{n,r}^N \\ B_{p,r}^{N'} &= \frac{p^2\rho^p(\rho - \rho^{-1}) - p\rho(\rho^p - \rho^{-p}) - 4p(\rho^{1-p} - \rho^{p-1}) - 4(p+2)\rho^{p-1}}{2N_p} L_{n,r}^N \\ \alpha_p^{N'} &= 2(1+p) \frac{(p-1)\rho^{-p}(\rho - \rho^{-1}) + 3(\rho^{p-1} - \rho^{1-p}) + 4\rho^{1-p}}{N_p} L_{n,r}^N \\ \beta_p^{N'} &= 2(p-1) \frac{p\rho^p(\rho - \rho^{-1}) - \rho^{-1}(\rho^p - \rho^{-p}) - 4\rho^{-1-p}}{N_p} L_{n,r}^N. \end{aligned}$$

We have to compute now

$$\begin{aligned} (-\partial_{\mathbf{n}} \mathbf{u}'_N + p'_N \mathbf{n})(\rho, \theta) &= \left(K_{|n+k|,r}^{N'} \cos(n+k)\theta + K_{|n-k|,r}^{N'} \cos(n-k)\theta \right) \vec{e}_r \\ &\quad - \left(L_{|n+k|,r}^{N'} \sin(n+k)\theta + L_{|n-k|,r}^{N'} \sin(n-k)\theta \right) \vec{e}_\theta \end{aligned}$$

where

$$\begin{aligned} K_{p,r}^{N'} &= A_{p,r}^{N'}(1-p)\rho^{p-2} + \rho^{-p-2}B_{p,r}^{N'}(1+p) + \alpha_p^{N'} \rho^n \left(1 - \frac{p}{4}\right) + \beta_p^{N'} \rho^{-p} \left(1 + \frac{p}{4}\right) \\ L_{p,r}^{N'} &= A_{p,r}^{N'}(1-p)\rho^{p-2} - \rho^{-p-2}B_{p,r}^{N'}(1+p) - \alpha_p^{N'} \rho^p \frac{p+2}{4} + \beta_p^{N'} \rho^{-p} \frac{2-p}{4}. \end{aligned}$$

Finally it comes

$$\begin{aligned} (-\partial_{\mathbf{n}}(\mathbf{u}'_N - \mathbf{u}'_D) + (p'_N - p'_D)\mathbf{n})(\rho, \theta) &= \left(\tilde{K}_{|n+k|,r} \cos(n+k)\theta + \tilde{K}_{|n-k|,r} \cos(n-k)\theta \right) \vec{e}_r \\ &\quad - \left(\tilde{L}_{|n+k|,r} \sin(n+k)\theta + \tilde{L}_{|n-k|,r} \sin(n-k)\theta \right) \vec{e}_\theta \end{aligned}$$

where

$$\tilde{K}_{p,r} = K_{p,r}^{D'} - K_{p,r}^{N'} \quad \text{and} \quad \tilde{L}_{p,r} = L_{p,r}^{D'} - L_{p,r}^{N'}. \quad (\text{II.2.22})$$

II.2.3.5 Proof of the explicit computation of the shape Hessian

We recall that $(\vec{e}_r, \vec{e}_\theta) := \left(\begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix} \right)$ represents the polar coordinates system.

Proof of Proposition II.2.6. Using the previous computations of the solution of $\mathbf{u}_D$, $\mathbf{u}_N$, $\mathbf{u}'_D$ and $\mathbf{u}'_N$, we will compute the elements of the Hessian $D^2 J_{KV}((\cos k\theta)\vec{e}_r, (\cos l\theta)\vec{e}_r)$ when the Dirichlet data on the exterior boundary is taken as $\mathbf{f} = \mathbf{f}^{(ext)} = \cos(n\theta)\vec{e}_r$. From the computations detailed above, we see that only the elements with $k = l$ or $|k - l| = 2n$ have to be computed; the other elements are zero.

Let us begin by the case $k = l$. We have

$$D^2 J_{KV}(\cos k\theta\vec{e}_r, \cos k\theta\vec{e}_r) = \langle -\partial_{\mathbf{n}}(\mathbf{u}'_N - \mathbf{u}'_D) + (p'_N - p'_D)\mathbf{n}(\rho, \theta), -V_{\mathbf{n}}\partial_{\mathbf{n}}\mathbf{u}_D \rangle = \int_{\partial\omega^*} (E + F),$$

with

$$E(\theta) = \left(\tilde{K}_{|n+k|,r} \cos(n+k)\theta + \tilde{K}_{|n-k|,r} \cos(n-k)\theta \right) \left(K_{n,r}^D \cos(n+k)\theta + K_{n,r}^D \cos(n-k)\theta \right)$$

and with

$$F(\theta) = \left(\tilde{L}_{|n+k|,r} \sin(n+k)\theta + \tilde{L}_{|n-k|,r} \sin(n-k)\theta \right) \left(L_{n,r}^D \sin(n+k)\theta + L_{n,r}^D \sin(n-k)\theta \right),$$

where $K_{n,r}^D$ and $L_{n,r}^D$ are defined by (II.2.21) and $\tilde{K}_{n,r}$ and $\tilde{L}_{n,r}$ by (II.2.22). Since $K_{n,r}^D = 0$ by computations, we have $E(\theta) = 0$ and then we have only to compute

$$D^2 J_{KV}((\cos k\theta)\vec{e}_r, (\cos k\theta)\vec{e}_r) = \rho \int_0^{2\pi} (E(\theta) + F(\theta)) d\theta = \pi \rho \left(\tilde{L}_{|n+k|,r} L_{n,r}^D + \tilde{L}_{|n-k|,r} L_{n,r}^D \right)$$

Set $p_1 = |n+k|$ and $p_2 = |n-k|$. Defining ξ_n , κ_n , A_{p_i} , B_{p_i} and R_{n,p_i} as in the statement of Proposition II.2.6, we obtain after some computations

$$D^2 J_{KV}((\cos k\theta)\vec{e}_r, (\cos k\theta)\vec{e}_r) = R_{n,p_1}(\rho) + R_{n,p_2}(\rho).$$

Similarly, we prove that

$$D^2 J_{KV}((\sin k\theta)\vec{e}_r, (\sin k\theta)\vec{e}_r) = R_{n,p_1}(\rho) + R_{n,p_2}(\rho).$$

Let us now study the case $|k - l| = 2n$. Set $\mathbf{V} := (\cos k\theta)\vec{e}_r$ and $\mathbf{W} := (\cos l\theta)\vec{e}_r$. We denote $(\mathbf{w}'_{\mathbf{W}}, \mathbf{q}'_{\mathbf{W}})$ the shape derivative in the direction $\mathbf{W}$ (the pair $(\mathbf{w}', \mathbf{q}')$ is defined in (II.2.11)). Then, using the linearity of the Stokes problem and the polarization formula for the quadratic form, we prove that

$$\begin{aligned} D^2 J_{KV}((\cos k\theta)\vec{e}_r, (\cos l\theta)\vec{e}_r) \\ = \frac{1}{2} \left(\int_{\partial\omega^*} (-\partial \mathbf{w}'_{\mathbf{W}} + \mathbf{q}'_{\mathbf{W}} \mathbf{n}) \cdot \partial_{\mathbf{n}} \mathbf{u}_D V_{\mathbf{n}} + \int_{\partial\omega^*} (-\partial \mathbf{w}'_{\mathbf{V}} + \mathbf{q}'_{\mathbf{V}} \mathbf{n}) \cdot \partial_{\mathbf{n}} \mathbf{u}_D W_{\mathbf{n}} \right). \end{aligned}$$

Let us focus on the first integral: the case of the second integral is obtained by reversing the roles of k and l . Defining

$$G(\theta) = \left(\tilde{K}_{|n+k|,r} \cos(n+k)\theta + \tilde{K}_{|n-k|,r} \cos(n-k)\theta \right) \left(K_{n,r}^D \cos(n+l)\theta + K_{n,r}^D \cos(n-l)\theta \right)$$

and

$$H(\theta) = \left(\tilde{L}_{|n+k|,r} \sin(n+k)\theta + \tilde{L}_{|n-k|,r} \sin(n-k)\theta \right) \left(L_{n,r}^D \sin(n+l)\theta + L_{n,r}^D \sin(n-l)\theta \right),$$

we have

$$\int_{\partial\omega^*} (-\partial\mathbf{w}'_{\mathbf{W}} + q'_{\mathbf{W}}\mathbf{n}) \cdot \partial_{\mathbf{n}}\mathbf{u}_D V_n = \int_{\partial\omega^*} (G + H).$$

Using the fact that $K_{n,r}^D = 0$, we then obtain

– if $k - l = 2n$

$$\int_{\partial\omega^*} (G + H) = \rho \tilde{L}_{|n-k|,r} L_{n,r}^D \int_0^{2\pi} (\sin(n-k)\theta) \sin((n+l)\theta) = -\rho\pi \tilde{L}_{|n-k|,r} L_{n,r}^D,$$

– if $l - k = 2n$

$$\int_{\partial\omega^*} (G + H) = \rho \tilde{L}_{|n+k|,r} L_{n,r}^D \int_0^{2\pi} (\sin(n+k)\theta) \sin((n-l)\theta) = -\rho\pi \tilde{L}_{|n+k|,r} L_{n,r}^D.$$

Thus,

$$D^2 J_{KV}(\omega^*) [(\cos k\theta)\mathbf{e}_r^*, (\cos l\theta)\mathbf{e}_r^*] = \begin{cases} -\frac{\rho\pi}{2} L_{n,r}^D (\tilde{L}_{n,|n-k|} + \tilde{L}_{n,|n+l|}) & \text{if } k - l = 2n \\ -\frac{\rho\pi}{2} L_{n,r}^D (\tilde{L}_{n,|n+k|} + \tilde{L}_{n,|n-l|}) & \text{if } l - k = 2n \\ \rho\pi L_{n,r}^D (\tilde{L}_{n,|n+k|} + \tilde{L}_{n,|n-l|}) & \text{if } k = l. \end{cases}$$

We proceed in the same way for $D^2 J_{KV}(\omega^*) [(\sin k\theta)\mathbf{e}_r^*, (\sin l\theta)\mathbf{e}_r^*]$. □

II.3 A least squares approach for the stationary Navier-Stokes case

II.3.1 The results

The problem setting. Let O be a non-empty connected open subset of $\partial\Omega$ and we denote by $\{\Gamma_i\}_{i=1,\dots,I}$ the connected components of $\partial\Omega$. We assume that there exists an open subset $\tilde{O} \subset \partial\Omega$ of class $C^{1,1}$ such that $O \subset \subset \tilde{O}$. Let $\mathbf{f}_b \in \mathbf{H}^{1/2}(O)$ be an admissible boundary measurement and $\mathbf{g} \in \mathbf{H}^{1/2}(\partial\Omega)$, $\mathbf{g} \neq \mathbf{0}$ such that $\mathbf{g}|_{\tilde{O}} \in \mathbf{H}^{3/2}(\tilde{O})$ and

$$\int_{\Gamma_i} \mathbf{g} \cdot \mathbf{n} = 0, \quad \forall i = 1, \dots, I. \quad (\text{II.3.1})$$

Let us consider, for $\omega \in \mathcal{O}_\delta$ (with $\mathcal{O}_\delta$ defined by (II.0.1)), the following overdetermined Navier-Stokes boundary values problem:

$$\begin{cases} -\nu\Delta\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{0} & \text{in } \Omega \setminus \bar{\omega} \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega \setminus \bar{\omega} \\ \mathbf{u} = \mathbf{g} & \text{on } \partial\Omega \\ \mathbf{u} = \mathbf{0} & \text{on } \partial\omega \\ \nu\partial_{\mathbf{n}}\mathbf{u} - p\mathbf{n} = \mathbf{f}_b & \text{on } O. \end{cases} \quad (\text{II.3.2})$$

We assume here that there exists $\omega^* \in \mathcal{O}_\delta$ such that (II.3.2) has a solution. This means that the measurement $\mathbf{f}_b$ is perfect, that is to say without error. Thus, we consider the following geometric inverse problem:

$$\text{find } \omega \in \mathcal{O}_\delta \text{ and a pair } (\mathbf{u}, p) \text{ which satisfy the overdetermined system (II.3.2).} \quad (\text{II.3.3})$$

To solve this inverse problem, we consider, for $\omega \in \mathcal{O}_\delta$, the least squares functional

$$J(\omega) := \int_O m^2 |\nu \partial_{\mathbf{n}} \mathbf{u}(\omega) - p(\omega) \mathbf{n} - \mathbf{f}_b|^2, \quad (\text{II.3.4})$$

where the function $m \in C_c^\infty(\partial\Omega)$ is such that $\text{supp}(m) = \bar{O}$ and the pair $(\mathbf{u}(\omega), p(\omega))$, which belongs to $\mathbf{H}^1(\Omega \setminus \bar{\omega}) \times L^2(\Omega \setminus \bar{\omega})$, is a solution of the Navier-Stokes problem

$$\begin{cases} -\nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = \mathbf{0} & \text{in } \Omega \setminus \bar{\omega} \\ \text{div } \mathbf{u} = 0 & \text{in } \Omega \setminus \bar{\omega} \\ \mathbf{u} = \mathbf{g} & \text{on } \partial\Omega \\ \mathbf{u} = \mathbf{0} & \text{on } \partial\omega. \end{cases} \quad (\text{II.3.5})$$

Remark II.3.1. Notice that using a local regularity result similar to Theorem A.2.6, we check that $(\nu \partial_{\mathbf{n}} \mathbf{u} - p \mathbf{n})|_O \in \mathbf{H}^{1/2}(O)$ (because $(\mathbf{u}, p)$ belongs to $\mathbf{H}^2 \times \mathbf{H}^1$ in a neighborhood of O , since $\tilde{O}$ is $C^{1,1}$). Hence, the functional J defined by (II.3.4) is well defined.

The results of existence, uniqueness and regularity of the solution of the Navier-Stokes equations are recalled in Appendix A.2 (we also refer to [31, 44, 61, 97]). Hence, the existence of a solution of Problem (II.3.5) is guaranteed by Theorem A.2.1 below since we imposed the compatibility condition (II.3.1). Note also that the solution is not necessarily unique and then the functional J may be multivalued. To overcome this difficulty, we assume ν big enough (see Theorem A.2.5) in order to guarantee the uniqueness of the solution of the previous problem (once a normalization condition on the pressure p is imposed): J is then single valued. More precisely, we assume that

$$\nu > c_b(\Omega) \left(2 \|\Delta \mathbf{G}\|_{\mathbf{W}'} + \frac{2}{\nu} \|(\mathbf{G} \cdot \nabla) \mathbf{G}\|_{\mathbf{W}'} + \|\mathbf{G}\|_{\mathbf{H}^1(\Omega \setminus \bar{\Omega}_\delta)} \right) \quad (\text{II.3.6})$$

where $\mathbf{W}'$ is the dual space of

$$\mathbf{W} := \{ \varphi \in \mathbf{H}^1(\Omega \setminus \bar{\Omega}_\delta), \text{div } \varphi = 0 \text{ in } \Omega \setminus \bar{\Omega}_\delta \text{ and } \varphi = \mathbf{0} \text{ on } \partial\Omega \},$$

where, according to [14, Lemma 3.3], $\mathbf{G} \in \mathbf{H}^1(\Omega)$ satisfies

$$\mathbf{G} = \mathbf{g} \text{ on } \partial\Omega, \quad \text{div } \mathbf{G} = 0 \text{ in } \Omega \quad \text{and} \quad \mathbf{G} = \mathbf{0} \text{ in } \Omega_\delta, \quad (\text{II.3.7})$$

and where

$$c_b(\Omega) := \begin{cases} \frac{|\Omega|^{1/2}}{2} & \text{if } d = 2 \\ \frac{2\sqrt{2}|\Omega|^{1/6}}{3} & \text{if } d = 3, \end{cases} \quad (\text{II.3.8})$$

(see [56, Vol. 2, Chap. VIII, page 6]). Moreover, we choose the normalization:

$$\int_O m^2 (\nu \partial_{\mathbf{n}} \mathbf{u} - p \mathbf{n}) \cdot \mathbf{n} = \int_O m^2 \mathbf{f}_b \cdot \mathbf{n}. \quad (\text{II.3.9})$$

Hence, according to Theorem A.2.5, this solution $(\mathbf{u}, p)$ is unique and is referred as the state in the rest of the paper. Notice that the terms in (II.3.9) have a sense using Remark II.3.1.

Remark II.3.2. The estimate (II.3.6) implies that

$$\nu > c_b(\Omega \setminus \bar{\omega}) \|\mathbf{u}\|_{\mathbf{H}^1(\Omega \setminus \bar{\omega})}, \quad (\text{II.3.10})$$

where $c_b(\Omega \setminus \bar{\omega})$ is defined analogously to (II.3.8). Indeed, if $\nu \geq 2c_b(\Omega) \|\mathbf{G}\|_{\mathbf{H}^1(\Omega \setminus \bar{\Omega}_\delta)}$, we know that $\|\mathbf{u} - \mathbf{G}\|_{\mathbf{H}^1(\Omega)} \leq \frac{2}{\nu} \|\nu \Delta \mathbf{G} - (\mathbf{G} \cdot \nabla) \mathbf{G}\|_{\mathbf{W}'}$ (see [97, Chapter II, equation (1.89)]) and we have obviously $c_b(\Omega) \geq c_b(\Omega \setminus \bar{\omega})$. Moreover, Inequality (II.3.6) implies the uniqueness of the solution of all the perturbed problems we will consider in this paper.

Then, we try to minimize the least squares criterion J :

$$\omega^* = \underset{\omega \in \mathcal{O}_\delta}{\operatorname{argmin}} J(\omega). \quad (\text{II.3.11})$$

Indeed, if ω^* solves the inverse problem (II.3.3), then $J(\omega^*) = 0$ and (II.3.11) holds. Conversely, if ω^* solves the optimization problem (II.3.11) with $J(\omega^*) = 0$, then it is a solution of (II.3.3).

Identifiability result We first state an identifiability result similar to Theorem 1.2 proved in [6] by Alvarez *et al.*: it states that given a fixed $\mathbf{g}$, two different geometries ω_0 and ω_1 in $\mathcal{O}_\delta$ yield two different measures $\mathbf{f}_{b1}$ and $\mathbf{f}_{b2}$. Hence problem (II.3.3) admits a unique solution. We also refer to [51, Theorem 1.2] proved by Doubova *et al.*

Theorem II.3.3 (Identifiability result). *Let $\Omega \subseteq \mathbb{R}^d$, $d = 2$ or $d = 3$, be a bounded Lipschitz domain, and O be a non-empty open subset of $\partial\Omega$ of class $C^{1,1}$. Let*

$$\omega_0, \omega_1 \in \{\omega \subset\subset \Omega; \omega \text{ is a Lipschitz open set and } \Omega \setminus \overline{\omega} \text{ is connected}\}$$

and $\mathbf{g} \in \mathbf{H}^{1/2}(\partial\Omega)$ with $\mathbf{g} \neq \mathbf{0}$ and $\mathbf{g}|_O \in \mathbf{H}^{3/2}(O)$, satisfying the flux condition (II.3.1). Let $(\mathbf{u}_j, p_j)$ for $j = 0, 1$, be a solution of

$$\begin{cases} -\nu \Delta \mathbf{u}_j + (\mathbf{u}_j \cdot \nabla) \mathbf{u}_j + \nabla p_j &= \mathbf{0} & \text{in } \Omega \setminus \overline{\omega_j} \\ \operatorname{div} \mathbf{u}_j &= 0 & \text{in } \Omega \setminus \overline{\omega_j} \\ \mathbf{u}_j &= \mathbf{g} & \text{on } \partial\Omega \\ \mathbf{u}_j &= \mathbf{0} & \text{on } \partial\omega_j. \end{cases}$$

Assume that $(\mathbf{u}_j, p_j)$ are such that

$$-\nu \partial_{\mathbf{n}} \mathbf{u}_0 + p_0 \mathbf{n} = -\nu \partial_{\mathbf{n}} \mathbf{u}_1 + p_1 \mathbf{n} \quad \text{on } O.$$

Then $\omega_0 \equiv \omega_1$.

Sensitivity with respect to the domain We aim to make a sensitivity (with respect to the shape) analysis. The Navier-Stokes problem on $\Omega \setminus \overline{\omega_t}$

$$\begin{cases} -\nu \Delta \mathbf{u}_t + (\mathbf{u}_t \cdot \nabla) \mathbf{u}_t + \nabla p_t &= \mathbf{0} & \text{in } \Omega \setminus \overline{\omega_t} \\ \operatorname{div} \mathbf{u}_t &= 0 & \text{in } \Omega \setminus \overline{\omega_t} \\ \mathbf{u}_t &= \mathbf{g} & \text{on } \partial\Omega \\ \mathbf{u}_t &= \mathbf{0} & \text{on } \partial\omega_t, \end{cases} \quad (\text{II.3.12})$$

admits a solution $(\mathbf{u}_t, p_t) \in \mathbf{H}^1(\Omega \setminus \overline{\omega_t}) \times L^2(\Omega \setminus \overline{\omega_t})$ satisfying the normalization condition

$$\int_O m^2 (\nu \partial_{\mathbf{n}} \mathbf{u}_t - p_t \mathbf{n}) \cdot \mathbf{n} = \int_O m^2 \mathbf{f}_b \cdot \mathbf{n}.$$

Remark II.3.4. *Using the same argument than in Remark II.3.1, we check that $(\nu \partial_{\mathbf{n}} \mathbf{u}_t - p_t \mathbf{n})|_O$ belongs to $\mathbf{H}^{1/2}(O)$. Thus, these two previous terms have a sense. We use again this argument to define the following normalization condition.*

Proposition II.3.5 (First order shape derivative of the state). *The solution $(\mathbf{u}, p)$ is differentiable with respect to the domain. The shape derivative $(\mathbf{u}', p')$ of $(\mathbf{u}, p)$ which belongs to $\mathbf{H}^1(\Omega \setminus \overline{\omega}) \times L^2(\Omega \setminus \overline{\omega})$ is the only solution of the following boundary values problem*

$$\begin{cases} -\nu \Delta \mathbf{u}' + (\mathbf{u}' \cdot \nabla) \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u}' + \nabla p' &= \mathbf{0} & \text{in } \Omega \setminus \overline{\omega} \\ \operatorname{div} \mathbf{u}' &= 0 & \text{in } \Omega \setminus \overline{\omega} \\ \mathbf{u}' &= \mathbf{0} & \text{on } \partial\Omega \\ \mathbf{u}' &= -\partial_{\mathbf{n}} \mathbf{u} V_{\mathbf{n}} & \text{on } \partial\omega, \end{cases} \quad (\text{II.3.13})$$

with the normalization condition

$$\langle m^2(\nu \partial_{\mathbf{n}} \mathbf{u}' - p' \mathbf{n}), \mathbf{n} \rangle_O = 0.$$

By introducing an adjoint problem, we then characterize the gradient of the least squares functional J defined by (II.3.4) in the following proposition.

Proposition II.3.6 (First order shape derivative of the functional). *For $\mathbf{V}$ in $\mathbf{U}$, the least squares functional J is differentiable at ω in the direction $\mathbf{V}$ with*

$$D J(\omega) \cdot \mathbf{V} = - \int_{\partial\omega} [(\nu \partial_{\mathbf{n}} \mathbf{w} - \pi \mathbf{n}) \cdot \partial_{\mathbf{n}} \mathbf{u}] V_{\mathbf{n}},$$

where $(\mathbf{w}, \pi) \in \mathbf{H}^1(\Omega \setminus \overline{\omega}) \times L^2(\Omega \setminus \overline{\omega})$ is the solution of the following boundary values problem:

$$\left\{ \begin{array}{ll} -\nu \Delta \mathbf{w} + {}^t \nabla \mathbf{u} \mathbf{w} - \nabla \mathbf{w} \mathbf{u} + \nabla \pi &= \mathbf{0} & \text{in } \Omega \setminus \overline{\omega} \\ \operatorname{div} \mathbf{w} &= 0 & \text{in } \Omega \setminus \overline{\omega} \\ \mathbf{w} &= 2m^2(\nu \partial_{\mathbf{n}} \mathbf{u} - p \mathbf{n} - \mathbf{f}_b) & \text{on } \partial\Omega \\ \mathbf{w} &= \mathbf{0} & \text{on } \partial\omega, \end{array} \right. \quad (\text{II.3.14})$$

with the normalization condition

$$\langle m^2(\nu \partial_{\mathbf{n}} \mathbf{w} - \pi \mathbf{n}), \mathbf{n} \rangle_O = 0.$$

Remark II.3.7. Note that the previous characterization of the gradient is well defined. Indeed, using a local regularity argument similar to Theorem A.2.6, we check $(\nu \partial_{\mathbf{n}} \mathbf{u} - p \mathbf{n})|_O$ belongs to $\mathbf{H}^{1/2}(O)$. Then the boundary condition $2m^2(\nu \partial_{\mathbf{n}} \mathbf{u} - p \mathbf{n} - \mathbf{f}_b)$ belongs to $\mathbf{H}^{1/2}(\partial\Omega)$. Notice also using the local regularity Theorems A.2.6 and A.2.8 that $\partial_{\mathbf{n}} \mathbf{u}$ and $\nu \partial_{\mathbf{n}} \mathbf{w} - \pi \mathbf{n}$ belong to $\mathbf{H}^{3/2}(\partial\omega)$.

Remark II.3.8. Note that for $\omega^* \in \mathcal{O}_\delta$ solution of the inverse problem (II.3.3), the boundary condition on $\partial\Omega$ of the adjoint problem (II.3.14) is homogeneous and then $\mathbf{w} = \mathbf{0}$ and $\pi = 0$. Hence, we check that for all $\mathbf{V} \in \mathbf{U}$, $D J(\omega^*) \cdot \mathbf{V} = 0$

Second order analysis: justification of the instability Let us consider $\omega^* \in \mathcal{O}_\delta$ (with $\mathcal{O}_\delta$ defined by (II.0.1)) a solution of the inverse problem (II.3.3) in order to study the stability of the optimization problem (II.3.11) at ω^* .

Proposition II.3.9 (Characterization of the shape Hessian at a critical shape). *The solutions $(\mathbf{u}, p)$ and $(\mathbf{w}, \pi)$ are respectively twice and once differentiable with respect to the domain. Moreover, for $\mathbf{V} \in \mathbf{U}$, we have*

$$D^2 J(\omega^*) \cdot \mathbf{V} \cdot \mathbf{V} = - \int_{\partial\omega^*} [(\nu \partial_{\mathbf{n}} \mathbf{w}' - \pi' \mathbf{n}) \cdot \partial_{\mathbf{n}} \mathbf{u}] V_{\mathbf{n}},$$

where $(\mathbf{w}', \pi') \in \mathbf{H}^1(\Omega \setminus \overline{\omega^*}) \times L^2(\Omega \setminus \overline{\omega^*})$ is the solution of the following problem:

$$\left\{ \begin{array}{ll} -\nu \Delta \mathbf{w}' + {}^t \nabla \mathbf{u} \mathbf{w}' - \nabla \mathbf{w}' \mathbf{u} + \nabla \pi' &= \mathbf{0} & \text{in } \Omega \setminus \overline{\omega^*} \\ \operatorname{div} \mathbf{w}' &= 0 & \text{in } \Omega \setminus \overline{\omega^*} \\ \mathbf{w}' &= 2m^2(\nu \partial_{\mathbf{n}} \mathbf{u}' - p' \mathbf{n}) & \text{on } \partial\Omega \\ \mathbf{w}' &= \mathbf{0} & \text{on } \partial\omega^*, \end{array} \right. \quad (\text{II.3.15})$$

with the normalization condition

$$\langle m^2(\nu \partial_{\mathbf{n}} \mathbf{w}' - \pi' \mathbf{n}), \mathbf{n} \rangle_O = 0.$$

Remark II.3.10. Note that $2m^2(\nu\partial_{\mathbf{n}}\mathbf{u}' - p'\mathbf{n} - \mathbf{f}_b)$ belongs to $\mathbf{H}^{1/2}(\partial\Omega)$ and $D^2J(\omega^*) \cdot \mathbf{V} \cdot \mathbf{V}$ is well defined.

Proposition II.3.11 (Compactness at a critical point). *The Riesz operator corresponding to $D^2J(\omega^*)$ defined from $\mathbf{H}^{1/2}(\partial\omega^*)$ to $\mathbf{H}^{-1/2}(\partial\omega^*)$ is compact.*

Remark II.3.12 (A remark on the nonsingular solutions of the Navier-Stokes problem). *All the results are still valid for a nonsingular solution $(\mathbf{u}, p)$ of the Navier-Stokes Problem (II.3.5) (see [61, Chapter IV, § 3] for details on this notion). However, without the uniqueness of the solution, the functional J may be multivalued. We then differentiate the functional J for a given nonsingular solution.*

We can also notice that, even if we do not assume the uniqueness of the solution of the Navier-Stokes Problem (II.3.5), the solution of the overdetermined Problem (II.3.2) is unique. Indeed, let us assume that $(\mathbf{u}_j, p_j)$, $j = 0, 1$, are two solutions of problem (II.3.2) corresponding to $\omega = \omega_j$. We know using the identifiability Theorem II.3.3 that the overdetermination permits to identify the domain ω . Then, using the unique continuation property (see Corollary II.3.14 below), we check, like what is done in the proof of the identifiability Theorem II.3.3, that $\mathbf{u} := \mathbf{u}_0 - \mathbf{u}_1 = \mathbf{0}$ and $p := p_0 - p_1 = 0$ in $\Omega \setminus \bar{\omega}$. In conclusion, the overdetermination of Problem (II.3.2) permits to identify the obstacle ω and to select, in the non-uniqueness case, a unique solution.

II.3.2 The proofs

The identifiability result. The proof is adapted from [6, Theorem 1.2] (see also [51, Theorem 1.2] proved by Doubova *et al.*). However, we detail the proof here since, in [6, 51], the case where $\omega_0 \setminus \bar{\omega}_1$ (with ω_0 and ω_1 two obstacles) is not Lipschitz is not treated. This case leads additional difficulties to conclude properly. As in [6, 51], the main ingredient is the following theorem (see [6, Theorem 2.3]) which is a direct adaptation of the unique continuation property stated in [55]:

Theorem II.3.13 (Alvarez *et al.* in [6]; Fabre, Lebeau in [55]). *Let $\Omega_0 \subset \mathbb{R}^d$, $d \geq 2$, be a connected open set and let D_0 be a non-empty open subset of Ω_0 . If $\mathbf{a} \in \mathbf{L}_{\text{loc}}^\infty(\Omega_0)$, $c \in L_{\text{loc}}^r(\Omega_0, \mathcal{M}_{d,d})$, with $r > d$, and $(\mathbf{u}, p) \in \mathbf{H}_{\text{loc}}^1(\Omega_0) \times L_{\text{loc}}^2(\Omega_0)$ is a solution of*

$$\begin{cases} -\Delta \mathbf{u} + (\mathbf{a} \cdot \nabla) \mathbf{u} + c \mathbf{u} + \nabla p &= \mathbf{0} & \text{in } \Omega_0, \\ \operatorname{div} \mathbf{u} &= 0 & \text{in } \Omega_0, \end{cases}$$

with $\mathbf{u} = \mathbf{0}$ in D_0 , then $\mathbf{u} = \mathbf{0}$ in Ω_0 and p is constant in Ω_0 .

We deduce from the above theorem the following corollary:

Corollary II.3.14 (Unique continuation property). *Let $\Omega_0 \subseteq \mathbb{R}^d$, $d \geq 2$, be a connected open set and $O \subset \partial\Omega_0$ be a non-empty Lipschitz relatively open subset. Let us assume that $(\mathbf{u}, p) \in \mathbf{H}^1(\Omega_0) \times L^2(\Omega_0)$ is a solution of*

$$\begin{cases} -\nu\Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{v}_1 + (\mathbf{v}_0 \cdot \nabla) \mathbf{u} + \nabla p &= \mathbf{0} & \text{in } \Omega_0 \\ \operatorname{div} \mathbf{u} &= 0 & \text{in } \Omega_0 \\ \mathbf{u} &= \mathbf{0} & \text{on } O \\ -\nu\partial_{\mathbf{n}} \mathbf{u} + p \mathbf{n} &= \mathbf{0} & \text{on } O, \end{cases}$$

where $\mathbf{v}_0 \in \mathbf{L}_{\text{loc}}^\infty(\Omega_0)$ and $\nabla \mathbf{v}_1 \in L_{\text{loc}}^r(\Omega_0, \mathcal{M}_{d,d})$, with $r > N$. Assume also that there exists a neighborhood $\mathcal{V}$ of O in $\bar{\Omega}$ such that $\mathbf{v}_0 \in \mathbf{L}^\infty(\mathcal{V})$ and $\nabla \mathbf{v}_1 \in L^r(\mathcal{V}, \mathcal{M}_{d,d})$. Then $\mathbf{u} = \mathbf{0}$ and $p = 0$ in Ω_0 .

Proof of Corollary II.3.14. The idea of the proof is to enlarge the domain Ω_0 at a part of O and to use the unique continuation Theorem II.3.13 in this enlarged domain. We choose $y \in O$ and a radius $\rho \leq \frac{d(y, \partial O)}{2}$ (or $\rho \leq \frac{|\partial\Omega|}{2}$ if $O = \partial\Omega$) such that $(\mathcal{B}(y, \rho) \cap \Omega_0) \subset \mathcal{V}$ and such that $\Omega_0 \cap \mathcal{B}(y, \rho)$

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and $\mathcal{B}(y, \rho) \setminus \overline{\Omega_0}$ are Lipschitz (where $\mathcal{B}(y, \rho)$ is the ball with center y and radius ρ). We then set $\Omega_\rho := \Omega_0 \cup \mathcal{B}(y, \rho)$ and we define

$$\begin{aligned} \tilde{\mathbf{u}} &= \begin{cases} \mathbf{u} & \text{in } \Omega_0 \\ \mathbf{0} & \text{in } \Omega_\rho \setminus \Omega_0 \end{cases} \quad \text{and} \quad \tilde{p} = \begin{cases} p & \text{in } \Omega_0 \\ 0 & \text{in } \Omega_\rho \setminus \Omega_0, \end{cases} \\ \widetilde{\mathbf{v}_0} &= \begin{cases} \mathbf{v}_0 & \text{in } \Omega_0 \\ \mathbf{0} & \text{in } \Omega_\rho \setminus \Omega_0 \end{cases} \quad \text{and} \quad \widetilde{\nabla \mathbf{v}_1} = \begin{cases} \nabla \mathbf{v}_1 & \text{in } \Omega_0 \\ 0 & \text{in } \Omega_\rho \setminus \Omega_0. \end{cases} \end{aligned}$$

Since $\mathbf{u} \in \mathbf{H}^1(\Omega_0)$ and $\mathbf{u} = \mathbf{0}$ on O , we check that $\tilde{\mathbf{u}} \in \mathbf{H}^1(\Omega_\rho)$. Moreover, $\tilde{p} \in L^2(\Omega_\rho)$, $\widetilde{\mathbf{v}_0} \in L_{\text{loc}}^\infty(\Omega_\rho)$ and $\widetilde{\nabla \mathbf{v}_1} \in L_{\text{loc}}^r(\Omega_\rho)$. By summing on $\Omega_0 \cap \mathcal{B}(y, \rho)$ and on $\Omega_\rho^{\text{ext}} := \Omega_\rho \setminus \overline{\Omega_0}$ and using the fact that $-\nu \partial_{\mathbf{n}} \mathbf{u} + p \mathbf{n} = \mathbf{0}$ on O , we get that for all φ in $\mathcal{D}(\mathcal{B}(y, \rho))$

$$\int_{\mathcal{B}(y, \rho)} (\nu \nabla \tilde{\mathbf{u}} : \nabla \varphi - \tilde{p} \operatorname{div} \varphi) + b_{\mathcal{B}(y, \rho)}(\tilde{\mathbf{u}}, \widetilde{\mathbf{v}_1}, \varphi) + b_{\mathcal{B}(y, \rho)}(\widetilde{\mathbf{v}_0}, \tilde{\mathbf{u}}, \varphi) = 0.$$

Then, $-\nu \Delta \tilde{\mathbf{u}} + (\tilde{\mathbf{u}} \cdot \nabla) \widetilde{\mathbf{v}_1} + (\widetilde{\mathbf{v}_0} \cdot \nabla) \tilde{\mathbf{u}} + \nabla \tilde{p} = \mathbf{0}$ in $\mathcal{D}'(\mathcal{B}(y, \rho))$ and then in $\mathcal{D}'(\Omega_\rho)$. Proceeding as above, we also get $\operatorname{div} \tilde{\mathbf{u}} = 0$ in $\mathcal{D}'(\Omega_\rho)$. Moreover, $\tilde{\mathbf{u}} = \mathbf{0}$ in $\Omega_\rho^{\text{ext}} \subset \Omega_\rho$. Thus, according to the unique continuation Theorem II.3.13, $\tilde{\mathbf{u}} = \mathbf{0}$ and $\tilde{p}$ is constant in Ω_ρ . Since $\tilde{p} = 0$ in Ω_ρ^{ext} , $\tilde{p} = 0$ in Ω_ρ and then $p = 0$ in Ω_0 by restriction. $\square$

The proof of Theorem II.3.3 is split into two steps. Firstly, we use the unique continuation Corollary II.3.14 to prove that $\mathbf{u}_0 = \mathbf{u}_1$ and $p_0 = p_1$ in $\Omega \setminus \overline{\omega}$ where ω is the smallest simply connected open set such that $(\omega_0 \cup \omega_1) \subset \omega$. Secondly, we proceed by contradiction: we assume $\omega_0 \neq \omega_1$ and we use the unique continuation Theorem II.3.13 to prove that $\mathbf{u}_1 = \mathbf{0}$ (respectively $\mathbf{u}_0 = \mathbf{0}$) on $\partial\Omega$ which contradicts the fact that $\mathbf{u}_1 = \mathbf{g} \neq \mathbf{0}$ (respectively $\mathbf{u}_0 = \mathbf{g} \neq \mathbf{0}$) on $\partial\Omega$.

Remark II.3.15. *The idea of the proof of Theorem II.3.3 is the same than the one detailed in the Stokes case with Neumann boundary conditions (see the proof of Theorem II.1.10). However, the part of the proof where we distinguish the Lipschitz case and the non Lipschitz case is here modified in order to only assume that ω_0 and ω_1 are Lipschitz (and not $C^{1,1}$ as in the Neumann case). This also permits to point out the difference between these two boundary conditions and the difficulties of the Neumann case.*

Proof of Theorem II.3.3. We define $\mathbf{u} := \mathbf{u}_0 - \mathbf{u}_1$, $p := p_0 - p_1$ and let ω be the smallest simply connected open set such that $(\omega_0 \cup \omega_1) \subset \omega$ (see Figure II.6). Notice that, in the particular case where

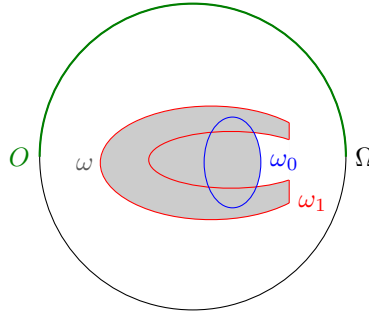


Figure II.6 – The set ω

the set $\omega_0 \cup \omega_1$ is simply connected, ω is exactly $\omega_0 \cup \omega_1$. Hence, the pair $(\mathbf{u}, p)$ satisfies

$$\begin{cases} -\nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u}_0 + (\mathbf{u}_1 \cdot \nabla) \mathbf{u} + \nabla p &= \mathbf{0} & \text{in } \Omega \setminus \overline{\omega} \\ \operatorname{div} \mathbf{u} &= 0 & \text{in } \Omega \setminus \overline{\omega} \\ \mathbf{u} &= \mathbf{0} & \text{on } O \\ -\nu \partial_{\mathbf{n}} \mathbf{u} + p \mathbf{n} &= \mathbf{0} & \text{on } O. \end{cases}$$

In order to use the previous Corollary II.3.14, we have to check that

$$\mathbf{u}_1 \in \mathbf{L}_{\text{loc}}^\infty(\Omega \setminus \bar{\omega}) \cap \mathbf{L}^\infty(\mathcal{V}), \quad \nabla \mathbf{u}_0 \in L_{\text{loc}}^r(\Omega \setminus \bar{\omega}, \mathcal{M}_{d,d}) \cap \mathbf{L}^r(\mathcal{V}), \quad \text{with } r > N,$$

where $\mathcal{V}$ is a neighborhood of O in $\overline{\Omega \setminus \bar{\omega}}$. This comes from a local regularity result similar to Theorem A.2.8: since O is $C^{1,1}$ and $\mathbf{g}|_O \in \mathbf{H}^{3/2}(O)$, we have, choosing a positive radius $\rho < d(y, \partial O)/2$ with $y \in O$ (or $\rho < |\partial \Omega|/2$ if $O = \partial \Omega$) in such a way that $\mathcal{B}(y, \rho) \cap \bar{\omega} = \emptyset$ and $\mathcal{B}(y, \rho) \cap ((\Omega \setminus \bar{\omega})^c)$ is simply connected,

$$\mathbf{u}_1 \in \mathbf{H}^2((\Omega \setminus \bar{\omega}) \cap \mathcal{B}(y, \rho)) \hookrightarrow \mathbf{L}^\infty((\Omega \setminus \bar{\omega}) \cap \mathcal{B}(y, \rho))$$

and

$$\nabla \mathbf{u}_0 \in \mathbf{H}^1((\Omega \setminus \bar{\omega}) \cap \mathcal{B}(y, \rho), \mathcal{M}_{d,d}) \hookrightarrow L^6((\Omega \setminus \bar{\omega}) \cap \mathcal{B}(y, \rho), \mathcal{M}_{d,d}).$$

Similarly, we prove that $\mathbf{u}_1 \in \mathbf{L}_{\text{loc}}^\infty(\Omega \setminus \bar{\omega})$ and $\nabla \mathbf{u}_0 \in L_{\text{loc}}^r(\Omega \setminus \bar{\omega}, \mathcal{M}_{d,d})$. Therefore, according to Corollary II.3.14, $\mathbf{u} = \mathbf{0}$ and $p = 0$ in $\Omega \setminus \bar{\omega}$.

We then proceed by contradiction assuming that $\omega_0 \setminus \bar{\omega}_1$ is non-empty. In the sequel, we assume that $\omega \setminus \bar{\omega}_1$ is connected. If not, it suffices to replace $\omega \setminus \bar{\omega}_1$ by one of its connected components in the following proof. We know that

$$-\nu \Delta \mathbf{u}_1 + (\mathbf{u}_1 \cdot \nabla) \mathbf{u}_1 + \nabla p_1 = \mathbf{0} \quad \text{in } \omega \setminus \bar{\omega}_1. \quad (\text{II.3.16})$$

In order to give a clear idea of the proof, let us first consider the particular case where $\omega = \omega_0 \cup \omega_1$.

Case where $\omega = \omega_0 \cup \omega_1$. Assume for the moment that $\omega_0 \setminus \bar{\omega}_1$ is Lipschitz. Then, multiplying Equation (II.3.16) by $\mathbf{u}_1$ and integrating by parts in $\omega_0 \setminus \bar{\omega}_1$, we obtain, since $\mathbf{u}_1 = \mathbf{0}$ on $\partial \omega_1$,

$$\int_{\omega_0 \setminus \bar{\omega}_1} \nu |\nabla \mathbf{u}_1|^2 + \int_{(\partial \omega_0) \setminus \bar{\omega}_1} (-\nu \partial_{\mathbf{n}} \mathbf{u}_1 + p_1 \mathbf{n}) \cdot \mathbf{u}_1 + \frac{1}{2} \int_{(\partial \omega_0) \setminus \bar{\omega}_1} \mathbf{u}_1 \cdot \mathbf{n} |\mathbf{u}_1|^2 = 0. \quad (\text{II.3.17})$$

Since $\mathbf{u}_0 = \mathbf{u}_1$ in $\Omega \setminus \bar{\omega}$, the boundary condition satisfied by $\mathbf{u}_0$ on $\partial \omega_0$ provides that $\mathbf{u}_1 = \mathbf{u}_0 = \mathbf{0}$ on $(\partial \omega_0) \setminus \bar{\omega}_1$. Hence, equality (II.3.17) is simply

$$\int_{\omega_0 \setminus \bar{\omega}_1} \nu |\nabla \mathbf{u}_1|^2 = 0. \quad (\text{II.3.18})$$

Hence, $\mathbf{u}_1$ is constant in $\omega_0 \setminus \bar{\omega}_1$. By Theorem II.3.13 (with $c \equiv 0$), $\mathbf{u}_1$ is then constant in $\Omega \setminus \bar{\omega}_1$. Since $\mathbf{u}_1 = \mathbf{0}$ on $\partial \omega_1$, then $\mathbf{u}_1 = \mathbf{0}$ in $\Omega \setminus \bar{\omega}_1$ which contradicts the boundary condition $\mathbf{u}_1 = \mathbf{g} \neq \mathbf{0}$ on $\partial \Omega$. Thus $\omega_0 \setminus \bar{\omega}_1 = \emptyset$.

Let us now turn to the general case where $\omega_0 \setminus \bar{\omega}_1$ is not necessary Lipschitz. In such a case, the integration by parts yielding (II.3.17) is not justified. Then, to overcome this difficulty and get (II.3.18), let us use a density argument. We define the space $\widetilde{\mathbf{H}}^1(\omega_0 \setminus \bar{\omega}_1)$ as the space of the functions which belong to $\mathbf{H}^1(\omega_0 \setminus \bar{\omega}_1)$ such that its extension by $\mathbf{0}$ in Ω belongs to $\mathbf{H}^1(\Omega)$, *i.e.*

$$\widetilde{\mathbf{H}}^1(\omega_0 \setminus \bar{\omega}_1) := \left\{ \mathbf{w} \in \mathbf{H}^1(\omega_0 \setminus \bar{\omega}_1), \tilde{\mathbf{w}} := \begin{cases} \mathbf{w} & \text{in } \omega_0 \setminus \bar{\omega}_1 \\ \mathbf{0} & \text{in } \omega_1 \\ \mathbf{0} & \text{in } \Omega \setminus \bar{\omega}_0 \end{cases} \text{ belongs to } \mathbf{H}^1(\Omega) \right\}.$$

Since $\partial(\omega_0 \setminus \bar{\omega}_1)$ is continuous, the space $\mathcal{D}(\omega_0 \setminus \bar{\omega}_1)$ is dense in $\widetilde{\mathbf{H}}^1(\omega_0 \setminus \bar{\omega}_1)$ according to [63, Theorem 1.4.2.2, page 24] (we also refer to the general definition of the space $\mathbf{H}_0^1$ given in [72, Definition 3.3.43]). We then multiply (II.3.16) by $\boldsymbol{\varphi} \in \mathcal{D}(\omega_0 \setminus \bar{\omega}_1)$ and, using an integration by parts, we get

$$\int_{\omega_0 \setminus \bar{\omega}_1} (\nu \nabla \mathbf{u}_1 : \nabla \boldsymbol{\varphi} - p_1 \operatorname{div} \boldsymbol{\varphi}) + \frac{1}{2} b_{\omega_0 \setminus \bar{\omega}_1}(\mathbf{u}_1, \mathbf{u}_1, \boldsymbol{\varphi}) - \frac{1}{2} b_{\omega_0 \setminus \bar{\omega}_1}(\mathbf{u}_1, \boldsymbol{\varphi}, \mathbf{u}_1) = 0, \quad (\text{II.3.19})$$

since $b_{\omega_0 \setminus \overline{\omega_1}}(\mathbf{u}_1, \mathbf{u}_1, \boldsymbol{\varphi}) = -b_{\omega_0 \setminus \overline{\omega_1}}(\mathbf{u}_1, \boldsymbol{\varphi}, \mathbf{u}_1)$. Then it suffices to show that $\mathbf{u}_1|_{\omega_0 \setminus \overline{\omega_1}}$ belongs to the space $\mathbf{H}^1(\omega_0 \setminus \overline{\omega_1})$. Indeed, writing Equality (II.3.19) with $\boldsymbol{\varphi} = \boldsymbol{\varphi}_n$, where $(\boldsymbol{\varphi}_n)_{n \in \mathbb{N}} \subset \mathcal{D}(\omega_0 \setminus \overline{\omega_1})$ is a sequence such that $\boldsymbol{\varphi}_n \rightarrow \mathbf{u}_1|_{\omega_0 \setminus \overline{\omega_1}}$ in $\mathbf{H}^1(\omega_0 \setminus \overline{\omega_1})$, we pass to the limit to get (II.3.18) and conclude as previously. Hence, let us prove that $\mathbf{u}_1|_{\omega_0 \setminus \overline{\omega_1}}$ belongs to $\widetilde{\mathbf{H}}^1(\omega_0 \setminus \overline{\omega_1})$. Then, we define $\widetilde{\mathbf{u}}_1$ the extension of $\mathbf{u}_1$ by $\mathbf{0}$ in ω_1 , *i.e.*

$$\widetilde{\mathbf{u}}_1 = \begin{cases} \mathbf{u}_1 & \text{in } \Omega \setminus \overline{\omega_1} \\ \mathbf{0} & \text{in } \omega_1. \end{cases}$$

Since $\mathbf{u}_1 = \mathbf{0}$ on $\partial\omega_1$, $\widetilde{\mathbf{u}}_1$ belongs to $\mathbf{H}^1(\Omega)$. We then consider $\widetilde{\mathbf{u}}_1|_{\omega_0} \in \mathbf{H}^1(\omega_0)$ and we extend it by $\mathbf{0}$ in $\Omega \setminus \overline{\omega_0}$. Since $\widetilde{\mathbf{u}}_1 = \mathbf{u}_0 = \mathbf{0}$ on $\partial\omega_0 \setminus \overline{\omega_1}$ and $\widetilde{\mathbf{u}}_1 = \mathbf{0}$ on $\partial\omega_0 \cap \omega_1$ by construction, this extension, noted again $\widetilde{\mathbf{u}}_1$, *i.e.*

$$\widetilde{\mathbf{u}}_1 = \begin{cases} \mathbf{u}_1 & \text{in } \omega_0 \setminus \overline{\omega_1} \\ \mathbf{0} & \text{in } \omega_1 \\ \mathbf{0} & \text{in } \Omega \setminus \overline{\omega_0}, \end{cases}$$

belongs to $\mathbf{H}^1(\Omega)$. Hence $\widetilde{\mathbf{u}}_1|_{\omega_0 \setminus \overline{\omega_1}} = \mathbf{u}_1|_{\omega_0 \setminus \overline{\omega_1}}$ belongs to $\widetilde{\mathbf{H}}^1(\omega_0 \setminus \overline{\omega_1})$ and we conclude.

Case where $\omega \neq \omega_0 \cup \omega_1$ (as in Figure II.6 for example). Let us now emphasize the differences and the difficulties of the proof in the general case where ω is not necessary equal to $\omega_0 \cup \omega_1$. The idea is to replace ω_0 by $\omega \setminus (\overline{\omega_1} \setminus \overline{\omega_0})$. Indeed, we can not work with ω_0 since we have no information on $\mathbf{u}_1$ on $\partial\omega_0 \setminus \partial\omega$: we have only proved that $\mathbf{u}_1 = \mathbf{u}_0$ in $\Omega \setminus \overline{\omega}$.

In the case where $\omega \setminus \overline{\omega_1}$ is Lipschitz, we proceed in exactly the same way as in the previous Lipschitz case noticing that $\omega_0 \setminus \overline{\omega_1}$ is then replaced by $\omega \setminus \overline{\omega_1}$.

In the general case where $\omega \setminus \overline{\omega_1}$ is not necessary Lipschitz, we can not prove that $\mathbf{u}_1|_{\omega \setminus \overline{\omega_1}}$ belongs to $\widetilde{\mathbf{H}}^1(\omega \setminus \overline{\omega_1})$ in the same manner than above. Indeed, after having extended $\mathbf{u}_1$ by $\mathbf{0}$ in ω_1 , we can not claim that the extension of $\widetilde{\mathbf{u}}_1|_{\omega \setminus (\overline{\omega_1} \setminus \overline{\omega_0})}$ by $\mathbf{0}$ in Ω belongs to $\mathbf{H}^1(\Omega)$. The idea is then to enlarge the domain $\omega \setminus \overline{\omega_1}$ inside ω_1 by a smooth (at least Lipschitz) domain $\widetilde{\omega}_0$ (see Figure II.7). We then

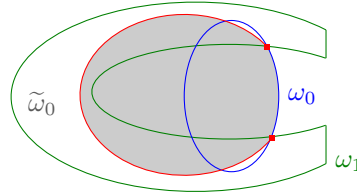


Figure II.7 – The set $\widetilde{\omega}_0$

make the same proof than in the first case using a density argument replacing ω_0 by $\widetilde{\omega}_0$.

In conclusion, we have proved that $\omega_0 \setminus \overline{\omega_1} = \emptyset$. By symmetry, $\omega_1 \setminus \overline{\omega_0} = \emptyset$ and finally we obtain $\omega_0 = \omega_1$. □

First order shape derivatives of the state. Let $(\mathbf{u}_t, p_t) \in \mathbf{H}^1(\Omega \setminus \overline{\omega_t}) \times L^2(\Omega \setminus \overline{\omega_t})$ be the solution of the perturbed Navier-Stokes problem (II.3.12). Then

$$\left\{ \begin{array}{ll} \int_{\Omega \setminus \overline{\omega_t}} \{ \nu \nabla \mathbf{u}_t : \nabla \boldsymbol{\varphi}_t - p_t \operatorname{div} \boldsymbol{\varphi}_t \} + b_{\Omega \setminus \overline{\omega_t}}(\mathbf{u}_t, \mathbf{u}_t, \boldsymbol{\varphi}_t) &= \mathbf{0} & \forall \boldsymbol{\varphi}_t \in \mathbf{H}^1(\Omega \setminus \overline{\omega_t}) \\ \int_{\Omega \setminus \overline{\omega_t}} \xi_t \operatorname{div} \mathbf{u}_t &= 0 & \forall \xi_t \in L^2(\Omega \setminus \overline{\omega_t}) \\ \int_O m^2 (\nu \partial_{\mathbf{n}} \mathbf{u}_t - p_t \mathbf{n}) \cdot \mathbf{n} &= \int_O m^2 \mathbf{f}_b \cdot \mathbf{n}. \end{array} \right. \quad (\text{II.3.20})$$

Proof of Proposition II.3.5. First step: first order shape differentiability. Using the Fréchet differentiability Lemma I.2.5, we obtain the Gâteaux differentiability in the direction $\mathbf{V}$: there exist $\mathbf{u}_t$ and p_t , respective extensions in Ω of $\mathbf{u}_t$ and p_t such that the functions, defined on $[0, T)$, $t \mapsto \mathbf{u}_t \in \mathbf{H}^1(\Omega)$ and $t \mapsto p_t \in L^2(\Omega)$ are differentiable at 0 by composition. We denote by $\mathbf{u}'$ and p' their respective derivative at 0.

Second step: derivative of the normalization condition. The function

$$\int_O m^2 (\nu \partial_{\mathbf{n}} \mathbf{u}_t - p_t \mathbf{n}) \cdot \mathbf{n} = \int_O m^2 \mathbf{f}_b \cdot \mathbf{n}$$

is constant on $[0, T)$. Thus, its derivative is 0 and we get $\int_O m^2 (\nu \partial_{\mathbf{n}} \mathbf{u}' - p' \mathbf{n}) \cdot \mathbf{n} = 0$.

Third step: derivative of the equalities on $\Omega \setminus \bar{\omega}_t$. Let $\varphi \in \mathcal{D}(\Omega \setminus \bar{\omega})$. As $\Omega \setminus \bar{\omega}$ is open, we have, for t small enough, $\varphi \in \mathcal{D}(\Omega \setminus \bar{\omega}_t)$. Therefore, using the first line of the variational formulation (II.3.20), we have for all $t \in [0, T)$

$$\begin{aligned} 0 &= \int_{\Omega \setminus \bar{\omega}_t} (\nu \nabla \mathbf{u}_t : \nabla \varphi - p_t \operatorname{div} \varphi) + b_{\Omega \setminus \bar{\omega}_t}(\mathbf{u}_t, \mathbf{u}_t, \varphi) \\ &= \int_{\Omega \setminus \bar{\omega}} (\nu \nabla \mathbf{u}_t : \nabla \varphi - p_t \operatorname{div} \varphi) + b_{\Omega \setminus \bar{\omega}}(\mathbf{u}_t, \mathbf{u}_t, \varphi). \end{aligned}$$

We differentiate with respect to t at $t = 0$ the equality

$$0 = \int_{\Omega \setminus \bar{\omega}} (\nu \nabla \mathbf{u}_t : \nabla \varphi + p_t \operatorname{div} \varphi) + b_{\Omega \setminus \bar{\omega}}(\mathbf{u}_t, \mathbf{u}_t, \varphi)$$

to obtain

$$\begin{aligned} 0 &= \int_{\Omega \setminus \bar{\omega}} (\nu \nabla \mathbf{u}' : \nabla \varphi + p' \operatorname{div} \varphi) + b_{\Omega \setminus \bar{\omega}}(\mathbf{u}', \mathbf{u}, \varphi) + b_{\Omega \setminus \bar{\omega}}(\mathbf{u}, \mathbf{u}', \varphi) \\ &= \langle -\nu \Delta \mathbf{u}' + (\mathbf{u}' \cdot \nabla) \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u}' + \nabla p', \varphi \rangle_{\mathcal{D}'(\Omega \setminus \bar{\omega}), \mathcal{D}(\Omega \setminus \bar{\omega})}. \end{aligned}$$

As it is true for all $\varphi \in \mathcal{D}(\Omega \setminus \bar{\omega})$, we get

$$-\nu \Delta \mathbf{u}' + (\mathbf{u}' \cdot \nabla) \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u}' + \nabla p' = \mathbf{0} \quad \text{in } \mathcal{D}'(\Omega \setminus \bar{\omega}).$$

Proceeding with the second line of (II.3.20) as previously, we check that $\operatorname{div} \mathbf{u}' = 0$ in $\mathcal{D}'(\Omega \setminus \bar{\omega})$.

Fourth step: derivatives of the boundary values condition. We differentiate with respect to t the relation $\mathbf{u}_t = \mathbf{g}$ on $\partial\Omega$, at $t = 0$. We obtain on $\partial\Omega$ that $\mathbf{u}' = \mathbf{0}$.

Let us set $\mathbf{v}_t := \mathbf{u}_t \circ \phi(t)$. By differentiation with respect to t (see [72, Corollary 5.2.5]), we obtain $\mathbf{u}' = \mathbf{v}' - \nabla \mathbf{u} \mathbf{V}$ in $\Omega_\delta \setminus \bar{\omega}$. Since $\mathbf{v}_t = \mathbf{0}$ on $\partial\omega$ for all $t \in [0, T)$, $\mathbf{v}' = \mathbf{0}$ on $\partial\omega$ and $\mathbf{u}' = -\nabla \mathbf{u} \mathbf{V}$. Finally, since $\mathbf{u} = \mathbf{0}$ on $\partial\omega$, $\nabla \mathbf{u} = \partial_{\mathbf{n}} \mathbf{u} \otimes \mathbf{n}$, where $\otimes$ is the tensorial product, and $\mathbf{u}' = -\partial_{\mathbf{n}} \mathbf{u} V_{\mathbf{n}}$. $\square$

First order shape derivatives of the functional. Define, for all $t \in [0, T)$,

$$J(\omega_t) := j(t) := \int_O m^2 |(\nu \partial_{\mathbf{n}} \mathbf{u}_t - p_t \mathbf{n}) - \mathbf{f}_b|^2.$$

Proof of Proposition II.3.6. First step: derivative of j and adjoint problem. By Proposition (II.3.5), the pair $(\mathbf{u}, p)$ is differentiable with respect to the shape. We denote the shape derivative by $(\mathbf{u}', p')$. Differentiating j with respect to t at $t = 0$, we obtain

$$j'(0) = 2 \int_O m^2 (\nu \partial_{\mathbf{n}} \mathbf{u}' - p' \mathbf{n}) \cdot ((\nu \partial_{\mathbf{n}} \mathbf{u} - p \mathbf{n}) - \mathbf{f}_b).$$

Then, we consider the adjoint problem (II.3.14). According to Theorem A.2.7, it admits a unique solution $(\mathbf{w}, \pi) \in \mathbf{H}^1(\Omega \setminus \bar{\omega}) \times L^2(\Omega \setminus \bar{\omega})$ with $\langle m^2(\nu \partial_{\mathbf{n}} \mathbf{w} - \pi \mathbf{n}), \mathbf{n} \rangle_O = 0$. Indeed, the compatibility condition $\int_O m^2(\nu \partial_{\mathbf{n}} \mathbf{u} - p \mathbf{n} - \mathbf{f}_b) \cdot \mathbf{n} = 0$ concerning the adjoint problem is true as consequence of our choice of compatibility condition for the state (II.3.9).

Second step: writing of $j'(0)$ as an integral on $\partial\omega$. We proceed by successive integrations by parts. We multiply the first equation of the adjoint problem (II.3.14) by $\mathbf{u}'$ to get:

$$\int_{\Omega \setminus \bar{\omega}} \nu \nabla \mathbf{w} : \nabla \mathbf{u}' + b_{\Omega \setminus \bar{\omega}}(\mathbf{u}', \mathbf{u}, \mathbf{w}) - b_{\Omega \setminus \bar{\omega}}(\mathbf{u}, \mathbf{w}, \mathbf{u}') = - \langle -\nu \partial_{\mathbf{n}} \mathbf{w} + \pi \mathbf{n}, \mathbf{u}' \rangle_{\partial(\Omega \setminus \bar{\omega})}, \quad (\text{II.3.21})$$

(since $\text{div } \mathbf{u}' = 0$ in $\Omega \setminus \bar{\omega}$). Then, we multiply the first equation of (II.3.13) by $\mathbf{w}$ to obtain

$$\int_{\Omega \setminus \bar{\omega}} \nu \nabla \mathbf{u}' : \nabla \mathbf{w} + b_{\Omega \setminus \bar{\omega}}(\mathbf{u}', \mathbf{u}, \mathbf{w}) + b_{\Omega \setminus \bar{\omega}}(\mathbf{u}, \mathbf{u}', \mathbf{w}) = - \langle -\nu \partial_{\mathbf{n}} \mathbf{u}' + p' \mathbf{n}, \mathbf{w} \rangle_{\partial(\Omega \setminus \bar{\omega})}, \quad (\text{II.3.22})$$

(since $\text{div } \mathbf{w} = 0$ in $\Omega \setminus \bar{\omega}$). Therefore, gathering Equalities (II.3.21) and (II.3.22) and using the boundary conditions of $(\mathbf{u}', p')$ and $(\mathbf{w}, \pi)$ (see Problems (II.3.13) and (II.3.14)) and the fact that $b(\mathbf{u}, \mathbf{w}, \mathbf{u}') = -b(\mathbf{u}, \mathbf{u}', \mathbf{w})$, we obtain

$$2 \int_O m^2 (\nu \partial_{\mathbf{n}} \mathbf{u}' - p' \mathbf{n}) \cdot ((\nu \partial_{\mathbf{n}} \mathbf{u} - p \mathbf{n}) - \mathbf{f}_b) = - \int_{\partial\omega} (\nu \partial_{\mathbf{n}} \mathbf{w} - \pi \mathbf{n}) \cdot \partial_{\mathbf{n}} \mathbf{u} V_{\mathbf{n}}.$$

□

Characterization of the shape Hessian at a critical point. We consider $\omega^* \in \mathcal{O}_\delta$ (with $\mathcal{O}_\delta$ defined by (II.0.1)) a solution of the inverse problem (II.3.3). Thanks to the local regularity results A.2.6 and A.2.8,

$$(\mathbf{u}, p) \in \mathbf{H}^3(\Omega_\delta \setminus \bar{\omega}^*) \times \mathbf{H}^2(\Omega_\delta \setminus \bar{\omega}^*) \quad \text{and} \quad (\mathbf{u}', p') \in \mathbf{H}^2(\Omega_\delta \setminus \bar{\omega}^*) \times \mathbf{H}^1(\Omega_\delta \setminus \bar{\omega}^*).$$

Proof of Proposition II.3.9. First step: second order shape differentiability. By Lemma I.2.9, $(\mathbf{u}, p)$ is twice differentiable with respect to the domain. We denote by $(\mathbf{u}'', p'')$ the second order shape derivatives.

Second step: second derivative of j and derivative of the adjoint problem. Consider $\mathbf{V} \in \mathbf{U}$. We differentiate the function j twice with respect to t . At $t = 0$, it holds

$$j''(0) = D^2 J(\omega) \cdot \mathbf{V} \cdot \mathbf{V} = 2 \int_O m^2 [(\nu \partial_{\mathbf{n}} \mathbf{u}'' - p'' \mathbf{n}) \cdot (\nu \partial_{\mathbf{n}} \mathbf{u} - p \mathbf{n} - \mathbf{f}_b) + |\nu \partial_{\mathbf{n}} \mathbf{u}' - p' \mathbf{n}|^2].$$

Since ω^* solves the inverse problem, $\nu \partial_{\mathbf{n}} \mathbf{u} - p \mathbf{n} = \mathbf{f}_b$ on O . Therefore

$$D^2 J(\omega^*) \cdot \mathbf{V} \cdot \mathbf{V} = 2 \int_O m^2 |\nu \partial_{\mathbf{n}} \mathbf{u}' - p' \mathbf{n}|^2.$$

We introduce $(\mathbf{w}, \pi) \in \mathbf{H}^1(\Omega \setminus \bar{\omega}) \times L^2(\Omega \setminus \bar{\omega})$ with $\langle m^2(\nu \partial_{\mathbf{n}} \mathbf{w} - \pi \mathbf{n}), \mathbf{n} \rangle_O = 0$ the solution of the adjoint system (II.3.14). According to Lemma I.2.10, $(\mathbf{w}, \pi)$ is differentiable with respect to the domain and we denote $(\mathbf{w}', \pi')$ its shape derivative. In the same manner that we characterized $\mathbf{u}'$ and p' (see Proposition II.3.5), we characterize $\mathbf{w}'$ and π' as a solution of

$$\left\{ \begin{array}{ll} -\nu \Delta \mathbf{w}' + {}^t \nabla \mathbf{u} \mathbf{w}' - \nabla \mathbf{w}' \mathbf{u} + \nabla \pi' &= \nabla \mathbf{w} \mathbf{u}' - {}^t \nabla \mathbf{u}' \mathbf{w} & \text{in } \Omega \setminus \bar{\omega} \\ \text{div } \mathbf{w}' &= 0 & \text{in } \Omega \setminus \bar{\omega} \\ \mathbf{w}' &= 2 m^2 (\nu \partial_{\mathbf{n}} \mathbf{u}' - p' \mathbf{n}) & \text{on } \partial\Omega \\ \mathbf{w}' &= -\partial_{\mathbf{n}} \mathbf{w} V_{\mathbf{n}} & \text{on } \partial\omega, \end{array} \right.$$

with $\langle m^2(\nu \partial_{\mathbf{n}} \mathbf{w}' - \pi' \mathbf{n}), \mathbf{n} \rangle_O = 0$. In particular, for $\omega = \omega^*$, $\nu \partial_{\mathbf{n}} \mathbf{u} - p \mathbf{n} = \mathbf{f}_b$ on O . Therefore, according to System (II.3.14) solved by $(\mathbf{w}, \pi)$, the uniqueness of the solution of the linearized Navier-Stokes problem enforces that $\mathbf{w} = \mathbf{0}$ in $\Omega \setminus \overline{\omega^*}$. Therefore $\partial_{\mathbf{n}} \mathbf{w} V_n = \mathbf{0}$ on $\partial \omega^*$ and we obtain System (II.3.15).

Third step: writing of $j''(0)$ as an integral on $\partial \omega$. We multiply the first equation of problem (II.3.15) by $\mathbf{u}'$ to get

$$\int_{\Omega \setminus \overline{\omega^*}} \nu \nabla \mathbf{w}' : \nabla \mathbf{u}' + b(\mathbf{u}', \mathbf{u}, \mathbf{w}') - b(\mathbf{u}, \mathbf{w}', \mathbf{u}') = - \langle -\nu \partial_{\mathbf{n}} \mathbf{w}' + \pi' \mathbf{n}, \mathbf{u}' \rangle_{\partial(\Omega \setminus \overline{\omega^*})}. \quad (\text{II.3.23})$$

We multiply the first equation of problem (II.3.13) by $\mathbf{w}'$ to get

$$\int_{\Omega \setminus \overline{\omega^*}} \nu \nabla \mathbf{u}' : \nabla \mathbf{w}' + b(\mathbf{u}', \mathbf{u}, \mathbf{w}') + b(\mathbf{u}, \mathbf{u}', \mathbf{w}') = - \langle -\nu \partial_{\mathbf{n}} \mathbf{u}' + p' \mathbf{n}, \mathbf{w}' \rangle_{\partial(\Omega \setminus \overline{\omega^*})}. \quad (\text{II.3.24})$$

Therefore, gathering (II.3.23) and (II.3.24), and using $b(\mathbf{u}, \mathbf{w}', \mathbf{u}') = -b(\mathbf{u}, \mathbf{u}', \mathbf{w})$, we obtain

$$2 \int_O m^2 |\nu \partial_{\mathbf{n}} \mathbf{u}' - p' \mathbf{n}|^2 = - \int_{\partial \omega^*} (\nu \partial_{\mathbf{n}} \mathbf{w}' - \pi' \mathbf{n}) \cdot \partial_{\mathbf{n}} \mathbf{u} V_n.$$

□

Justifying the ill-posedness of the problem. We recall that we assumed that $\omega^* \in \mathcal{O}_\delta$ (with $\mathcal{O}_\delta$ defined by (II.0.1)) is a solution of the inverse problem (II.3.3). To prove Proposition II.3.11, we first decompose the shape Hessian at ω^* : for $\mathbf{V} \in \mathbf{U}$,

$$D^2 J(\omega^*) \cdot \mathbf{V} \cdot \mathbf{V} = \langle M_2 \circ M_1(\mathbf{V}), T_{u,p}(\mathbf{V}) \rangle_{\partial \omega^*}.$$

Here, $T_{u,p} : \mathbf{H}^{1/2}(\partial \omega^*) \rightarrow \mathbf{H}^{1/2}(\partial \omega^*)$ is defined by

$$T_{u,p}(\mathbf{V}) := -\partial_{\mathbf{n}} \mathbf{u} V_n,$$

and $M_1 : \mathbf{H}^{1/2}(\partial \omega^*) \rightarrow \mathbf{H}^{1/2}(\partial \Omega)$ by $M_1(\mathbf{V}) := 2m^2(\nu \partial_{\mathbf{n}} \mathbf{u}' - p' \mathbf{n})$, with $(\mathbf{u}', p')$ the solution of (II.3.13). Finally, $M_2 : \mathbf{H}^{1/2}(\partial \Omega) \rightarrow \mathbf{H}^{-1/2}(\partial \omega^*)$ satisfies $M_2(\boldsymbol{\varphi}) := \nu \partial_{\mathbf{n}} \boldsymbol{\psi} - \chi \mathbf{n}$, where $(\boldsymbol{\psi}, \chi)$ is the solution of the following problem

$$\begin{cases} -\nu \Delta \boldsymbol{\psi} + {}^t \nabla \mathbf{u} \boldsymbol{\psi} - \nabla \boldsymbol{\psi} \mathbf{u} + \nabla \chi &= \mathbf{0} & \text{in } \Omega \setminus \overline{\omega^*} \\ \operatorname{div} \boldsymbol{\psi} &= 0 & \text{in } \Omega \setminus \overline{\omega^*} \\ \boldsymbol{\psi} &= \boldsymbol{\varphi} & \text{on } \partial \Omega \\ \boldsymbol{\psi} &= \mathbf{0} & \text{on } \partial \omega^*. \end{cases} \quad (\text{II.3.25})$$

We study each operator in the following two lemmas: Lemma II.3.16 states that $T_{u,p}$ and M_1 are linear continuous and Lemma II.3.17 claims that M_2 is compact. Hence, compactness is obtained by composition of linear continuous operator with a compact one.

Lemma II.3.16. *The operators $T_{u,p}$ and M_1 are linear continuous.*

Proof. Since $\mathbf{u}$ does not depend on $\mathbf{V}$ and using the continuity of the gradient operator ∇ , the operator $T_{u,p}$ is linear continuous as multiplier by a smooth function (see [82]).

Let us consider $\mathbf{V} \in \mathbf{H}^{1/2}(\partial \omega^*)$. Using a local regularity result similar to Theorem A.2.8, we prove that $(\mathbf{v}, q)$ belongs to $\mathbf{H}^2 \times \mathbf{H}^1$ in a neighborhood of O (since $\bar{O}$ is $C^{1,1}$) and, since $\operatorname{supp}(m) = \bar{O}$, the operator

$$\Phi : \mathbf{B} \in \mathbf{H}^{1/2}(\partial \omega^*) \mapsto 2m^2(\nu \partial_{\mathbf{n}} \mathbf{v} - q \mathbf{n}) \in \mathbf{H}^{1/2}(\partial \Omega),$$

II.4 Remarque concernant l'instabilité du problème

where $(\mathbf{v}, q)$ is the solution of the problem

$$\begin{cases} -\nu \Delta \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{v} + \nabla q &= \mathbf{0} & \text{in } \Omega \setminus \overline{\omega^*} \\ \operatorname{div} \mathbf{v} &= 0 & \text{in } \Omega \setminus \overline{\omega^*} \\ \mathbf{v} &= \mathbf{0} & \text{on } \partial\Omega \\ \mathbf{v} &= \mathbf{B} & \text{on } \partial\omega^*, \end{cases}$$

with $\int_O m^2(\nu \partial_{\mathbf{n}} \mathbf{v} - q \mathbf{n}) \cdot \mathbf{n} = 0$, is linear continuous. Finally, by composition, $M_1 = \Phi \circ T_{u,p}$ is linear continuous from $\mathbf{H}^{1/2}(\partial\omega^*)$ into $\mathbf{H}^{1/2}(\partial\Omega)$. $\square$

Lemma II.3.17. *The operator $M_2 : \mathbf{H}^{1/2}(\partial\Omega) \rightarrow \mathbf{H}^{-1/2}(\partial\omega^*)$ is compact.*

Proof. According to a local regularity result similar to Theorem A.2.8, the operator

$$\varphi_1 : \boldsymbol{\varphi} \in \mathbf{H}^{1/2}(\partial\Omega) \mapsto (\boldsymbol{\psi}, \chi) \in \mathbf{H}^3(\Omega_\delta \setminus \overline{\omega^*}) \times \mathbf{H}^2(\Omega_\delta \setminus \overline{\omega^*}),$$

where $(\boldsymbol{\psi}, \chi)$ is the solution of Problem (II.3.25), is linear continuous. Moreover,

$$\varphi_2 : (\boldsymbol{\psi}, \chi) \in \mathbf{H}^3(\Omega_\delta \setminus \overline{\omega^*}) \times \mathbf{H}^2(\Omega_\delta \setminus \overline{\omega^*}) \mapsto \nu \partial_{\mathbf{n}} \boldsymbol{\psi} - \chi \mathbf{n} \in \mathbf{H}^{3/2}(\partial\omega^*)$$

is linear continuous and the embedding of $\mathbf{H}^{3/2}(\partial\omega^*)$ in $\mathbf{H}^{-1/2}(\partial\omega^*)$, denoted by φ_3 , is compact. Thus, we get, by composition, that the operator $M_2 = \varphi_3 \circ \varphi_2 \circ \varphi_1$ is compact. $\square$

Remark II.3.18. *We see in this proof that if $\nu \partial_{\mathbf{n}} \boldsymbol{\psi} - \chi \mathbf{n} \in \mathbf{H}^{k+1/2}(\partial\omega^*)$, for a $k \in \mathbb{N}^*$, we use the compactness of the embedding of $\mathbf{H}^{k+1/2}(\partial\omega^*)$ in $\mathbf{H}^{-1/2}(\partial\omega^*)$. Thus, the more ω^* is smooth (with a $C^{k+1,1}$ boundary), the more $\nu \partial_{\mathbf{n}} \boldsymbol{\psi} - \chi \mathbf{n}$ is regular (it belongs to $\mathbf{H}^{k+1/2}(\partial\omega^*)$), the more the functional J is degenerate (see the comment after the statement of Proposition II.3.11) and the more the object is difficult to detect. For example, a circle (for $N = 2$) is very difficult to detect, if we do not know that we are looking for a circle.*

II.4 Remark on the instability of the problem

Statements II.1.7, II.1.17, II.2.4 and II.3.11 explain the difficulties encountered to solve numerically this problem. Indeed, the gradient has not a uniform sensitivity with respect to the deformation directions. Since the functional J is twice differentiable, it behaves about ω^* as its second order approximation. the compactness result means, roughly speaking, that in a neighborhood of ω^* (i.e. for t small) one cannot expect an estimate of the kind $C t \leq \sqrt{J(\omega_t)}$ with a constant C uniform in $\mathbf{V}$.

However we prove that, for a finite dimensional space of deformation fields, the discrete shape Hessian is coercive: our identifiability result proves that the domain to be recovered is a local strict minimum of the least squares functional. Let us take the example of a starshaped domain ω in dimension two. Assume that $\partial\omega$ is parametrized by

$$\partial\omega = \left\{ \begin{pmatrix} g_0 \\ g_1 \end{pmatrix} + \left(g_2 + \sum_{k=1}^{\infty} (g_{2k+1} \cos(kt) + g_{2k+2} \sin(kt)) \right) \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} = \sum_{k=0}^{\infty} g_k \mathbf{V}_k(t); t \in (0, 2\pi) \right\},$$

where $g_k \in \mathbb{R}$. Then, for all $n \in \mathbb{N}$, we have an estimate of the kind

$$\forall \mathbf{V} \in \operatorname{Span}(\mathbf{V}_k)_{0 \leq k \leq 2n+2}, \quad D^2 J(\omega^*) \cdot (\mathbf{V}, \mathbf{V}) \geq C_n \|\mathbf{V}\|^2,$$

where C_n is a positive constant. But this constant C_n tends to 0 when n tends to $+\infty$. We even expect that this decreasing is exponential as observed in [3] for the Laplacian case. Therefore, our functional is

degenerate for the highly oscillating deformations, *i.e.* for the deformation directions $\mathbf{V}_k$ with $k \gg 1$. Thus, for a numerical resolution, one has to exclude these highly oscillating deformations. We therefore explain the result of Conca *et al.* in [42]. This instability can be avoided by regularization, for example, by adding to the least squares functional a penalization in terms of the perimeter. This term leads to well posed problems (see [33] and [45]). This lack of real stability is also pointed out in the recent paper of Ballerini ([19]).

Chapitre III

Reconstruction numérique avec la méthode de variation frontières en deux dimensions

Dans ce chapitre, nous reconstruisons numériquement un (ou des) objet(s) immergé(s) dans un fluide lorsque le mouvement du fluide est supposé régi par les équations de Stokes incompressibles. Nous utilisons pour cela l'approche de Kohn-Vogelius détaillée dans le chapitre précédent.

Afin de minimiser cette fonctionnelle de forme, nous adoptons un algorithme d'optimisation de descente utilisant le gradient de forme caractérisé dans la proposition II.2.2 du chapitre II précédent. Cependant, notre approche théorique a permis d'établir que le problème inverse de détection d'obstacle dans un fluide est sévèrement mal posé au sens où la fonctionnelle de forme est dégénérée pour les hautes fréquences. Nous avons ainsi besoin de méthodes de régularisation pour le résoudre numériquement. Une solution est par exemple l'ajout d'un terme de pénalisation sur le périmètre à la fonctionnelle. En effet, ce terme entraîne alors des problèmes bien posés (voir [33] ou [45]) : la fonctionnelle n'est alors plus dégénérée. Ici, nous choisissons d'utiliser une méthode de régularisation paramétrique en utilisant un modèle paramétrique de variations de forme. Dans un premier temps, nous mettons alors en évidence le mauvais conditionnement de la matrice Hessienne de forme, puis nous réalisons des simulations numériques efficaces à l'aide d'une méthode adaptative.

Ce chapitre se décompose en six parties. Après avoir exposé le cadre des simulations numériques que nous réalisons, nous effectuons de premiers tests montrant la dégénérescence de la fonctionnelle pour les hautes fréquences. Nous proposons alors une méthode adaptative qui semble efficace pour éviter l'apparition d'oscillations du bord lors de la reconstruction et permet ainsi de travailler avec ces hautes fréquences. Nous utilisons ensuite cette méthode pour reconstruire des objets non réguliers, voire même plusieurs objets. L'influence du domaine de mesure O est également testé et nous montrons que, naturellement, il est difficile de reconstruire des objets (ou des parties d'un objet) éloignés du domaine de mesure.

III.1 Framework for the numerical simulations

Let Ω be a bounded, connected and Lipschitz open subset of $\mathbb{R}^2$ and $\omega \subset\subset \Omega$ be an open subset such that $\Omega \setminus \bar{\omega}$ is connected. Let $\mathbf{f} \in \mathbf{H}^{1/2}(\partial\Omega)$ such that $\mathbf{f} \neq \mathbf{0}$ satisfying the compatibility condition

$$\int_{\partial\Omega} \mathbf{f} \cdot \mathbf{n} = 0. \quad (\text{III.1.1})$$

Let O be a non-empty subset of $\partial\Omega$ and $\mathbf{g} \in \mathbf{H}^{-1/2}(O)$ be an admissible boundary measurement. We recall that, here, we want to solve the original inverse problem of reconstructing ω by minimizing the

following Kohn-Vogelius functional:

$$J_{KV}(\omega) := \frac{1}{2} \int_{\Omega \setminus \bar{\omega}} \nu |\nabla(\mathbf{u}_D - \mathbf{u}_N)|^2,$$

where $(\mathbf{u}_D, p_D) \in \mathbf{H}^1(\Omega \setminus \bar{\omega}) \times L_0^2(\Omega \setminus \bar{\omega})$ is the unique solution of the Stokes problem with Dirichlet boundary conditions

$$\begin{cases} -\nu \Delta \mathbf{u}_D + \nabla p_D &= \mathbf{0} & \text{in } \Omega \setminus \bar{\omega} \\ \operatorname{div} \mathbf{u}_D &= 0 & \text{in } \Omega \setminus \bar{\omega} \\ \mathbf{u}_D &= \mathbf{f} & \text{on } \partial\Omega \\ \mathbf{u}_D &= \mathbf{0} & \text{on } \partial\omega \end{cases} \quad (\text{III.1.2})$$

and $(\mathbf{u}_N, p_N) \in \mathbf{H}^1(\Omega \setminus \bar{\omega}) \times L^2(\Omega \setminus \bar{\omega})$ is the unique solution of the Stokes problem with mixed boundary conditions

$$\begin{cases} -\nu \Delta \mathbf{u}_N + \nabla p_N &= \mathbf{0} & \text{in } \Omega \setminus \bar{\omega} \\ \operatorname{div} \mathbf{u}_N &= 0 & \text{in } \Omega \setminus \bar{\omega} \\ -\nu \partial_{\mathbf{n}} \mathbf{u}_N + p_N \mathbf{n} &= \mathbf{g} & \text{on } O \\ \mathbf{u}_N &= \mathbf{f} & \text{on } \partial\Omega \setminus \bar{O} \\ \mathbf{u}_N &= \mathbf{0} & \text{on } \partial\omega. \end{cases} \quad (\text{III.1.3})$$

We denote by ω^* the actual obstacle, *i.e.* the unique inclusion such that $J_{KV}(\omega^*) = 0$.

The numerical simulations presented are made in dimension two using the finite elements library MÉLINA (see [78]) and the mesh generator TRIANGLE (see [89]). We use a $P3$ - $P2$ finite elements discretization to solve the Stokes equations (III.1.2) and (III.1.3). The framework is the following: we assume the kinematic viscosity ν is equal to 1, the exterior boundary is assumed to be the unit circle centered at the origin and we consider the exterior Dirichlet boundary condition

$$\mathbf{f} := \begin{pmatrix} n_2 \\ -n_1 \end{pmatrix} = \left\{ \begin{pmatrix} \sin \theta \\ -\cos \theta \end{pmatrix}, \theta \in [0, 2\pi) \right\},$$

where $\mathbf{n} = (n_1, n_2)$ is the exterior unit normal. Notice that $\mathbf{f}$ is such that the compatibility condition (III.1.1) is satisfied. In order to have a suitable pair (*measure* $\mathbf{g}$, *domain* ω^*), we use a synthetic data: we fix a shape ω^* , solve the Stokes problem (III.1.2) in $\Omega \setminus \bar{\omega}^*$ using another finite elements method (here a $P4$ - $P3$ finite elements discretization) and extract the measurement $\mathbf{g}$ by computing $-\nu \partial_{\mathbf{n}} \mathbf{u} + p \mathbf{n}$ on $\partial\Omega$. Except when mentioned, the simulations are performed in the case where $O = \partial\Omega$.

Here, we restrict ourselves to star-shaped domains and use polar coordinates for parametrization: the boundary $\partial\omega$ of the object can be then parametrized by

$$\partial\omega = \left\{ \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + r(\theta) \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \theta \in [0, 2\pi) \right\},$$

where $x_0, y_0 \in \mathbb{R}$ and where r is a $C^{2,1}$ function 2π -periodic. Taking into account of our main compactness result (Proposition II.2.4), we approximate the polar radius r by its truncated Fourier series

$$r_N(\theta) := a_0^N + \sum_{k=1}^N a_k^N \cos(k\theta) + b_k^N \sin(k\theta),$$

for the numerical simulations. Indeed this regularization by projection permits to remove *high frequencies* generated by $\cos(k\theta)$ and $\sin(k\theta)$ for $k \gg 1$, for which the functional is degenerated.

Then, the unknown shape is entirely defined by the coefficients (a_i, b_i) . Hence, for $k = 1, \dots, N$, the corresponding deformation directions are respectively,

$$\mathbf{V}_1 := \mathbf{V}_{x_0} := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{V}_2 := \mathbf{V}_{y_0} := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathbf{V}_3(\theta) := \mathbf{V}_{a_0}(\theta) := \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix},$$

$$\mathbf{V}_{2k+2}(\theta) := \mathbf{V}_{a_k}(\theta) := \cos(k\theta) \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \quad \mathbf{V}_{2k+3}(\theta) := \mathbf{V}_{b_k}(\theta) := \sin(k\theta) \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix},$$

$\theta \in [0, 2\pi)$. The gradient is then compute component by component using its characterization (see Proposition II.2.2) that we recall it:

$$\mathrm{D}J_{KV}(\omega) \cdot \mathbf{V} = - \int_{\partial\omega} (\nu \partial_{\mathbf{n}} \mathbf{w} - q \mathbf{n}) \cdot \partial_{\mathbf{n}} \mathbf{u}_D V_n + \frac{1}{2} \nu \int_{\partial\omega} |\nabla \mathbf{w}|^2 V_n, \quad (\text{III.1.4})$$

where $\mathbf{w} := \mathbf{u}_D - \mathbf{u}_N$ and $q = p_D - p_N$. Hence

$$\left(\nabla J_{KV}(\omega) \right)_k = \mathrm{D}J_{KV}(\omega) \cdot \mathbf{V}_k, \quad k = 1, \dots, 2N + 3.$$

This equality is simply that

$$\lim_{t \rightarrow 0} \frac{J_{KV}((\mathbf{I} + t\mathbf{V}_k)(\omega)) - J_{KV}(\omega)}{t} = \mathrm{D}J_{KV}(\omega) \cdot \mathbf{V}_k.$$

Notice that we have to solve only two Stokes problems (Problems (III.1.2) and (III.1.3)) to compute the gradient. Indeed, the directional perturbations $\mathbf{V}$ dependance is explicit in Formula (III.1.4). However, if we did not write the gradient in this way and use the first expression obtained by shape derivative, *i.e.*

$$\mathrm{D}J(\omega) \cdot \mathbf{V} = \nu \int_{\Omega \setminus \bar{\omega}} \nabla(\mathbf{u}_D - \mathbf{u}_N) : \nabla(\mathbf{u}'_D - \mathbf{u}'_N) + \frac{1}{2} \nu \int_{\partial\omega} |\nabla(\mathbf{u}_D - \mathbf{u}_N)|^2 V_n,$$

then the $\mathbf{V}$ dependance would be implicit ($\mathbf{u}'_D$ and $\mathbf{u}'_N$ depend on $\mathbf{V}$). Thus, we would have to solve $2 + 2 * k$ Stokes problems to compute the k coordinates of the gradient.

The optimization method used for the numerical simulations is here the classical gradient algorithm with a line search (using the Wolfe conditions: see for example [85, eq. (3.6) page 34]). Moreover, we here use $N_{ext} := 100$ discretization points for the exterior boundary and $N_{int} := 75$ for the interior boundary. In order to be completely explicit, we detail this algorithm:

Algorithm 1

1. fix a number of iterations M and an initial shape ω_0 ,
2. mesh $\Omega \setminus \bar{\omega}_i$ using TRIANGLE (where ω_i denotes the i^{th} iterate of the approximate shape),
3. solve problems (III.1.2) and (III.1.3) with $\omega = \omega_i$ using MÉLINA,
4. extract $\nabla \mathbf{u}_D$, $\nabla \mathbf{u}_N$, p_D and p_N on $\partial\omega_i$ and compute $\nabla J_{KV}(\omega_i)$ using Formula (III.1.4),
5. use the Wolfe conditions to compute a satisfying step length α_i ,
6. move the coefficients associated to the shape: $\omega_{i+1} = \omega_i - \alpha_i \nabla J_{KV}(\omega_i)$,
7. get back to the step 2. while $i < M$.

Remark III.1.1. Here, we do not use an optimization method of order two due to the expression of the shape Hessian (II.2.12). Indeed, notice that $(\mathbf{u}'_D, p'_D)$ and $(\mathbf{u}'_N, p'_N)$ (and so $(\mathbf{w}', q')$) depend on the perturbation direction $\mathbf{V}$. Therefore, if we want to compute this shape Hessian (for a shape parametrized by k parameters), then we will have to solve $2 + 2 * k$ Stokes problems, which would be too costly.

III.2 Highlight of the degenerescence of the functional

First, we want to detect an obstacle ω_1^* contained in the class of objects with which we work. We want to detect the obstacle which boundary is parametrized by:

$$\partial\omega_1^* = \left\{ \begin{pmatrix} 0.15 \\ 0.15 \end{pmatrix} + (0.65 - 0.25 \cos \theta - 0.15 \sin \theta) \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \theta \in [0, 2\pi) \right\},$$

that is to say with five parameters. In order to reconstruct this object, we work with shapes parametrized respectively by five and fifteen parameters and we stop each experiment when we obtain a residual value of the cost function J_{KV} : $\varepsilon \approx 10^{-3}$. Figure III.1 shows that when we work with five

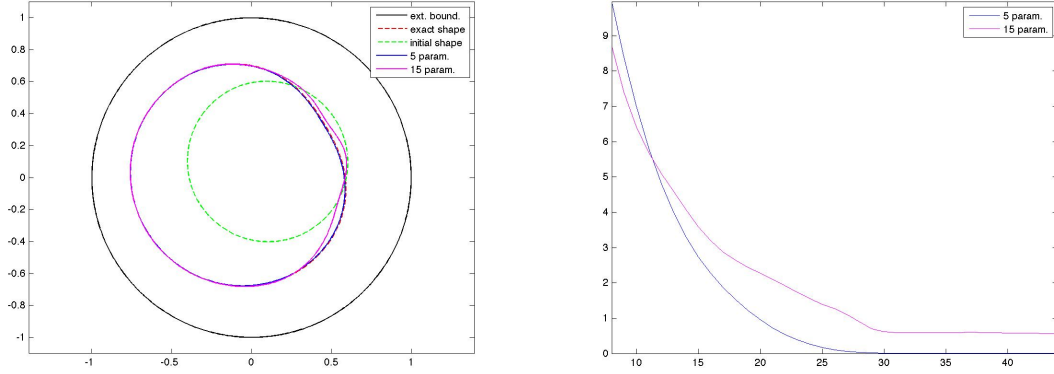


Figure III.1 – Reconstruction of ω_1^* with different numbers of parameters

parameters, the residual ε corresponds to a very good approximation of the object. But the more we add parameters, the more the functional flattens and the more the approximation corresponding to the residual ε is poor. Indeed, we see oscillations of the final object when we work with *high frequencies*. We can also notice that it is longer to obtain the residual ε with many parameters.

The same experiment was conducted to detect a more complicated obstacle which does not belong to the discretized set of objects. It is parametrized by:

$$\partial\omega_2^* = \left\{ \begin{pmatrix} 0.1 + 0.5 \cos \theta + 0.1 \cos 4\theta \\ 0.5 \sin \theta + 0.1 \cos 4\theta \end{pmatrix}, \theta \in [0, 2\pi] \right\}. \quad (\text{III.2.1})$$

Here again, the conclusion is the same (see Figure III.2): *high frequencies* lead oscillations of the

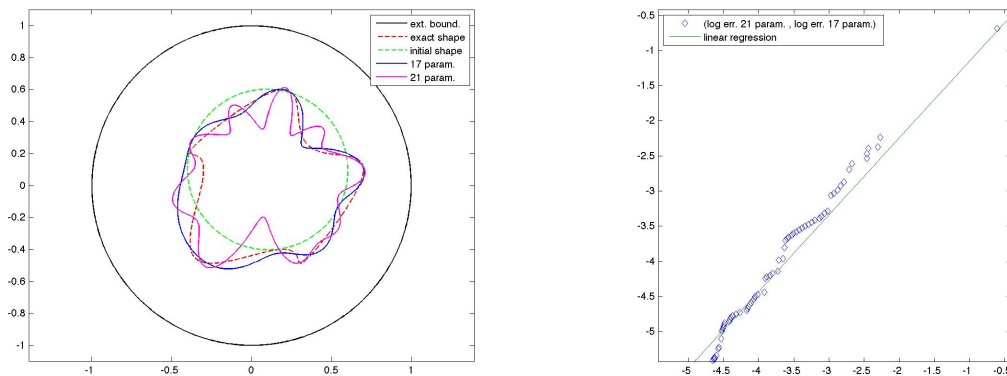


Figure III.2 – Reconstruction of ω_2^* with different numbers of parameters

boundary and then a poorer approximation.

Here again, more iterations are needed to obtain the residual ε . In order to compare the errors, we represent on the second graph of Figure III.2 the logarithm of the residual obtained with 17 parameters (noted $\log(J_{17})$) versus the logarithm of the residual obtained with 21 parameters (noted $\log(J_{21})$). We

then obtain a regression line whose equation is given by $y = 1.091x - 0.064$. Hence $\frac{J_{17}}{J_{21}} = 0.938 J_{21}^{0.091}$ and this points out that when the number of iterations is big, the gap between J_{17} and J_{21} is big too (J_{17} is much smaller than J_{21}).

The problem is: *how to know the number of parameters with which we must work to well approximate the shape?* Indeed, if we work with too few parameters we cannot detect a non-trivial shape and if we work with too many parameters, degeneracy of the functional leads problems. From now, we present some numerical illustrations.

III.3 An adaptive method

A solution which seems to be efficient is to use an *adaptive method*. It consists in increasing gradually the number of parameters during the algorithm to a fixed final number of parameters. For example, if we want to work with twenty-one parameters, we begin by working with two parameters during five iterations, then with three parameters (we add the radius) during five more iterations, and then we add two search parameters every fifteen iterations.

We also adapt the number of discretization points for the interior boundary (we conserve the same for the exterior boundary). We fix a minimum number of discretization points (here 40). Then, when we consider the perturbation directions associated to $\cos(k\theta)$ and $\sin(k\theta)$, we discretize the interior boundary using $N_{int} = 9 * k$ points. This method permits to discretize half of a period using five points, which seems suitable.

The algorithm is the same than Algorithm 1 described above. However, the step 6. is replaced by

$$\omega_{i+1}(1 : m) = \omega_i(1 : m) - \alpha_i \nabla J_{KV}(\omega_i)(1 : m),$$

where $\omega_i(1 : m)$ represents the m first coefficients parametrizing the shape ω_i (the same notation holds for $\nabla J_{KV}(\omega_i)(1 : m)$). The number m grows to the fixed final number of parameters following the procedure described previously. This process permits to avoid the oscillations of the boundary as what is shown in Figure III.3. Indeed, we saw in Figure III.2 that if we work with twenty-one parameters,

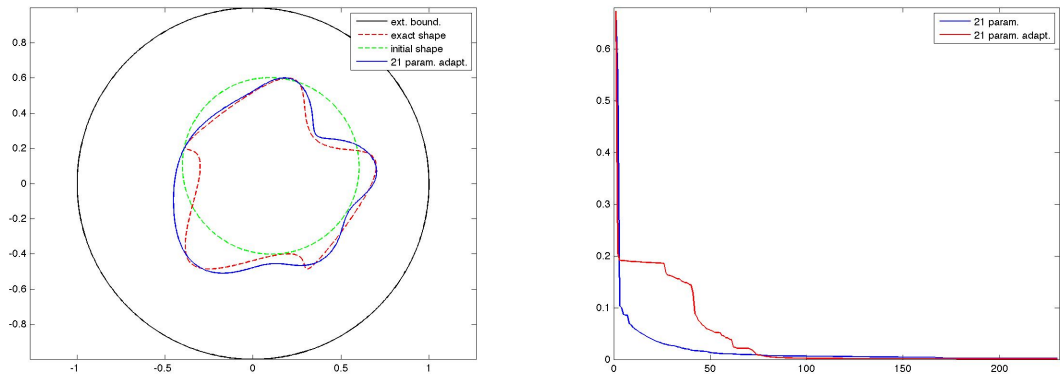


Figure III.3 – Reconstruction of ω_2^* using the *adaptive method* with 21 parameters

oscillations appear but we see in Figure III.3 that it is not the case using this *adaptive method*. We can notice that this *adaptive method* leads to steps in the evolution of the residual. Hence this *adaptive method* seems to be efficient and permits to reconstruct the obstacle removing the oscillations due to the *high frequencies*.

III.4 Detecting objects with corners

In order to test the performances of our algorithm and of our *adaptive method*, we want now to detect more complicated obstacles and particularly, we wonder if it is effective to reconstruct objects containing straight lines and corners.

The first objective is then to detect the square ω_3^* whose vertices are the points $(-0.55, -0.55)$, $(0.55, -0.55)$, $(0.55, 0.55)$ and $(-0.55, 0.55)$. We see in Figure III.4 that this reconstruction is quite

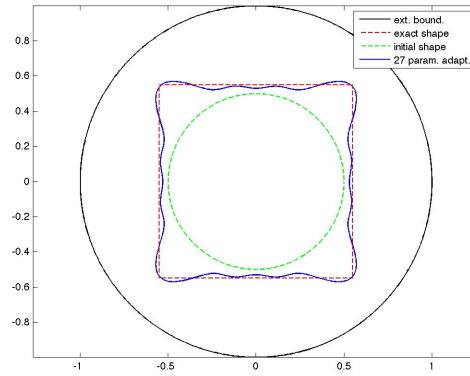


Figure III.4 – Reconstruction of ω_3^* using the *adaptive method* with 27 parameters

efficient. However, if we want to reconstruct a smaller object, the results are significantly worse. Indeed, in Figure III.5, we reconstruct the square ω_4^* whose vertices are the points $(-0.2, -0.2)$, $(0.2, -0.2)$, $(0.2, 0.2)$ and $(-0.2, 0.2)$.

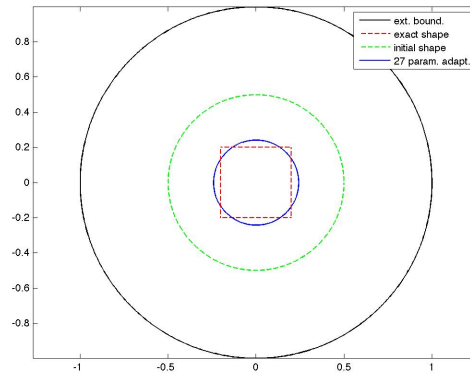


Figure III.5 – Reconstruction of ω_4^* using the *adaptive method* with 27 parameters

$(0.2, 0.2)$ and $(-0.2, 0.2)$. We see that the detection is not good. This phenomenon concerning the size of the object was underlined using the explicit calculus of the shape Hessian in Table II.1 (in Example II.2.8).

Thus, the parametrization method coupling with our *adaptive method* seems to be efficient to reconstruct obstacles, even if the shapes are not trivial. However, the regularizing behavior complicate the detection of small objects or of distant parts of the measurement domain.

III.5 Influence of the size of the domain where measurements are made

Let us consider the influence of the size of O , the part of the boundary where the measurements are assumed to be made. We now restrict the domain O to some part of $\partial\Omega$. We precise that we use here exactly the same algorithm and the same parameters and data that the ones used in Section III.3. In particular we use the previously described *adaptive method* and we want to reconstruct the object ω_2^* which is given by (III.2.1).

In Figure III.6, we make the measurement only on a semicircle. On the left of Figure III.6, O is

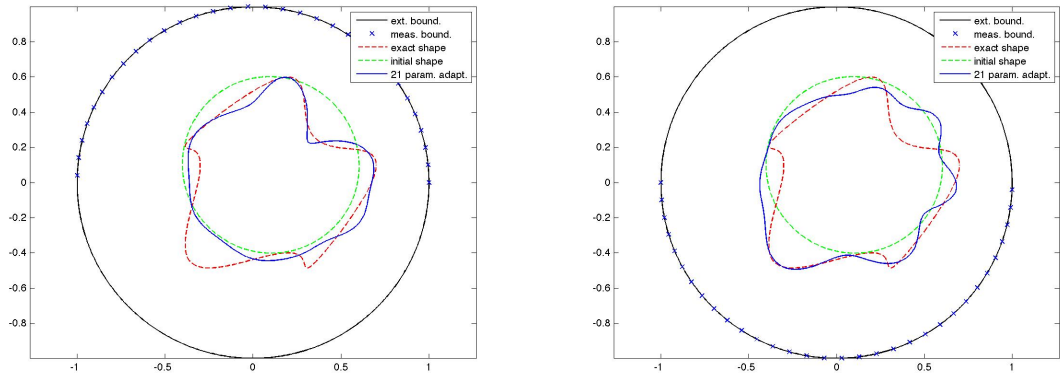


Figure III.6 – Reconstruction of ω_2^* using the *adaptive method* with 21 parameters restricting the measurement domain (O is a half of $\partial\Omega$)

the upper semicircle. As we could expect, the reconstruction of the object ω_2^* is less efficient than in the case $O = \partial\Omega$ (see Figure III.3). In particular, the bottom of the obstacle (*i.e.* the part exposed to the part of the exterior boundary where we do not make the measurement) is poorly detected. We precise that here, we do not obtain a residual $\varepsilon \approx 10^{-3}$ as in the case $O = \partial\Omega$ but only $\varepsilon \approx 6.8 \cdot 10^{-3}$. On the right of Figure III.6, O is the lower semicircle. We obtain similar results. As we expected, we see that in this case only the bottom of the obstacle is well detected.

In Figure III.7, O is the right superior quart-circle. Here again, the reconstruction is less efficient than in the case $O = \partial\Omega$ and even in the previous case $O = \frac{1}{2}\partial\Omega$. We can notice again that only the part exposed to the measurement domain O is well determined. Here, we only obtain a residual $\varepsilon \approx 8.4 \cdot 10^{-3}$.

These two simulations emphasize an intuitive idea: the more the measurement domain $O \subset \partial\Omega$ is small, the more it is difficult to well reconstruct the object ω .

III.6 Detecting more than one object

Theoretically, besides the regularity assumptions, the main assumption is that $\Omega \setminus \bar{\omega}$ is connected. This assumption does not exclude the case of two or more inclusions in Ω . Thus, we want now to detect numerically two objects: a square ω_5^* whose vertices are the points $(-0.6, 0.1)$, $(-0.1, 0.1)$, $(-0.1, 0.6)$ and $(-0.6, 0.6)$ and a circle ω_6^* centered at $(0.35, -0.35)$ with radius 0.35. Here again we use the *adaptive method* described in Section III.3. Since the computation is significantly longer than the one with only one object, we stop the experiment when the residual is $\varepsilon \approx 1.5e-02$. We then obtain Figure III.8.

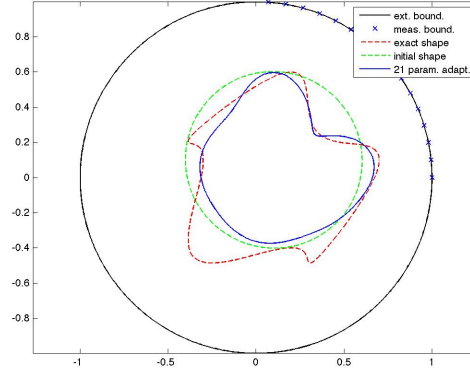


Figure III.7 – Reconstruction of ω_2^* using the *adaptive method* with 21 parameters restricting the measurement domain (O is a quarter of $\partial\Omega$)

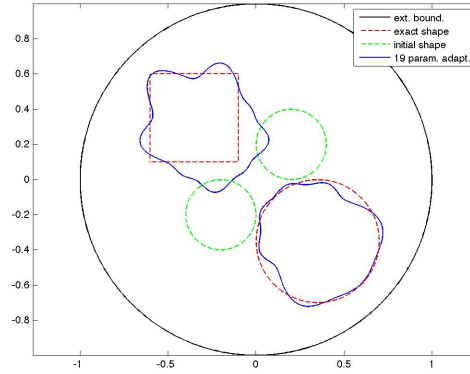


Figure III.8 – Reconstruction of ω_5^* and ω_6^* using the *adaptive method* with 19 parameters

This result is satisfying and we can hope that it would be more efficient if we increase the accuracy of the computation (using a better finite elements discretization for example) and/or if we increase the computation time. Notice that the non-smooth square is not as well reconstructed as the circle: this is caused by the polar representation we used.

Notice an important remark. The parametric method that we adopted here to detect the objects does not permit to modify the topology of the object ω . Thus, in order to detect two or more inclusions, we have to know how many objects are included in Ω . A solution could be to initialize the algorithm using the notion of *topological gradient* (see for example [23], [60] or [93]) which could give us the number of inclusions and their rough location, providing initial shapes for our optimization method.

Chapitre IV

Cas de petites inclusions : analyse asymptotique

Dans ce chapitre, nous traitons le cas de petites inclusions immergées dans un fluide visqueux, incompressible et stationnaire lorsque le mouvement du fluide est régi par les équations de Stokes. En effet, l'hypothèse de petitesse des inclusions permet d'utiliser des formules asymptotiques qui nous permettent de changer de point de vue (voir les travaux d'Ammari et al. [8–12]). Nous allons alors utiliser la notion de gradient topologique afin de déterminer le nombre d'objets présents et leur localisation approximative.

Nous voulons à l'aide de cette technique détecter de petits obstacles immergés dans un fluide vivant dans un domaine plus grand Ω . Nous nous concentrons ici sur la dimension trois. Nous effectuons une mesure sur une partie O du bord extérieur $\partial\Omega$ et étudions alors une fonctionnelle de type Kohn-Vogelius. L'objectif est de minimiser cette fonctionnelle à l'aide d'un algorithme d'optimisation de type gradient. Afin de pouvoir obtenir une expression du gradient topologique de cette fonctionnelle, nous démontrons dans un premier temps le développement asymptotique de la solution des équations de Stokes considérées lorsque l'on ajoute un obstacle à l'intérieur. Nous obtenons finalement un développement asymptotique topologique de la fonctionnelle de Kohn-Vogelius, en particulier lorsque l'on ajoute des objets sphériques. En effet, dans ce cas très particulier, nous avons une formule explicite du gradient topologique. Nous disposons ainsi d'un critère nous permettant d'ajouter des objets à l'intérieur d'un domaine afin de diminuer le coût de la fonctionnelle.

Ce chapitre se décompose en quatre parties. Nous commençons par définir le problème étudié et rappeler la notion de dérivée topologique. Nous énonçons alors nos résultats principaux caractérisant le gradient topologique de la fonctionnelle de Kohn-Vogelius. Nous démontrons ensuite le développement asymptotique de la solution du problème de Stokes, en particulier avec des conditions mixtes sur le bord extérieur et de Dirichlet sur le bord intérieur. Enfin, nous démontrons nos résultats principaux à l'aide de ce résultat.

Nous nous inspirons principalement des travaux [1, 23, 66, 68]. Le fait de considérer ici des conditions de Dirichlet sur le bord intérieur et des conditions mixtes sur le bord extérieur (du fait de notre approche par une fonctionnelle de Kohn-Vogelius) ajoute des difficultés pour montrer ce développement asymptotique. Nous reprenons donc l'ensemble des démonstrations des références précédentes en détails.

IV.1 The problem setting

Let Ω be a bounded Lipschitz open set of $\mathbb{R}^3$ containing a Newtonian and incompressible fluid with coefficient of kinematic viscosity $\nu > 0$. Let $\omega \subset \mathbb{R}^3$ a fixed bounded Lipschitz domain containing the

origin. For $z \in \Omega$ and $\varepsilon > 0$, we denote

$$\omega_{z,\varepsilon} := z + \varepsilon\omega.$$

The aim of this work is to detect some unknown objects included in Ω . We assume that a finite number m^* of obstacles $\omega_{z,\varepsilon}^* \subset \Omega$ have to be detected. Moreover, we assume that they are well separated and have the geometry form

$$\omega_{z_k,\varepsilon}^* = z_k^* + \varepsilon\omega_k^*, \quad 1 \leq k \leq m^*,$$

where ε is the shared diameter and $\omega_k^* \subset \mathbb{R}^3$ are bounded Lipschitz domains containing the origin. The points $z_k^* \in \Omega$, $1 \leq k \leq m^*$, determine the location of the objects. Finally, we assume that, for all $1 \leq k \leq m^*$, $\omega_{z_k,\varepsilon}^*$ is far from the boundary $\partial\Omega$.

Let $\mathbf{f} \in \mathbf{H}^{1/2}(\partial\Omega)$ such that $\mathbf{f} \neq \mathbf{0}$ satisfying the compatibility condition

$$\int_{\partial\Omega} \mathbf{f} \cdot \mathbf{n} = 0. \quad (\text{IV.1.1})$$

In order to determine the location of the objects, we make a measurement $\mathbf{g} \in \mathbf{H}^{-1/2}(O)$ on a part O of the exterior boundary $\partial\Omega$ with $O \subsetneq \partial\Omega$. Then, we notice $\omega_\varepsilon^* := \bigcup_{k=1}^{m^*} \omega_{z_k,\varepsilon}^*$ and consider the following overdetermined Stokes problem

$$\left\{ \begin{array}{ll} -\nu\Delta \mathbf{u} + \nabla p &= \mathbf{0} \quad \text{in } \Omega \setminus \overline{\omega_\varepsilon^*} \\ \operatorname{div} \mathbf{u} &= 0 \quad \text{in } \Omega \setminus \overline{\omega_\varepsilon^*} \\ \mathbf{u} &= \mathbf{f} \quad \text{on } \partial\Omega \\ \mathbf{u} &= \mathbf{0} \quad \text{on } \partial\omega_\varepsilon^* \\ \sigma(\mathbf{u}, p)\mathbf{n} &= \mathbf{g} \quad \text{on } O \subset \partial\Omega. \end{array} \right. \quad (\text{IV.1.2})$$

Here $\sigma(\mathbf{u}, p)$ represents the stress tensor defined by

$$\sigma(\mathbf{u}, p) := \nu (\nabla \mathbf{u} + {}^t\nabla \mathbf{u}) - p\mathbf{I}.$$

Notice that, if $\operatorname{div} \mathbf{u} = 0$ in Ω , we have

$$-\nu\Delta \mathbf{u} + \nabla p = -\operatorname{div}(\nu\mathcal{D}(\mathbf{u})) + \nabla p = -\operatorname{div}(\sigma(\mathbf{u}, p)) \quad \text{in } \Omega,$$

with $\mathcal{D}(\mathbf{u}) := (\nabla \mathbf{u} + {}^t\nabla \mathbf{u})$. Therefore, we can consider two problems:

$$(\mathcal{P}_D^\varepsilon) \left\{ \begin{array}{ll} \text{Find } (\mathbf{u}_D^\varepsilon, p_D^\varepsilon) \in \mathbf{H}^1(\Omega \setminus \overline{\omega_\varepsilon}) \times L_0^2(\Omega \setminus \overline{\omega_\varepsilon}) \text{ such that} \\ -\nu\Delta \mathbf{u}_D^\varepsilon + \nabla p_D^\varepsilon &= \mathbf{0} \quad \text{in } \Omega \setminus \overline{\omega_\varepsilon} \\ \operatorname{div} \mathbf{u}_D^\varepsilon &= 0 \quad \text{in } \Omega \setminus \overline{\omega_\varepsilon} \\ \mathbf{u}_D^\varepsilon &= \mathbf{f} \quad \text{on } \partial\Omega \\ \mathbf{u}_D^\varepsilon &= \mathbf{0} \quad \text{on } \partial\omega_\varepsilon \end{array} \right. \quad (\text{IV.1.3})$$

and

$$(\mathcal{P}_N^\varepsilon) \left\{ \begin{array}{ll} \text{Find } (\mathbf{u}_N^\varepsilon, p_N^\varepsilon) \in \mathbf{H}^1(\Omega \setminus \overline{\omega_\varepsilon}) \times L^2(\Omega \setminus \overline{\omega_\varepsilon}) \text{ such that} \\ -\nu\Delta \mathbf{u}_N^\varepsilon + \nabla p_N^\varepsilon &= \mathbf{0} \quad \text{in } \Omega \setminus \overline{\omega_\varepsilon} \\ \operatorname{div} \mathbf{u}_N^\varepsilon &= 0 \quad \text{in } \Omega \setminus \overline{\omega_\varepsilon} \\ \sigma(\mathbf{u}_N^\varepsilon, p_N^\varepsilon)\mathbf{n} &= \mathbf{g} \quad \text{on } O \\ \mathbf{u}_N^\varepsilon &= \mathbf{f} \quad \text{on } \partial\Omega \setminus \overline{O} \\ \mathbf{u}_N^\varepsilon &= \mathbf{0} \quad \text{on } \partial\omega_\varepsilon, \end{array} \right. \quad (\text{IV.1.4})$$

where $\omega_\varepsilon := \bigcup_{k=1}^m \omega_{z_k, \varepsilon}$ for a finite number m of objects located in $z_1, \dots, z_m$. These two problems are classically well-defined.

We refer, for example, to [13, 31, 44, 56, 61, 97], for the results of existence, uniqueness and regularity of the solution of the Stokes system with Dirichlet boundary conditions $(\mathcal{P}_D^\varepsilon)$. Notice that we assume the compatibility condition (IV.1.1) associated to Problem (IV.1.3) is satisfied. Hence, the solution $(\mathbf{u}_D^\varepsilon, p_D^\varepsilon)$ exists and is unique. Moreover, the existence and the uniqueness of the couple $(\mathbf{u}_N^\varepsilon, p_N^\varepsilon)$ is guaranteed by Theorem A.1.5 and other results on the mixed boundary value problem $(\mathcal{P}_N^\varepsilon)$ are recalled in Appendix A.1.2.2.

One can remark that if ω_ε coincides with the actual domain ω_ε^* , then $\mathbf{u}_D^\varepsilon = \mathbf{u}_N^\varepsilon$ in $\Omega \setminus \overline{\omega_\varepsilon}$. According to this observation, we propose a resolution of the inverse problem of reconstructing ω_ε^* based on the minimization of the following Kohn-Vogelius functional

$$\mathcal{F}_\varepsilon^{KV}(\mathbf{u}_D^\varepsilon, \mathbf{u}_N^\varepsilon) := \frac{1}{2} \int_{\Omega \setminus \overline{\omega_\varepsilon}} \nu |\mathcal{D}(\mathbf{u}_D^\varepsilon) - \mathcal{D}(\mathbf{u}_N^\varepsilon)|^2.$$

We then define

$$\mathcal{J}_{KV}(\Omega \setminus \overline{\omega_\varepsilon}) := \mathcal{F}_\varepsilon^{KV}(\mathbf{u}_D^\varepsilon, \mathbf{u}_N^\varepsilon).$$

We assume that there exists ω_ε^* such that the overdetermined problem (IV.1.2) has a solution. Hence, according to the identifiability result [6, Theorem 1.2] proved by Alvarez *et al.*, the domain ω_ε^* is unique. Hence, if we find ω_ε such that $\mathcal{J}(\Omega \setminus \overline{\omega_\varepsilon}) = 0$, then $\mathbf{u}_D^\varepsilon = \mathbf{u}_N^\varepsilon$ in $\Omega \setminus \overline{\omega_\varepsilon}$, *i.e.* $\mathbf{u}_D^\varepsilon$ satisfies (IV.1.2) and thus $\omega_\varepsilon = \omega_\varepsilon^*$ is the real domain.

Remark IV.1.1. In the following $\mathbf{u}_D^0, \mathbf{u}_N^0, \Omega_0$, etc. will denote respectively the solutions, the domain, etc. when $\omega_\varepsilon = \emptyset$ and not $\omega_\varepsilon = \omega_0 = \bigcup_{k=1}^m \{z_k\}$.

IV.2 The main results

From now, we consider that we seek a single obstacle $\omega_{z, \varepsilon} := z + \varepsilon\omega$, located at a point $z \in \Omega$. Notice that in the case of several inclusions, we proceed by detecting the objects one by one. Thus, after detecting a first obstacle $\omega_{z_1, \varepsilon_1}$, we work replacing the whole domain Ω by $\Omega \setminus \overline{\omega_{z_1, \varepsilon_1}}$ (and then we have $\partial\omega_{z_1, \varepsilon_1} \subset \partial(\Omega \setminus \overline{\omega_{z_1, \varepsilon_1}}) \setminus \overline{O}$) and the results presented below (in particular the topological derivative) are still valid for a new inclusion $\omega_{z, \varepsilon}$. Note that, the asymptotic expansion of the solution of elliptic boundary value problem in multiply perforated domains is studied in [25, 80].

IV.2.1 The needed functional tools

In order to be more clear on the notion of topological gradient, we begin by presenting it briefly. Let us consider a shape functional $\mathcal{J}(\Omega) = J_\Omega(u_\Omega)$, where u_Ω solves a given PDE defined in $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. For a small parameter $\varepsilon > 0$ and $z \in \Omega$, we consider the perturbed domain $\Omega_{z, \varepsilon}$ created by inserting a small hole (or a small obstacle in our case) inside the initial domain $\Omega_0 = \Omega$ at the point z , *i.e.*

$$\Omega_{z, \varepsilon} := \Omega \setminus \overline{\omega_{z, \varepsilon}},$$

with $\omega_{z, \varepsilon} := z + \varepsilon\omega$ and where ω is a fixed bounded open subset of $\mathbb{R}^N$ containing the origin. Then, we want to know the influence of this modification of topology on the variation of $\mathcal{J}$. We can generally obtain an asymptotic expansion of the functional $\mathcal{J}$ in the following form:

$$\mathcal{J}(\Omega_{z, \varepsilon}) = \mathcal{J}(\Omega) + \xi(\varepsilon) G(z) + o(\xi(\varepsilon)) \quad \forall z \in \Omega,$$

where ξ is a positive scalar function which going to zero with ε and where G is called the *topological gradient* (or *topological derivative*). This expansion is called the *topological asymptotic* and the topological gradient G is denoted by $\delta\mathcal{J}$. This topological derivative provides information for inserting small holes at the point z . Indeed, if $\delta\mathcal{J}(z) < 0$, then $\mathcal{J}(\Omega_{z,\varepsilon}) < \mathcal{J}(\Omega)$ for ε small. Thus, the best location to create the hole is the point where $\delta\mathcal{J}$ reaches its minimum. A difficulty is then to know the size of the hole we have to add. We summarize the notations concerning the domains in Figure IV.1.

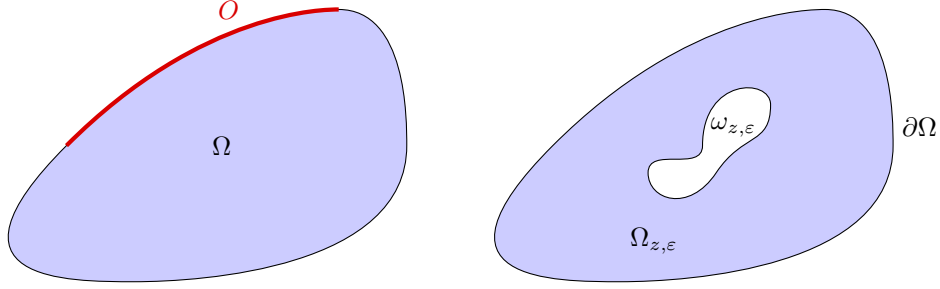


Figure IV.1 – The initial domain and the same domain after inclusion of an object

The computation of the topological gradient $\delta\mathcal{J}$ exposed below is strongly influenced by the paper of Ben Abda *et al.* [23]. We were also inspired by the papers of Guillaume *et al.* [66], of Abdelwahed *et al.* [1] and of Hassine [68]. In [66], they use the Masmoudi's approach based on an adaptation of the adjoint method and a domain truncation technique that provides an equivalent formulation of the PDE in a fixed functional space. Here, we do not use this truncation technique but we follow the simplified approach presented in [23] (see also [1] and [68]). However, contrary to the problem studied in [23], we impose Dirichlet boundary conditions on the interior boundary. Moreover, contrary to [1] and [68], we have here to consider mixed boundary conditions on the exterior boundary (due to our Kohn-Vogelius approach). These modifications lead additional difficulties.

We recall the expression of the fundamental solution $(E, \mathbf{P})$ to the Stokes system in $\mathbb{R}^3$ given by

$$E(x) = \frac{1}{8\pi\nu\|x\|} (\mathbf{I} + \mathbf{e}_r^t \mathbf{e}_r), \quad \mathbf{P}(x) = \frac{x}{4\pi\|x\|^3}, \quad (\text{IV.2.1})$$

with $\mathbf{e}_r = \frac{x}{\|x\|}$; that is

$$-\nu\Delta E_j + \nabla \mathbf{P}_j = \delta \mathbf{e}_j,$$

where E_j denotes the j^{th} column of E , $(\mathbf{e}_j)_{j=1}^3$ is the canonical basis of $\mathbb{R}^3$ and δ is the Dirac distribution.

IV.2.2 The results

The following theorem give us the expression of the topological gradient of the Kohn-Vogelius functional $\mathcal{J}_{KV}$:

Theorem IV.2.1. *For $z \in \Omega$, the functional $\mathcal{J}_{KV}$ admits the following topological asymptotic expansion*

$$\mathcal{J}_{KV}(\Omega_{z,\varepsilon}) - \mathcal{J}_{KV}(\Omega) = \varepsilon \left[- \left(\int_{\partial\omega} \boldsymbol{\eta}_D \right) \cdot \mathbf{u}_D^0(z) + \left(\int_{\partial\omega} \boldsymbol{\eta}_N \right) \cdot \mathbf{u}_N^0(z) \right] + o(\varepsilon),$$

where $\mathbf{u}_D^0 \in \mathbf{H}^1(\Omega_{z,\varepsilon})$ and $\mathbf{u}_N^0 \in \mathbf{H}^1(\Omega_{z,\varepsilon})$ solve respectively Problems (IV.1.3) and (IV.1.4) with $\omega_\varepsilon = \emptyset$ and where $\boldsymbol{\eta}_D \in \mathbf{H}^{-1/2}(\partial\omega)/\mathbb{R}\mathbf{n}$ and $\boldsymbol{\eta}_N \in \mathbf{H}^{-1/2}(\partial\omega)/\mathbb{R}\mathbf{n}$ are respectively the unique solution

of

$$\int_{\partial\omega} E(y-x)\boldsymbol{\eta}_D(x)ds(x) = -\mathbf{u}_D^0(z) \quad \forall y \in \partial\omega \quad (\text{IV.2.2})$$

and

$$\int_{\partial\omega} E(y-x)\boldsymbol{\eta}_N(x)ds(x) = -\mathbf{u}_N^0(z) \quad \forall y \in \partial\omega. \quad (\text{IV.2.3})$$

In the case of spherical objects $\omega = B(0,1)$, we get a more explicit formula for the topological gradient of $\mathcal{J}_{KV}$. Indeed,

$$\int_{\partial\omega} \boldsymbol{\eta}_D = -6\pi\nu\mathbf{u}_D^0(z) \quad \text{and} \quad \int_{\partial\omega} \boldsymbol{\eta}_N = -6\pi\nu\mathbf{u}_N^0(z)$$

holds in this very particular case (see for example [66, Proof of Corollary 4.2]). Then, we obtain the following corollary:

Corollary IV.2.2. *If $\omega = B(0,1)$ then, for $z \in \Omega$, the functional $\mathcal{J}_{KV}$ admits the following topological asymptotic expansion*

$$\mathcal{J}_{KV}(\Omega_{z,\varepsilon}) - \mathcal{J}_{KV}(\Omega) = 6\pi\nu\varepsilon \left(|\mathbf{u}_D^0(z)|^2 - |\mathbf{u}_N^0(z)|^2 \right) + o(\varepsilon), \quad (\text{IV.2.4})$$

where $\mathbf{u}_D^0 \in \mathbf{H}^1(\Omega_{z,\varepsilon})$ and $\mathbf{u}_N^0 \in \mathbf{H}^1(\Omega_{z,\varepsilon})$ solve respectively Problems (IV.1.3) and (IV.1.4) with $\omega_\varepsilon = \emptyset$.

IV.3 Asymptotic expansion of the solution of the Stokes system

In order to provide an asymptotic expansion of the Kohn-Vogelius functional $\mathcal{J}_{KV}$, we need first an asymptotic expansion of the solution of the Stokes problems (IV.1.3) and (IV.1.4). We precise that, in this section, we will work with exterior Stokes problems: the theoretical tools concerning these problems are recalled in Appendix A.3 (in particular the definition of weighted Sobolev space $\mathbf{W}_0^{1,2}$).

This section is devoted to the proof of the following proposition:

Proposition IV.3.1. *The respective solutions $\mathbf{u}_D^\varepsilon \in \mathbf{H}^1(\Omega_{z,\varepsilon})$ and $\mathbf{u}_N^\varepsilon \in \mathbf{H}^1(\Omega_{z,\varepsilon})$ of Problems (IV.1.3) and (IV.1.4) admit the following asymptotic expansion (with the subscript $\mathfrak{h} = D$ and $\mathfrak{h} = N$ respectively):*

$$\mathbf{u}_\mathfrak{h}^\varepsilon(x) = \mathbf{u}_\mathfrak{h}^0(x) + \mathbf{U}_\mathfrak{h} \left(\frac{x-z}{\varepsilon} \right) + O_{\mathbf{H}^1(\Omega_{z,\varepsilon})}(\varepsilon)$$

where $(\mathbf{U}_\mathfrak{h}, P_\mathfrak{h}) \in \mathbf{W}_0^{1,2}(\mathbb{R}^3 \setminus \overline{\omega}) \times L^2(\mathbb{R}^3 \setminus \overline{\omega})$ solves the following Stokes exterior problem

$$\begin{cases} -\nu\Delta\mathbf{U}_\mathfrak{h} + \nabla P_\mathfrak{h} &= \mathbf{0} & \text{in } \mathbb{R}^3 \setminus \overline{\omega} \\ \operatorname{div} \mathbf{U}_\mathfrak{h} &= 0 & \text{in } \mathbb{R}^3 \setminus \overline{\omega} \\ \mathbf{U}_\mathfrak{h} &= -\mathbf{u}_\mathfrak{h}^0(z) & \text{on } \partial\omega. \end{cases} \quad (\text{IV.3.1})$$

The notation $O_{\mathbf{H}^1(\Omega_{z,\varepsilon})}(\varepsilon)$ means that there exist a constant $c > 0$ (independent of ε) and $\varepsilon_1 > 0$ such that for all $0 < \varepsilon < \varepsilon_1$

$$\left\| \mathbf{u}_\mathfrak{h}^\varepsilon(x) - \mathbf{u}_\mathfrak{h}^0(x) - \mathbf{U}_\mathfrak{h} \left(\frac{x-z}{\varepsilon} \right) \right\|_{1,\Omega_{z,\varepsilon}} \leq c\varepsilon.$$

We will only detail the Neumann case $\mathfrak{h} = N$ in this section. The proof of the Dirichlet case $\mathfrak{h} = D$ is a direct adaptation of this Neumann case (we also refer to [68, Proposition 3.1] and [1, Proposition 3.1]). The case $\mathfrak{h} = N$ is not treated in [1] and [68] and the mixed boundary condition on the exterior boundary leads modifications in the following proof.

Here we prove this result using uniform *a priori* estimates as what is done in [66], [68] or [1]. An alternative method to obtain the same result consists to use *multi-scale expansion* and the notion of *profile* (see [26], [46] or [81]).

IV.3.1 Some notations and préliminaries

In the following, we will use the fact that, for $\mathbf{u} \in \mathbf{H}^1(\Omega)$, the norm

$$|||\mathbf{u}||| := \left(\int_{\Omega} |\mathcal{D}(\mathbf{u})|^2 + |\mathbf{u}|^2 \right)^{1/2}$$

is equivalent to the norm $\|\mathbf{u}\|_{1,\Omega}$. Indeed, using the Cauchy-Schartz inequality

$$|||\mathbf{u}|||^2 \leq \int_{\Omega} \left(2|\nabla \mathbf{u}|^2 + 2\nabla \mathbf{u} : {}^t \nabla \mathbf{u} \right) + \int_{\Omega} |\mathbf{u}|^2 \leq 4\|\mathbf{u}\|_{0,\Omega}^2 + \|\mathbf{u}\|_{0,\Omega}^2 \leq 4\|\mathbf{u}\|_{1,\Omega}^2,$$

and using Korn's inequality (see for example [77, Lemma 5.4.4]), there exists a constant $c > 0$ such that

$$|||\mathbf{u}||| \geq \frac{c}{c+1} \|\mathbf{u}\|_{1,\Omega}.$$

It is also important to notice that, for $\mathbf{u}, \mathbf{v} \in \mathbf{H}^1(\Omega)$, we have

$$\int_{\Omega} \mathcal{D}(\mathbf{u}) : \mathcal{D}(\mathbf{v}) = 2 \int_{\Omega} \mathcal{D}(\mathbf{u}) : \nabla \mathbf{v}. \quad (\text{IV.3.2})$$

Moreover, we will use the following change of variables: for a given function $u \in H^1(\Omega)$, we define the function $\tilde{u}$ on $\tilde{\Omega} := \frac{\Omega}{\varepsilon}$ by

$$\tilde{u}(y) = u(x), \quad y = \frac{x}{\varepsilon}.$$

Using that $\nabla_x u(x) = \frac{\nabla_y \tilde{u}(y)}{\varepsilon}$, we obtain

$$|u|_{1,\Omega}^2 = \int_{\Omega} |\nabla_x u(x)|^2 dx = \varepsilon \int_{\tilde{\Omega}} |\nabla_y \tilde{u}(y)|^2 dy.$$

Hence,

$$|u|_{1,\Omega} = \varepsilon^{1/2} |\tilde{u}|_{1,\tilde{\Omega}}. \quad (\text{IV.3.3})$$

Similarly, we obtain

$$\|u\|_{0,\Omega} = \varepsilon^{3/2} \|\tilde{u}\|_{0,\tilde{\Omega}}. \quad (\text{IV.3.4})$$

Notice that we have the same equalities with the change of variables $y = \frac{x-z}{\varepsilon}$, for $z \in \Omega$.

Finally, let us introduce some other domains. Let $R > 0$ be such that the closed ball $\overline{B(z, R)}$ is included in Ω and $\omega_{z,\varepsilon} \subset B(z, R)$. We define the domains

$$\Omega_R^z := \Omega \setminus \overline{B(z, R)} \quad \text{and} \quad D_{\varepsilon}^z := B(z, R) \setminus \overline{\omega_{z,\varepsilon}}$$

(see Figure IV.2). Thus, in particular, we denote $\Omega_R^0 := \Omega \setminus \overline{B(0, R)}$ and $D_{\varepsilon}^0 := B(0, R) \setminus \overline{\omega_{\varepsilon}}$.

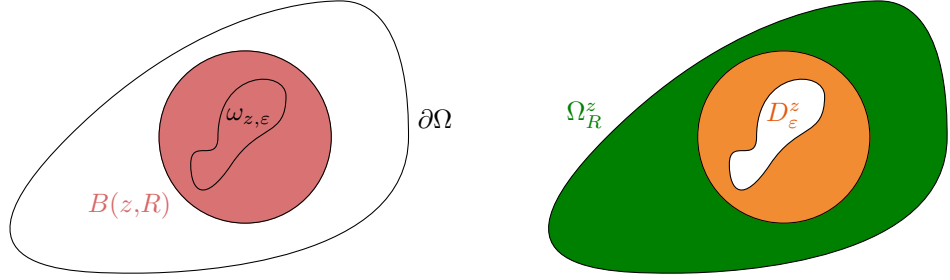


Figure IV.2 – The truncated domain

IV.3.2 Uniform *a priori* estimates

In order to prove Proposition IV.3.1, we will use uniform *a priori* estimates given by the two following lemmas.

This first lemma is quoted in [66, Lemma 6.2] (see also [68, Lemma 3.1] or [69, Lemma 7.1]) but we recall the proof for the reader's convenience:

Lemma IV.3.2 (Guillaume *et al.*, [66]). *Let $\mathbf{g} \in \mathbf{H}^{1/2}(\partial\omega)$ such that $\int_{\partial\omega} \mathbf{g} \cdot \mathbf{n} = 0$. We consider $(\mathbf{w}, s) \in \mathbf{W}_0^{1,2}(\mathbb{R}^3 \setminus \bar{\omega}) \times L^2(\mathbb{R}^3 \setminus \bar{\omega})$ the solution of the Stokes exterior problem*

$$\begin{cases} -\nu \Delta \mathbf{w} + \nabla s &= \mathbf{0} & \text{in } \mathbb{R}^3 \setminus \bar{\omega} \\ \operatorname{div} \mathbf{w} &= 0 & \text{in } \mathbb{R}^3 \setminus \bar{\omega} \\ \mathbf{w} &= \mathbf{g} & \text{on } \partial\omega. \end{cases}$$

Then there exists a constant $c > 0$ (independent of ε and $\mathbf{g}$) and $\varepsilon_1 > 0$ such that for all $0 < \varepsilon < \varepsilon_1$

$$\begin{aligned} \|\mathbf{w}\|_{0, D_\varepsilon^0/\varepsilon} &\leq c \varepsilon^{-1/2} \|\mathbf{g}\|_{1/2, \partial\omega}, & \|\mathbf{w}\|_{0, \Omega_R^0/\varepsilon} &\leq c \varepsilon^{-1/2} \|\mathbf{g}\|_{1/2, \partial\omega}, \\ |\mathbf{w}|_{1, D_\varepsilon^0/\varepsilon} &\leq c \|\mathbf{g}\|_{1/2, \partial\omega} & \text{and } |\mathbf{w}|_{1, \Omega_R^0/\varepsilon} &\leq c \varepsilon^{1/2} \|\mathbf{g}\|_{1/2, \partial\omega}. \end{aligned}$$

Proof. According to the representation formula (see Theorem A.3.3), since $\int_{\partial\omega} \mathbf{g} \cdot \mathbf{n} = 0$, the velocity $\mathbf{w}$ can be written as

$$\mathbf{w}(y) = S\boldsymbol{\eta}(y) = \int_{\partial\omega} E(y-x)\boldsymbol{\eta}(x)ds(x), \quad y \in \mathbb{R}^3 \setminus \bar{\omega}, \quad (\text{IV.3.5})$$

where $\boldsymbol{\eta} \in \mathbf{H}^{-1/2}(\partial\omega)/\mathcal{R}$ is the unique solution of

$$S\boldsymbol{\eta}(y) = \mathbf{g}(y) \quad \forall y \in \partial\omega \quad (\text{IV.3.6})$$

(the simple layer hydrodynamic potential operator S and the relation $\mathcal{R}$ being defined in Theorem A.3.3 and the Green function associated to the Stokes system E by (IV.2.1)). Then, using a first order Taylor expansion of E , we have, for x bounded,

$$E(y-x) = E(y) + O\left(\frac{1}{\|y\|^2}\right) \quad \text{and} \quad \nabla E(y-x) = \nabla E(y) + O\left(\frac{1}{\|y\|^3}\right)$$

and using (IV.3.5),

$$\mathbf{w}(y) = E(y) \langle \boldsymbol{\eta}, 1 \rangle_{-1/2, 1/2, \partial\omega} + O\left(\frac{1}{\|y\|^2}\right) \langle \boldsymbol{\eta}, 1 \rangle_{-1/2, 1/2, \partial\omega}$$

(with $O\left(\frac{1}{\|y\|^2}\right)$ uniform in x). Moreover, S is an isomorphism from $\mathbf{H}^{-1/2}(\partial\omega)/\mathcal{R}$ into the space $\left\{ \mathbf{g} \in \mathbf{H}^{1/2}(\partial\omega), \int_{\partial\omega} \mathbf{g} \cdot \mathbf{n} = 0 \right\}$ (see Theorem A.3.3). Thus, according to (IV.3.6),

$$\|\boldsymbol{\eta}\|_{-1/2, \partial\omega} \leq c \|\mathbf{g}\|_{1/2, \partial\omega}.$$

Then, since $|E(y)| \leq \frac{c}{\|y\|}$, there exists $M > 0$ such that, for $\|y\| > M$,

$$|\mathbf{w}(y)| \leq \frac{c}{\|y\|} \|\mathbf{g}\|_{1/2, \partial\omega} + \frac{c}{\|y\|^2} \|\mathbf{g}\|_{1/2, \partial\omega}. \quad (\text{IV.3.7})$$

We differentiate under the sum sign in Equality (IV.3.5) and proceed as above using the fact that $|\nabla E(y)| \leq \frac{c}{\|y\|^2}$ to obtain that for $\|y\| > M$ (even if we have to increase M)

$$|\nabla \mathbf{w}(y)| \leq \frac{c}{\|y\|^2} \|\mathbf{g}\|_{1/2, \partial\omega} + \frac{c}{\|y\|^3} \|\mathbf{g}\|_{1/2, \partial\omega}. \quad (\text{IV.3.8})$$

Notice that since the upper bounds for $\mathbf{w}$ and its gradient (see (IV.3.7) and (IV.3.8)) are valid only for $\|y\| > M$, that is far away of $\partial\omega$, we can not obtain directly the estimates on $D_\varepsilon^0/\varepsilon$. We split them in two estimates: one on $B(0, M) \setminus \overline{\omega}$ and one on $B(0, \frac{R}{\varepsilon}) \setminus \overline{B(0, M)}$ (where we can use the above inequalities).

Using the change of variables $x = \varepsilon y$, the inequalities (IV.3.7) and (IV.3.8) are valid for $\varepsilon < \frac{R}{M}$ and we have

$$\begin{aligned} & \|\mathbf{w}\|_{0, B(0, \frac{R}{\varepsilon}) \setminus \overline{B(0, M)}} \\ & \leq c \left(\int_{B(0, R) \setminus \overline{B(0, \varepsilon M)}} \frac{\varepsilon^2}{\|x\|^2} \frac{1}{\varepsilon^3} \right)^{1/2} \|\mathbf{g}\|_{1/2, \partial\omega} + c \left(\int_{B(0, R) \setminus \overline{B(0, \varepsilon M)}} \frac{\varepsilon^4}{\|x\|^4} \frac{1}{\varepsilon^3} \right)^{1/2} \|\mathbf{g}\|_{1/2, \partial\omega} \\ & \leq c \left(\int_{B(0, R)} \frac{\varepsilon^2}{\|x\|^2} \frac{1}{\varepsilon^3} \right)^{1/2} \|\mathbf{g}\|_{1/2, \partial\omega} + c \left(\int_{B(0, R)} \frac{\varepsilon^4}{\|x\|^4} \frac{1}{\varepsilon^3} \right)^{1/2} \|\mathbf{g}\|_{1/2, \partial\omega} \\ & \leq c \varepsilon^{-1/2} \|\mathbf{g}\|_{1/2, \partial\omega} + c \varepsilon^{1/2} \|\mathbf{g}\|_{1/2, \partial\omega}. \end{aligned} \quad (\text{IV.3.9})$$

Moreover, since the weight ρ defined in Appendix A.3 is bounded from above and below in $B(0, M) \setminus \overline{\omega}$ by non-negative constants and according to the definition of the norm $\|\cdot\|_{\mathbf{W}_0^{1,2}}$, there exists a constant $c > 0$ such that

$$\|\mathbf{w}\|_{1, B(0, M) \setminus \overline{\omega}} \leq c \|\mathbf{w}\|_{\mathbf{W}_0^{1,2}(B(0, M) \setminus \overline{\omega})}.$$

Using the *a priori* estimate in Theorem A.3.2, we also obtain $\|\mathbf{w}\|_{\mathbf{W}_0^{1,2}(\mathbb{R}^3 \setminus \overline{\omega})} \leq c \|\mathbf{g}\|_{1/2, \partial\omega}$. Then,

$$\|\mathbf{w}\|_{1, B(0, M) \setminus \overline{\omega}} \leq c \|\mathbf{g}\|_{1/2, \partial\omega}. \quad (\text{IV.3.10})$$

Hence, gathering the estimates on each subdomains (IV.3.9) and (IV.3.10), we obtain the global estimate

$$\|\mathbf{w}\|_{0, \frac{D_\varepsilon^0}{\varepsilon}} \leq c \varepsilon^{-1/2} \|\mathbf{g}\|_{1/2, \partial\omega}.$$

Similarly, we obtain $|\mathbf{w}|_{1, B(0, \frac{R}{\varepsilon}) \setminus \overline{B(0, M)}} \leq c \varepsilon^{1/2} \|\mathbf{g}\|_{1/2, \partial\omega} + c \varepsilon^{3/2} \|\mathbf{g}\|_{1/2, \partial\omega}$ and then, using Inequality (IV.3.10),

$$|\mathbf{w}|_{1, \frac{D_\varepsilon^0}{\varepsilon}} \leq c \|\mathbf{g}\|_{1/2, \partial\omega}.$$

IV.3 Développement asymptotique de la solution du problème de Stokes

Finally, without splitting the norm in two, we obtain

$$\begin{aligned} \|\mathbf{w}\|_{0, \frac{\Omega_R^0}{\varepsilon}} &\leq c\varepsilon^{-1/2} \|\mathbf{g}\|_{1/2, \partial\omega} + c\varepsilon^{1/2} \|\mathbf{g}\|_{1/2, \partial\omega} \\ |\mathbf{w}|_{1, \frac{\Omega_R^0}{\varepsilon}} &\leq c\varepsilon^{1/2} \|\mathbf{g}\|_{1/2, \partial\omega} + c\varepsilon^{3/2} \|\mathbf{g}\|_{1/2, \partial\omega}, \end{aligned}$$

which concludes the proof. $\square$

Next we want to prove the following lemma:

Lemma IV.3.3. *For $\varphi \in \mathbf{H}^{-1/2}(O)$, $\Phi \in \mathbf{H}^{1/2}(\partial\Omega \setminus \overline{O})$, $\Psi \in \mathbf{H}^1(D_0)$ such that $\operatorname{div} \Psi = 0$ and $\varepsilon > 0$, let $(\mathbf{v}_\varepsilon, q_\varepsilon) \in \mathbf{H}^1(\Omega_{z,\varepsilon}) \times L^2(\Omega_{z,\varepsilon})$ be the solution of the Stokes problem*

$$\begin{cases} -\nu \Delta \mathbf{u}_\varepsilon + \nabla p_\varepsilon = \mathbf{0} & \text{in } \Omega_{z,\varepsilon} \\ \operatorname{div} \mathbf{u}_\varepsilon = 0 & \text{in } \Omega_{z,\varepsilon} \\ \sigma(\mathbf{u}_\varepsilon, p_\varepsilon) \mathbf{n} = \varphi & \text{on } O \\ \mathbf{u}_\varepsilon = \Phi & \text{on } \partial\Omega \setminus \overline{O} \\ \mathbf{u}_\varepsilon = \Psi & \text{on } \partial\omega_{z,\varepsilon}. \end{cases} \quad (\text{IV.3.11})$$

Then there exist a constant $c > 0$ (independent of ε and $\mathbf{g}$) and $\varepsilon_1 > 0$ such that for all $0 < \varepsilon < \varepsilon_1$

$$\|\mathbf{u}_\varepsilon\|_{0, \Omega_{z,\varepsilon}} \leq c \left(\|\varphi\|_{-1/2, O} + \|\Phi\|_{1/2, \partial\Omega \setminus \overline{O}} + \varepsilon^{1/2} \|\Psi(z + \varepsilon y)\|_{1/2, \partial\omega_{z,\varepsilon}} \right).$$

To prove this result, we split it in two lemmas. First, we prove the following result inspired by [69, Lemma 7.2] and [66, Lemma 6.3]:

Lemma IV.3.4. *Let $\varepsilon > 0$. For $\varphi \in \mathbf{H}^{-1/2}(O)$ and $\Phi \in \mathbf{H}^{1/2}(\partial\Omega \setminus \overline{O})$, let $(\mathbf{v}_\varepsilon, q_\varepsilon) \in \mathbf{H}^1(\Omega_{z,\varepsilon}) \times L^2(\Omega_{z,\varepsilon})$ be the solution of the Stokes problem*

$$\begin{cases} -\nu \Delta \mathbf{v}_\varepsilon + \nabla q_\varepsilon = \mathbf{0} & \text{in } \Omega_{z,\varepsilon} \\ \operatorname{div} \mathbf{v}_\varepsilon = 0 & \text{in } \Omega_{z,\varepsilon} \\ \sigma(\mathbf{v}_\varepsilon, q_\varepsilon) \mathbf{n} = \varphi & \text{on } O \\ \mathbf{v}_\varepsilon = \Phi & \text{on } \partial\Omega \setminus \overline{O} \\ \mathbf{v}_\varepsilon = \mathbf{0} & \text{on } \partial\omega_{z,\varepsilon}. \end{cases} \quad (\text{IV.3.12})$$

Then there exist a constant $c > 0$ (independent of ε and φ) and $\varepsilon_1 > 0$ such that for all $0 < \varepsilon < \varepsilon_1$

$$\|\mathbf{v}_\varepsilon\|_{1, \Omega_{z,\varepsilon}} \leq c \left(\|\varphi\|_{-1/2, O} + \|\Phi\|_{1/2, \partial\Omega \setminus \overline{O}} \right).$$

Remark IV.3.5. *In the two following proofs, the brackets $\langle \cdot, \cdot \rangle_O$ represents the duality product between $\mathbf{H}^{-1/2}(O)$ and the space $\mathbf{H}_{00}^{1/2}(O)$ ($\mathbf{H}^{-1/2}(O)$ is the dual of $\mathbf{H}_{00}^{1/2}(O)$). This space $\mathbf{H}_{00}^{1/2}(O)$ is the space of restrictions to O of functions $\beta \in \mathbf{H}^{1/2}(\partial\Omega)$ such that the function $\tilde{\beta} = \beta$ on O and $\tilde{\beta} = 0$ on $\partial\Omega \setminus \overline{O}$ belongs to $\mathbf{H}^1(\Omega_{z,\varepsilon})$ (see for example [47, VII.§2.4, page 1247]). Then, the norm $\|\cdot\|_{-1/2, O}$ is defined as a duality norm:*

$$\|\alpha\|_{-1/2, O} = \sup_{\beta \in \mathbf{H}_{00}^{1/2}(O)} \frac{\langle \alpha, \beta \rangle_O}{\|\beta\|_{1/2, O}}.$$

Proof. Let $\varepsilon > 0$ and $(\mathbf{v}_\varepsilon, q_\varepsilon) \in \mathbf{H}^1(\Omega_{z,\varepsilon}) \times L^2(\Omega_{z,\varepsilon})$ be the solution of Problem (IV.3.12). Let $(\mathbf{V}_\varepsilon, Q_\varepsilon) \in \mathbf{H}^1(\Omega_{z,\varepsilon}) \times L^2(\Omega_{z,\varepsilon})$ be the solution of

$$\begin{cases} -\nu \Delta \mathbf{V}_\varepsilon + \nabla Q_\varepsilon = \mathbf{0} & \text{in } \Omega_{z,\varepsilon} \\ \operatorname{div} \mathbf{V}_\varepsilon = 0 & \text{in } \Omega_{z,\varepsilon} \\ \sigma(\mathbf{V}_\varepsilon, Q_\varepsilon) \mathbf{n} = \mathbf{0} & \text{on } O \\ \mathbf{V}_\varepsilon = \Phi & \text{on } \partial\Omega \setminus \overline{O} \\ \mathbf{V}_\varepsilon = \mathbf{0} & \text{on } \partial\omega_{z,\varepsilon}. \end{cases} \quad (\text{IV.3.13})$$

Let $\widetilde{\mathbf{v}}_\varepsilon$ and $\widetilde{\mathbf{V}}_\varepsilon$ the respective extensions of $\mathbf{v}_\varepsilon$ and $\mathbf{V}_\varepsilon$ to Ω by $\mathbf{0}$. Therefore the following equalities hold for all $\Psi \in \left\{ \Psi \in \mathbf{H}^1(\Omega_{z,\varepsilon}), \operatorname{div} \Psi = 0, \Psi|_{\partial\omega_{z,\varepsilon}} = \mathbf{0}, \Psi|_{\partial\Omega \setminus \overline{O}} = \mathbf{0} \right\}$:

$$\frac{1}{2}\nu \int_{\Omega_{z,\varepsilon}} \mathcal{D}(\mathbf{v}_\varepsilon - \mathbf{V}_\varepsilon) : \mathcal{D}(\Psi) = \langle \varphi, \Psi \rangle_O$$

and then, taking $\Psi = \mathbf{v}_\varepsilon - \mathbf{V}_\varepsilon$,

$$\frac{1}{2}\nu \left\| \mathcal{D}(\widetilde{\mathbf{v}}_\varepsilon - \widetilde{\mathbf{V}}_\varepsilon) \right\|_{0,\Omega}^2 = \langle \varphi, \mathbf{v}_\varepsilon - \mathbf{V}_\varepsilon \rangle_O.$$

Thus, there exists a constant (independent of ε) such that

$$\left\| \mathcal{D}(\widetilde{\mathbf{v}}_\varepsilon - \widetilde{\mathbf{V}}_\varepsilon) \right\|_{0,\Omega}^2 \leq c \|\varphi\|_{-1/2,O} \left\| \widetilde{\mathbf{v}}_\varepsilon - \widetilde{\mathbf{V}}_\varepsilon \right\|_{1,\Omega}.$$

Moreover, since $\mathbf{v}_\varepsilon - \mathbf{V}_\varepsilon = \mathbf{0}$ on $\partial\Omega \setminus \overline{O}$, Korn's inequality (see for example [86, eq. (2.14) page 19]) leads

$$\left\| \widetilde{\mathbf{v}}_\varepsilon - \widetilde{\mathbf{V}}_\varepsilon \right\|_{1,\Omega} \leq c \left\| \mathcal{D}(\widetilde{\mathbf{v}}_\varepsilon - \widetilde{\mathbf{V}}_\varepsilon) \right\|_{0,\Omega}$$

(with a constant c independent of ε). Hence,

$$\begin{aligned} \|\mathbf{v}_\varepsilon - \mathbf{V}_\varepsilon\|_{1,\Omega_{z,\varepsilon}}^2 &= \left\| \widetilde{\mathbf{v}}_\varepsilon - \widetilde{\mathbf{V}}_\varepsilon \right\|_{1,\Omega}^2 \leq c \|\varphi\|_{-1/2,O} \left\| \mathcal{D}(\widetilde{\mathbf{v}}_\varepsilon - \widetilde{\mathbf{V}}_\varepsilon) \right\|_{0,\Omega} \\ &\leq c \|\varphi\|_{-1/2,O} \left\| \widetilde{\mathbf{v}}_\varepsilon - \widetilde{\mathbf{V}}_\varepsilon \right\|_{1,\Omega} \leq c \|\varphi\|_{-1/2,O} \|\mathbf{v}_\varepsilon - \mathbf{V}_\varepsilon\|_{1,\Omega_{z,\varepsilon}}. \end{aligned}$$

Thus,

$$\|\mathbf{v}_\varepsilon - \mathbf{V}_\varepsilon\|_{1,\Omega_{z,\varepsilon}} \leq c \|\varphi\|_{-1/2,O}.$$

Now, let us prove that $\|\mathbf{V}_\varepsilon\|_{1,\Omega_{z,\varepsilon}} \leq c \|\Phi\|_{1/2,\partial\Omega \setminus \overline{O}}$. For a fixed $\varepsilon_0 > 0$, Problem (IV.3.13) is well-posed and admits a unique solution $(\mathbf{V}_{\varepsilon_0}, Q_{\varepsilon_0}) \in \mathbf{H}^1(\Omega_{z,\varepsilon_0}) \times L^2(\Omega_{z,\varepsilon_0})$ and there exists $c > 0$ such that

$$\|\mathbf{V}_{\varepsilon_0}\|_{1,\Omega_{z,\varepsilon_0}} \leq c \|\Phi\|_{1/2,\partial\Omega \setminus \overline{O}}.$$

Let $0 < \varepsilon_1 < \varepsilon_0$ such that $\Omega_{z,\varepsilon_0} \subset \Omega_{z,\varepsilon}$ for all $0 < \varepsilon < \varepsilon_1$. Let $\widetilde{\mathbf{V}}_{\varepsilon_0}$ the extension of $\mathbf{V}_{\varepsilon_0}$ to Ω by 0. The solution $\mathbf{V}_\varepsilon$ of (IV.3.13) can be considered as the solution of the following minimization problem:

$$\min_{\mathbf{V} \in \mathcal{U}} \left\{ \nu |\mathbf{V}|_{1,\Omega_{z,\varepsilon}} \right\},$$

where

$$\mathcal{U} := \left\{ \mathbf{V} \in \mathbf{H}^1(\Omega_{z,\varepsilon}), \operatorname{div} \mathbf{V} = 0 \text{ in } \Omega_{z,\varepsilon}, \mathbf{V} = \mathbf{0} \text{ on } \partial\omega_{z,\varepsilon}, \mathbf{V} = \Phi \text{ on } \partial\Omega \setminus \overline{O} \right\}.$$

Hence, for all $0 < \varepsilon < \varepsilon_1$, we have

$$|\mathbf{V}_\varepsilon|_{1,\Omega_{z,\varepsilon}} \leq c \left| \widetilde{\mathbf{V}}_{\varepsilon_0} \right|_{1,\Omega_{z,\varepsilon}} = c |\mathbf{V}_{\varepsilon_0}|_{1,\Omega_{z,\varepsilon_0}} \leq c \|\mathbf{V}_{\varepsilon_0}\|_{1,\Omega_{z,\varepsilon_0}} \leq c \|\Phi\|_{1/2,\partial\Omega \setminus \overline{O}}.$$

Notice that $\|\mathbf{V}_0\|_{1,\Omega} \leq c \|\Phi\|_{1/2,\partial\Omega \setminus \overline{O}}$. Hence, using Poincaré's inequality,

$$\begin{aligned} \|\mathbf{V}_\varepsilon\|_{0,\Omega_{z,\varepsilon}} &= \left\| \widetilde{\mathbf{V}}_\varepsilon \right\|_{0,\Omega} \leq \left\| \widetilde{\mathbf{V}}_\varepsilon - \mathbf{V}_0 \right\|_{0,\Omega} + \|\mathbf{V}_0\|_{0,\Omega} \leq c \left| \widetilde{\mathbf{V}}_\varepsilon - \mathbf{V}_0 \right|_{1,\Omega} + \|\mathbf{V}_0\|_{0,\Omega} \\ &\leq c \left| \widetilde{\mathbf{V}}_\varepsilon \right|_{1,\Omega} + c \|\mathbf{V}_0\|_{1,\Omega} \leq c |\mathbf{V}_\varepsilon|_{1,\Omega_{z,\varepsilon}} + c \|\mathbf{V}_0\|_{1,\Omega} \leq c \|\Phi\|_{1/2,\partial\Omega \setminus \overline{O}}. \end{aligned}$$

Hence, we have the announced result. $\square$

Remark IV.3.6. In the case $O = \partial\Omega$, we cannot adapt the proof of Lemma IV.3.4. This is due to the fact that, in this case, Korn's inequality used is not valid. Notice that in this case, the solution of the limit problem (for $\varepsilon = 0$) is defined to an additive constant whereas it is not the case for the problems with obstacles. This is the reason why we assume $O \neq \partial\Omega$ and let the case $O = \partial\Omega$ as an open problem.

Next we prove the following lemma which is a direct adaptation of Lemma 7.3 in [69]:

Lemma IV.3.7. Let $\varepsilon > 0$. For $\Psi \in \mathbf{H}^1(\Omega)$ such that $\operatorname{div} \Psi = 0$ in Ω , we consider the solution $(\mathbf{u}_\varepsilon, p_\varepsilon) \in \mathbf{H}^1(\Omega_{z,\varepsilon}) \times L^2(\Omega_{z,\varepsilon})$ of the Stokes boundary value problem

$$\begin{cases} -\nu \Delta \mathbf{u}_\varepsilon + \nabla p_\varepsilon &= \mathbf{0} & \text{in } \Omega_{z,\varepsilon} \\ \operatorname{div} \mathbf{u}_\varepsilon &= 0 & \text{in } \Omega_{z,\varepsilon} \\ \sigma(\mathbf{u}_\varepsilon, p_\varepsilon) \mathbf{n} &= \mathbf{0} & \text{on } O \\ \mathbf{u}_\varepsilon &= \mathbf{0} & \text{on } \partial\Omega \setminus \overline{O} \\ \mathbf{u}_\varepsilon &= \Psi & \text{on } \partial\omega_{z,\varepsilon}. \end{cases}$$

Then there exist a constant $c > 0$ (independent of ε and $\mathbf{g}$) and $\varepsilon_1 > 0$ such that for all $0 < \varepsilon < \varepsilon_1$

$$\begin{aligned} \|\mathbf{u}_\varepsilon\|_{0,\Omega_R^z} &\leq c\varepsilon \|\Psi(z + \varepsilon y)\|_{1/2,\partial\Omega}, & \|\mathbf{u}_\varepsilon\|_{1,\Omega_R^z} &\leq c\varepsilon \|\Psi(z + \varepsilon y)\|_{1/2,\partial\Omega}, \\ \|\mathbf{u}_\varepsilon\|_{0,D_\varepsilon^z} &\leq c\varepsilon \|\Psi(z + \varepsilon y)\|_{1/2,\partial\Omega} & \text{and} & \|\mathbf{u}_\varepsilon\|_{1,D_\varepsilon^z} \leq c\varepsilon^{1/2} \|\Psi(z + \varepsilon y)\|_{1/2,\partial\Omega}. \end{aligned}$$

Proof. Let us consider $(\mathbf{V}_\varepsilon, Q_\varepsilon) \in \mathbf{W}_0^{1,2}(\mathbb{R}^3 \setminus \overline{\omega}) \times L^2(\mathbb{R}^3 \setminus \overline{\omega})$ the solution of the Stokes exterior problem

$$\begin{cases} -\nu \Delta \mathbf{V}_\varepsilon + \nabla Q_\varepsilon &= \mathbf{0} & \text{in } \mathbb{R}^3 \setminus \overline{\omega} \\ \operatorname{div} \mathbf{V}_\varepsilon &= 0 & \text{in } \mathbb{R}^3 \setminus \overline{\omega} \\ \mathbf{V}_\varepsilon &= \frac{1}{\varepsilon} \Psi(z + \varepsilon y) & \text{on } \partial\omega. \end{cases}$$

Let us define $\mathbf{v}_\varepsilon := \varepsilon \mathbf{V}_\varepsilon \left(\frac{x-z}{\varepsilon}\right)$ and $q_\varepsilon := Q_\varepsilon \left(\frac{x-z}{\varepsilon}\right)$. Hence $(\mathbf{w}_\varepsilon := \mathbf{v}_\varepsilon - \mathbf{u}_\varepsilon, s_\varepsilon := q_\varepsilon - p_\varepsilon)$ solves

$$\begin{cases} -\nu \Delta \mathbf{w}_\varepsilon + \nabla s_\varepsilon &= \mathbf{0} & \text{in } \Omega_{z,\varepsilon} \\ \operatorname{div} \mathbf{w}_\varepsilon &= 0 & \text{in } \Omega_{z,\varepsilon} \\ \sigma(\mathbf{w}_\varepsilon, s_\varepsilon) \mathbf{n} &= \sigma(\mathbf{v}_\varepsilon, q_\varepsilon) \mathbf{n} & \text{on } O \\ \mathbf{w}_\varepsilon &= \mathbf{v}_\varepsilon & \text{on } \partial\Omega \setminus \overline{O} \\ \mathbf{w}_\varepsilon &= \mathbf{0} & \text{on } \partial\omega_{z,\varepsilon}. \end{cases}$$

Using Lemma IV.3.4, we know that there exists a constant $c > 0$ and $\varepsilon_1 > 0$ such that for all $0 < \varepsilon < \varepsilon_1$

$$\|\mathbf{w}_\varepsilon\|_{1,\Omega_{z,\varepsilon}} \leq c \left(\|\sigma(\mathbf{v}_\varepsilon, q_\varepsilon) \mathbf{n}\|_{-1/2,O} + \|\mathbf{v}_\varepsilon\|_{1/2,\partial\Omega \setminus \overline{O}} \right). \quad (\text{IV.3.14})$$

We also have

$$\|\mathbf{v}_\varepsilon\|_{1/2,\partial\Omega \setminus \overline{O}} \leq \|\mathbf{v}_\varepsilon\|_{1/2,\partial\Omega} \leq \|\mathbf{v}_\varepsilon\|_{1,\Omega_R^z}. \quad (\text{IV.3.15})$$

Moreover, we have

$$\|\sigma(\mathbf{v}_\varepsilon, q_\varepsilon) \mathbf{n}\|_{-1/2,O} \leq \|\mathbf{v}_\varepsilon\|_{1,\Omega_R^z}. \quad (\text{IV.3.16})$$

Indeed, for all $\varphi \in \mathbf{H}^{1/2}(O)$ and all $\Phi \in \mathbf{H}^1(\Omega_R^z)$ extension of φ such that $\Phi|_{\partial\Omega \setminus \overline{O}} = \mathbf{0}$ and $\Phi|_{\partial(B(0,R))} = \mathbf{0}$, we have

$$\langle \sigma(\mathbf{v}_\varepsilon, q_\varepsilon) \mathbf{n}, \varphi \rangle_{-1/2,1/2,O} = \nu \int_{\Omega_R^z} \mathcal{D}(\mathbf{v}_\varepsilon) : \nabla(\Phi) \leq c \|\mathcal{D}(\mathbf{v}_\varepsilon)\|_{0,\Omega_R^z} \|\Phi\|_{1,\Omega_R^z}$$

and, choosing Φ such that $\|\Phi\|_{1,\Omega_R^z} = \|\varphi\|_{1/2,O}$, we obtain that

$$\|\sigma(\mathbf{v}_\varepsilon, q_\varepsilon) \mathbf{n}\|_{-1/2,O} \leq c \|\mathcal{D}(\mathbf{v}_\varepsilon)\|_{0,\Omega_R^z}.$$

Finally, using the change of variables $x = z + \varepsilon y$ (see (IV.3.3) and (IV.3.4)) and Lemma IV.3.2, we obtain

$$\begin{aligned} \|\mathbf{v}_\varepsilon\|_{0,\Omega_R^z} &\leq \varepsilon^{3/2} \|\varepsilon \mathbf{V}_\varepsilon\|_{0,\frac{\Omega_R^0}{\varepsilon}} \leq c\varepsilon \|\Psi(z + \varepsilon y)\|_{1/2,\partial\omega} \\ |\mathbf{v}_\varepsilon|_{1,\Omega_R^z} &\leq \varepsilon^{1/2} |\varepsilon \mathbf{V}_\varepsilon|_{1,\frac{\Omega_R^0}{\varepsilon}} \leq c\varepsilon \|\Psi(z + \varepsilon y)\|_{1/2,\partial\omega}. \end{aligned} \quad (\text{IV.3.17})$$

Hence, gathering Inequalities (IV.3.14), (IV.3.15), (IV.3.16) and (IV.3.17),

$$\|\mathbf{w}_\varepsilon\|_{1,\Omega_{z,\varepsilon}} \leq c\varepsilon \|\Psi(z + \varepsilon y)\|_{1/2,\partial\omega}.$$

Proceeding as above (using the change of variables $x = z + \varepsilon y$ and Lemma IV.3.2), we also have

$$\|\mathbf{v}_\varepsilon\|_{0,D_\varepsilon^z} \leq c\varepsilon \|\Psi(z + \varepsilon y)\|_{1/2,\partial\omega} \quad \text{and} \quad |\mathbf{v}_\varepsilon|_{1,D_\varepsilon^z} \leq c\varepsilon^{1/2} \|\Psi(z + \varepsilon y)\|_{1/2,\partial\omega}.$$

Therefore, we have

$$\begin{aligned} |\mathbf{u}_\varepsilon|_{1,\Omega_R^z} &\leq |\mathbf{v}_\varepsilon|_{1,\Omega_R^z} + |\mathbf{w}_\varepsilon|_{1,\Omega_R^z} \leq c\varepsilon \|\Psi(z + \varepsilon y)\|_{1/2,\partial\omega} \\ \|\mathbf{u}_\varepsilon\|_{0,\Omega_R^z} &\leq \|\mathbf{v}_\varepsilon\|_{0,\Omega_R^z} + \|\mathbf{w}_\varepsilon\|_{0,\Omega_R^z} \leq c\varepsilon \|\Psi(z + \varepsilon y)\|_{1/2,\partial\omega}, \end{aligned}$$

and similarly

$$\|\mathbf{u}_\varepsilon\|_{0,D_\varepsilon^z} \leq c\varepsilon \|\Psi(z + \varepsilon y)\|_{1/2,\partial\omega} \quad \text{and} \quad |\mathbf{u}_\varepsilon|_{1,D_\varepsilon^z} \leq c\varepsilon^{1/2} \|\Psi(z + \varepsilon y)\|_{1/2,\partial\omega}.$$

□

Proof of Lemma IV.3.3. Lemma IV.3.3 is a direct consequence of the two previous results splitting Problem (IV.3.11) into the two problems intervening in Lemmas IV.3.4 and IV.3.7. □

IV.3.3 Proof of Proposition IV.3.1

In order to simplify the notations, let us define

$$\mathbf{r}_N^\varepsilon(x) := \mathbf{u}_N^\varepsilon(x) - \mathbf{u}_N^0(x) - \mathbf{U}_N\left(\frac{x-z}{\varepsilon}\right) \quad \text{and} \quad s_N^\varepsilon(x) := p_N^\varepsilon(x) - p_N^0(x) - \frac{1}{\varepsilon}P_N\left(\frac{x-z}{\varepsilon}\right),$$

where $(\mathbf{U}_N, P_N) \in \mathbf{W}_0^{1,2}(\mathbb{R}^3 \setminus \bar{\omega}) \times L^2(\mathbb{R}^3 \setminus \bar{\omega})$ solves (IV.3.1) with $\mathfrak{d} = N$. Then the pair $(\mathbf{r}_N^\varepsilon, s_N^\varepsilon)$, which belongs to $\mathbf{H}^1(\Omega_{z,\varepsilon}) \times L^2(\Omega_{z,\varepsilon})$, solves

$$\begin{cases} -\nu \Delta \mathbf{r}_N^\varepsilon + \nabla s_N^\varepsilon &= \mathbf{0} & \text{in } \Omega_{z,\varepsilon} \\ \operatorname{div} \mathbf{r}_N^\varepsilon &= 0 & \text{in } \Omega_{z,\varepsilon} \\ \sigma(\mathbf{r}_N^\varepsilon, s_N^\varepsilon) \mathbf{n} &= -\frac{1}{\varepsilon} \sigma(\mathbf{U}_N, P_N) \left(\frac{x-z}{\varepsilon}\right) \mathbf{n} & \text{on } O \\ \mathbf{r}_N^\varepsilon &= -\mathbf{U}_N\left(\frac{x-z}{\varepsilon}\right) & \text{on } \partial\Omega \setminus \bar{O} \\ \mathbf{r}_N^\varepsilon &= -\mathbf{u}_N^0(x) + \mathbf{u}_N^0(z) & \text{on } \partial\omega_{z,\varepsilon}. \end{cases}$$

Then we know by Lemma IV.3.3 that there exists a constant $c > 0$ independent of ε such that

$$\begin{aligned} \|\mathbf{r}_N^\varepsilon\|_{1,\Omega_{z,\varepsilon}} &\leq c \left(\frac{1}{\varepsilon} \left\| \sigma(\mathbf{U}_N, P_N) \left(\frac{x-z}{\varepsilon}\right) \mathbf{n} \right\|_{-1/2,O} \right. \\ &\quad \left. + \left\| \mathbf{U}_N \left(\frac{x-z}{\varepsilon}\right) \right\|_{1/2,\partial\Omega \setminus \bar{O}} + \varepsilon^{1/2} \left\| -\mathbf{u}_N^0(z + \varepsilon y) + \mathbf{u}_N^0(z) \right\|_{1/2,\partial\omega} \right). \end{aligned} \quad (\text{IV.3.18})$$

Mimicking the proof of (IV.3.16), we have

$$\left\| \sigma(\mathbf{U}_N, P_N) \left(\frac{x-z}{\varepsilon}\right) \mathbf{n} \right\|_{-1/2,O} \leq c \left\| (\nabla \mathbf{U}_N) \left(\frac{x-z}{\varepsilon}\right) \right\|_{0,\Omega_R^z}.$$

Using the change of variables $y = x - z$, we obtain

$$\left\| \sigma(\mathbf{U}_N, P_N) \left(\frac{x-z}{\varepsilon} \right) \mathbf{n} \right\|_{-1/2, O} \leq c \left\| (\nabla \mathbf{U}_N) \left(\frac{y}{\varepsilon} \right) \right\|_{0, \Omega_R^0} = c\varepsilon \left\| \nabla \left(\mathbf{U}_N \left(\frac{y}{\varepsilon} \right) \right) \right\|_{0, \Omega_R^0}.$$

Then, using the change of variables $y = \varepsilon x$ (see (IV.3.4)),

$$\left\| \sigma(\mathbf{U}_N, P_N) \left(\frac{x-z}{\varepsilon} \right) \mathbf{n} \right\|_{-1/2, O} \leq c\varepsilon^{5/2} \|\nabla \mathbf{U}_N\|_{0, \frac{\Omega_R^0}{\varepsilon}}.$$

Thus, Lemma IV.3.2 gives

$$\left\| \sigma(\mathbf{U}_N, P_N) \left(\frac{x-z}{\varepsilon} \right) \mathbf{n} \right\|_{-1/2, O} \leq c\varepsilon^3 \|\mathbf{u}_N^0(z)\|_{1/2, \partial\omega} \leq c\varepsilon^3. \quad (\text{IV.3.19})$$

Proceeding as for $\|\sigma(\mathbf{U}_N, P_N) \left(\frac{x-z}{\varepsilon} \right) \mathbf{n}\|_{-1/2, O}$, we have, using the change of variables $y = \frac{x-z}{\varepsilon}$ (see (IV.3.3) and (IV.3.4)),

$$\begin{aligned} \left\| \mathbf{U}_N \left(\frac{x-z}{\varepsilon} \right) \right\|_{1/2, \partial\Omega \setminus \overline{O}} &\leq c \left(\left\| \mathbf{U}_N \left(\frac{x-z}{\varepsilon} \right) \right\|_{0, \Omega_R^z} + \left\| \nabla (\mathbf{U}_N) \left(\frac{x-z}{\varepsilon} \right) \right\|_{0, \Omega_R^z} \right) \\ &\leq c \left(\varepsilon^{3/2} \|\mathbf{U}_N(x)\|_{0, \frac{\Omega_R^0}{\varepsilon}} + \varepsilon^{5/2} \|\nabla \mathbf{U}_N\|_{0, \frac{\Omega_R^0}{\varepsilon}} \right). \end{aligned}$$

Thus, according to Lemma IV.3.2, we obtain

$$\left\| \mathbf{U}_N \left(\frac{x-z}{\varepsilon} \right) \right\|_{1/2, \partial\Omega \setminus \overline{O}} \leq c\varepsilon \|\mathbf{u}_N^0(z)\|_{1/2, \partial\omega} \leq c\varepsilon. \quad (\text{IV.3.20})$$

Using a Taylor expansion, we obtain $\mathbf{u}_N^0(z + \varepsilon y) = \mathbf{u}_N^0(z) + \varepsilon \nabla \mathbf{u}_N^0(\xi_y) y$ with $\xi_y \in \omega_{z, \varepsilon}$. Since $\nabla \mathbf{u}_N^0$ is uniformly bounded in $\omega_{z, \varepsilon}$, we have

$$\|\mathbf{u}_N^0(z + \varepsilon y) - \mathbf{u}_N^0(z)\|_{1/2, \partial\omega} \leq c\varepsilon. \quad (\text{IV.3.21})$$

Hence, relations (IV.3.18), (IV.3.19), (IV.3.20) and (IV.3.21) implies

$$\left\| \mathbf{u}_N^\varepsilon(x) - \mathbf{u}_N^0(x) - \mathbf{U}_N \left(\frac{x-z}{\varepsilon} \right) \right\|_{1, \Omega_{z, \varepsilon}} \leq c\varepsilon.$$

IV.4 Proof of Theorem IV.2.1

IV.4.1 A preliminary lemma

First we need an estimate of the norm $\|\cdot\|_{1/2, \partial\omega_{z, \varepsilon}}$ of an uniformly bounded function. Here $\|\cdot\|_{1/2, \partial\omega_{z, \varepsilon}}$ has to be seen as the trace norm

$$\|\mathbf{f}\|_{1/2, \partial\omega_{z, \varepsilon}} := \inf \left\{ \|\mathbf{u}\|_{\mathbf{H}^1(\Omega \setminus \overline{\omega_{z, \varepsilon}})}, \mathbf{u} \in \mathbf{H}^1(\Omega \setminus \overline{\omega_{z, \varepsilon}}), \mathbf{u}|_{\partial\omega_{z, \varepsilon}} = \mathbf{f} \right\}.$$

Lemma IV.4.1. *Let $\varepsilon \in (0, 1/2)$. If $\mathbf{u} \in \mathbf{H}^1(\Omega)$ is such that its restriction to $\overline{\omega_{z, 1}}$ is C^1 , then there exists a constant $c > 0$ independent of ε such that*

$$\|\mathbf{u}\|_{1/2, \partial\omega_{z, \varepsilon}} \leq c\varepsilon^{1/2}.$$

Proof. From Theorem B.5.1, there exists a constant $c > 0$ independent of ε such that

$$\|\mathbf{u}\|_{1/2, \partial\omega_{z,\varepsilon}} \leq c\varepsilon^{-1/2} \|\mathbf{u}\|_{\mathbf{L}^2(\partial\omega_{z,\varepsilon})} + c \left(\iint_{\partial\omega_{z,\varepsilon} \times \partial\omega_{z,\varepsilon}} \frac{|\mathbf{u}(x) - \mathbf{u}(y)|^2}{|x - y|^3} ds(x) ds(y) \right)^{1/2}.$$

Since $\mathbf{u}$ is uniformly bounded on $\partial\omega_{z,\varepsilon}$, we use the change of variables $y = z + \varepsilon x$ to prove that there exists a constant $c > 0$ independent of ε such that

$$\|\mathbf{u}\|_{\mathbf{L}^2(\partial\omega_{z,\varepsilon})} \leq c\varepsilon.$$

Moreover, using the changes of variables $x = z + \varepsilon X$ and $y = z + \varepsilon Y$ and the fact that we have $\mathbf{u}(z + \varepsilon X) = \mathbf{u}(z) + \varepsilon \nabla(\mathbf{u})(\xi_X)X$, $\xi_X \in \omega_{z,\varepsilon}$ and $\mathbf{u}(z + \varepsilon Y) = \mathbf{u}(z) + \varepsilon \nabla(\mathbf{u})(\xi_Y)Y$, $\xi_Y \in \omega_{z,\varepsilon}$, then there exists a constant $c > 0$ independent of ε such that

$$\begin{aligned} & \left(\iint_{\partial\omega_{z,\varepsilon} \times \partial\omega_{z,\varepsilon}} \frac{|\mathbf{u}(x) - \mathbf{u}(y)|^2}{|x - y|^3} ds(x) ds(y) \right)^{1/2} \\ &= \left(\iint_{\partial\omega \times \partial\omega} \varepsilon^4 \frac{|\varepsilon (\nabla(\mathbf{u})(\xi_X)X - \nabla(\mathbf{u})(\xi_Y)Y)|^2}{\varepsilon^3 |X - Y|^3} ds(x) ds(y) \right)^{1/2} \leq c\varepsilon^{3/2}. \end{aligned}$$

□

IV.4.2 Splitting the variations of the objective

In order to make a sensibility analysis for the Kohn-Vogelius functional $\mathcal{J}_{KV}$, we recall that

$$\mathcal{J}_{KV}(\Omega_{z,\varepsilon}) = \frac{1}{2}\nu \int_{\Omega_{z,\varepsilon}} |\mathcal{D}(\mathbf{u}_D^\varepsilon) - \mathcal{D}(\mathbf{u}_N^\varepsilon)|^2.$$

Lemma IV.4.2. *We have*

$$\mathcal{J}_{KV}(\Omega_{z,\varepsilon}) - \mathcal{J}_{KV}(\Omega) = A_D + A_N, \quad (\text{IV.4.1})$$

where

$$A_D := \frac{1}{2}\nu \int_{\Omega_{z,\varepsilon}} \mathcal{D}(\mathbf{u}_D^\varepsilon - \mathbf{u}_D^0) : \mathcal{D}(\mathbf{u}_D^\varepsilon - \mathbf{u}_D^0) + \nu \int_{\Omega_{z,\varepsilon}} \mathcal{D}(\mathbf{u}_D^\varepsilon - \mathbf{u}_D^0) : \mathcal{D}(\mathbf{u}_D^0) - \frac{1}{2}\nu \int_{\omega_{z,\varepsilon}} |\mathcal{D}(\mathbf{u}_D^0)|^2$$

and

$$A_N := \int_{\partial\omega_{z,\varepsilon}} [\sigma(\mathbf{u}_N^\varepsilon - \mathbf{u}_N^0, p_N^\varepsilon - p_N^0)\mathbf{n}] \cdot \mathbf{u}_N^0 - \frac{1}{2}\nu \int_{\omega_{z,\varepsilon}} |\mathcal{D}(\mathbf{u}_N^0)|^2.$$

Proof. The functional $\mathcal{J}_{KV}$ can be expanded in

$$\begin{aligned} \mathcal{J}_{KV}(\Omega_{z,\varepsilon}) &= \underbrace{\mathcal{J}_{DD}(\Omega_{z,\varepsilon})}_{:= \frac{1}{2}\nu \int_{\Omega_{z,\varepsilon}} |\mathcal{D}(\mathbf{u}_D^\varepsilon)|^2} + \underbrace{\mathcal{J}_{DN}(\Omega_{z,\varepsilon})}_{:= -\nu \int_{\Omega_{z,\varepsilon}} \mathcal{D}(\mathbf{u}_D^\varepsilon) : \mathcal{D}(\mathbf{u}_N^\varepsilon)} + \underbrace{\mathcal{J}_{NN}(\Omega_{z,\varepsilon})}_{:= \frac{1}{2}\nu \int_{\Omega_{z,\varepsilon}} |\mathcal{D}(\mathbf{u}_N^\varepsilon)|^2}. \end{aligned}$$

Variation of $\mathcal{J}_{DD}$. We have

$$\begin{aligned} \mathcal{J}_{DD}(\Omega_{z,\varepsilon}) - \mathcal{J}_{DD}(\Omega) &= \frac{1}{2}\nu \int_{\Omega_{z,\varepsilon}} |\mathcal{D}(\mathbf{u}_D^\varepsilon)|^2 - \frac{1}{2}\nu \int_{\Omega} |\mathcal{D}(\mathbf{u}_D^0)|^2 \\ &= \frac{1}{2}\nu \int_{\Omega_{z,\varepsilon}} \mathcal{D}(\mathbf{u}_D^\varepsilon - \mathbf{u}_D^0) : \mathcal{D}(\mathbf{u}_D^\varepsilon - \mathbf{u}_D^0) \\ &\quad + \nu \int_{\Omega_{z,\varepsilon}} \mathcal{D}(\mathbf{u}_D^\varepsilon - \mathbf{u}_D^0) : \mathcal{D}(\mathbf{u}_D^0) - \frac{1}{2}\nu \int_{\omega_{z,\varepsilon}} |\mathcal{D}(\mathbf{u}_D^0)|^2. \quad (\text{IV.4.2}) \end{aligned}$$

Variation of $\mathcal{J}_{DN}$. We have, using Green's formula on (IV.1.4) and (IV.3.2),

$$\begin{aligned}\mathcal{J}_{DN}(\Omega_{z,\varepsilon}) - \mathcal{J}_{DN}(\Omega) &= -\nu \int_{\Omega_{z,\varepsilon}} \mathcal{D}(\mathbf{u}_N^\varepsilon) : \mathcal{D}(\mathbf{u}_D^\varepsilon) + \nu \int_{\Omega} \mathcal{D}(\mathbf{u}_N^0) : \mathcal{D}(\mathbf{u}_D^0) \\ &= 2 \int_{\partial\Omega} [\sigma(\mathbf{u}_N^0 - \mathbf{u}_N^\varepsilon, p_N^0 - p_N^\varepsilon) \mathbf{n}] \cdot \mathbf{f}.\end{aligned}\quad (\text{IV.4.3})$$

Variation of $\mathcal{J}_{NN}$. We have

$$\begin{aligned}\mathcal{J}_{NN}(\Omega_{z,\varepsilon}) - \mathcal{J}_{NN}(\Omega) &= \frac{1}{2}\nu \int_{\Omega_{z,\varepsilon}} |\mathcal{D}(\mathbf{u}_N^\varepsilon)|^2 - \frac{1}{2}\nu \int_{\Omega} |\mathcal{D}(\mathbf{u}_N^0)|^2 \\ &= \frac{1}{2}\nu \int_{\Omega_{z,\varepsilon}} \mathcal{D}(\mathbf{u}_N^\varepsilon - \mathbf{u}_N^0) : \mathcal{D}(\mathbf{u}_N^\varepsilon) + \frac{1}{2}\nu \int_{\Omega_{z,\varepsilon}} \mathcal{D}(\mathbf{u}_N^\varepsilon - \mathbf{u}_N^0) : \mathcal{D}(\mathbf{u}_N^0) \\ &\quad - \frac{1}{2}\nu \int_{\omega_{z,\varepsilon}} |\mathcal{D}(\mathbf{u}_N^0)|^2.\end{aligned}$$

Then, using Green's formula and (IV.3.2) in the two first integrals, we obtain

$$\begin{aligned}\mathcal{J}_{NN}(\Omega_{z,\varepsilon}) - \mathcal{J}_{NN}(\Omega) &= \int_{\partial\Omega} [\sigma(\mathbf{u}_N^\varepsilon - \mathbf{u}_N^0, p_N^\varepsilon - p_N^0) \mathbf{n}] \cdot \mathbf{u}_N^\varepsilon + \int_{\partial\Omega} [\sigma(\mathbf{u}_N^\varepsilon - \mathbf{u}_N^0, p_N^\varepsilon - p_N^0) \mathbf{n}] \cdot \mathbf{u}_N^0 \\ &\quad + \int_{\partial\omega_{z,\varepsilon}} [\sigma(\mathbf{u}_N^\varepsilon - \mathbf{u}_N^0, p_N^\varepsilon - p_N^0) \mathbf{n}] \cdot \mathbf{u}_N^0 - \frac{1}{2}\nu \int_{\omega_{z,\varepsilon}} |\mathcal{D}(\mathbf{u}_N^0)|^2.\end{aligned}\quad (\text{IV.4.4})$$

Variation of $\mathcal{J}_{KV}$. In order to obtain the variation $\mathcal{J}_{KV}$, we will sum the variations of $\mathcal{J}_{DD}$, $\mathcal{J}_{DN}$ and $\mathcal{J}_{NN}$. First, we remark that $\mathbf{f} - \mathbf{u}_N^0$ and $\mathbf{f} - \mathbf{u}_N^\varepsilon$ belong to $\mathbf{H}_{00}^{1/2}(O)$ since they belong to $\mathbf{H}^{1/2}(\partial\Omega)$ and vanish on $\partial\Omega \setminus \overline{O}$. Then

$$\begin{aligned}&2 \int_{\partial\Omega} [\sigma(\mathbf{u}_N^\varepsilon - \mathbf{u}_N^0, p_N^\varepsilon - p_N^0) \mathbf{n}] \cdot \mathbf{f} \\ &\quad - \int_{\partial\Omega} [\sigma(\mathbf{u}_N^\varepsilon - \mathbf{u}_N^0, p_N^\varepsilon - p_N^0) \mathbf{n}] \cdot \mathbf{u}_N^0 - \int_{\partial\omega_{z,\varepsilon}} [\sigma(\mathbf{u}_N^\varepsilon - \mathbf{u}_N^0, p_N^\varepsilon - p_N^0) \mathbf{n}] \cdot \mathbf{u}_N^\varepsilon \\ &= \int_{\partial\Omega} [\sigma(\mathbf{u}_N^\varepsilon - \mathbf{u}_N^0, p_N^\varepsilon - p_N^0) \mathbf{n}] \cdot (\mathbf{f} - \mathbf{u}_N^0) + \int_{\partial\Omega} [\sigma(\mathbf{u}_N^\varepsilon - \mathbf{u}_N^0, p_N^\varepsilon - p_N^0) \mathbf{n}] \cdot (\mathbf{f} - \mathbf{u}_N^\varepsilon) \\ &= \langle \sigma(\mathbf{u}_N^\varepsilon - \mathbf{u}_N^0, p_N^\varepsilon - p_N^0) \mathbf{n}, \mathbf{f} - \mathbf{u}_N^0 \rangle_O + \langle \sigma(\mathbf{u}_N^\varepsilon - \mathbf{u}_N^0, p_N^\varepsilon - p_N^0) \mathbf{n}, \mathbf{f} - \mathbf{u}_N^\varepsilon \rangle_O \\ &\quad + \int_{\partial\Omega \setminus \overline{O}} [\sigma(\mathbf{u}_N^\varepsilon - \mathbf{u}_N^0, p_N^\varepsilon - p_N^0) \mathbf{n}] \cdot (\mathbf{f} - \mathbf{u}_N^0) + \int_{\partial\Omega \setminus \overline{O}} [\sigma(\mathbf{u}_N^\varepsilon - \mathbf{u}_N^0, p_N^\varepsilon - p_N^0) \mathbf{n}] \cdot (\mathbf{f} - \mathbf{u}_N^\varepsilon).\end{aligned}$$

Since $\sigma(\mathbf{u}_N^\varepsilon, p_N^\varepsilon) \mathbf{n} = \sigma(\mathbf{u}_N^0, p_N^0) \mathbf{n} = \mathbf{g}$ on O and $\mathbf{u}_N^0 = \mathbf{u}_N^\varepsilon = \mathbf{f}$ on $\partial\Omega \setminus \overline{O}$, we prove that this expression is null. Finally, we obtain the expansion (IV.4.1) using the variations of $\mathcal{J}_{DD}$, $\mathcal{J}_{DN}$ and $\mathcal{J}_{NN}$ (see Equalities (IV.4.2), (IV.4.3) and (IV.4.4)). $\square$

IV.4.3 Asymptotic expansion of A_N

We know using elliptic regularity that $\nabla \mathbf{u}_N^0$ is uniformly bounded on $\omega_{z,\varepsilon}$. Thus

$$-\frac{1}{2}\nu \int_{\omega_{z,\varepsilon}} |\mathcal{D}(\mathbf{u}_N^0)|^2 \leq c \int_{\omega} \varepsilon^3 = O(\varepsilon^3).\quad (\text{IV.4.5})$$

Let us recall that

$$\mathbf{r}_N^\varepsilon(x) := \mathbf{u}_N^\varepsilon(x) - \mathbf{u}_N^0(x) - \mathbf{U}_N\left(\frac{x-z}{\varepsilon}\right) \quad \text{and} \quad s_N^\varepsilon(x) := p_N^\varepsilon(x) - p_N^0(x) - \frac{1}{\varepsilon}P_N\left(\frac{x-z}{\varepsilon}\right),$$

where $(U_N, P_N) \in \mathbf{W}_0^{1,2}(\mathbb{R}^3 \setminus \bar{\omega}) \times L^2(\mathbb{R}^3 \setminus \bar{\omega})$ solves (IV.3.1) with $\mathfrak{h} = N$. Then the following equality holds

$$\int_{\partial\omega_{z,\varepsilon}} [\sigma(\mathbf{u}_N^\varepsilon - \mathbf{u}_N^0, p_N^\varepsilon - p_N^0)\mathbf{n}] \cdot \mathbf{u}_N^0 = \int_{\partial\omega_{z,\varepsilon}} [\sigma(\mathbf{r}_N^\varepsilon, s_N^\varepsilon)\mathbf{n}] \cdot \mathbf{u}_N^0 + \frac{1}{\varepsilon} \int_{\partial\omega_{z,\varepsilon}} \left[\sigma(U_N, P_N) \left(\frac{x-z}{\varepsilon} \right) \mathbf{n} \right] \cdot \mathbf{u}_N^0. \quad (\text{IV.4.6})$$

Let us first focus on the first term in the right-hand side of Equality (IV.4.6). We recall that for all $\boldsymbol{\varphi} \in \mathbf{H}^{1/2}(\partial\omega_{z,\varepsilon})$ and all $\boldsymbol{\Phi} \in \mathbf{H}^1(\Omega_{z,\varepsilon})$ extension of $\boldsymbol{\varphi}$ such that $\boldsymbol{\Phi}|_{\partial\Omega} = \mathbf{0}$, we have

$$\langle \sigma(\mathbf{r}_N^\varepsilon, s_N^\varepsilon)\mathbf{n}, \boldsymbol{\varphi} \rangle_{-1/2, 1/2, \partial\omega_{z,\varepsilon}} = \frac{1}{2} \nu \int_{\Omega_{z,\varepsilon}} \mathcal{D}(\mathbf{r}_N^\varepsilon) : \mathcal{D}(\boldsymbol{\Phi}) \leq c \|\mathcal{D}(\mathbf{r}_N^\varepsilon)\|_{0, \Omega_{z,\varepsilon}} \|\boldsymbol{\Phi}\|_{1, \Omega_{z,\varepsilon}}.$$

Hence, choosing $\boldsymbol{\Phi}$ such that $\|\boldsymbol{\Phi}\|_{1, \Omega_{z,\varepsilon}} = \|\boldsymbol{\varphi}\|_{1, \partial\omega_{z,\varepsilon}}$, we obtain that

$$\|\sigma(\mathbf{r}_N^\varepsilon, s_N^\varepsilon)\mathbf{n}\|_{-1/2, \partial\omega_{z,\varepsilon}} \leq c \|\mathcal{D}(\mathbf{r}_N^\varepsilon)\|_{0, \Omega_{z,\varepsilon}}. \quad (\text{IV.4.7})$$

Therefore, using the explicit upper bound of $\|\mathbf{u}_N^0\|_{1/2, \partial\omega_{z,\varepsilon}}$ given by Lemma IV.4.1, we have

$$\left| \int_{\partial\omega_{z,\varepsilon}} [\sigma(\mathbf{r}_N^\varepsilon, s_N^\varepsilon)\mathbf{n}] \cdot \mathbf{u}_N^0 \right| \leq \|\sigma(\mathbf{r}_N^\varepsilon, s_N^\varepsilon)\mathbf{n}\|_{-1/2, \partial\omega_{z,\varepsilon}} \|\mathbf{u}_N^0\|_{1/2, \partial\omega_{z,\varepsilon}} \leq c \varepsilon^{1/2} \|\mathbf{r}_N^\varepsilon\|_{1, \Omega_{z,\varepsilon}}.$$

Then, using the explicit upper bound of $\|\mathbf{r}_N^\varepsilon\|_{1, \Omega_{z,\varepsilon}}$ given by Proposition IV.3.1, we obtain

$$\left| \int_{\partial\omega_{z,\varepsilon}} [\sigma(\mathbf{r}_N^\varepsilon, s_N^\varepsilon)\mathbf{n}] \cdot \mathbf{u}_N^0 \right| \leq c \varepsilon^{3/2} = O(\varepsilon^{3/2}). \quad (\text{IV.4.8})$$

Let us now focus on the second term in the right-hand side of (IV.4.6). Using the change of variables $x = z + \varepsilon y$, we have

$$\int_{\partial\omega_{z,\varepsilon}} \left[[\sigma(U_N, P_N)] \left(\frac{x-z}{\varepsilon} \right) \mathbf{n} \right] \cdot \mathbf{u}_N^0 \, ds(x) = \varepsilon^2 \int_{\partial\omega} [\sigma(U_N, P_N)(y)\mathbf{n}] \cdot \mathbf{u}_N^0(z + \varepsilon y) \, ds(y).$$

Since $\mathbf{u}_N^0(z + \varepsilon y) = \mathbf{u}_N^0(z) + \varepsilon \nabla(\mathbf{u}_N^0)(\xi_y)y$, $\xi_y \in \omega_{z,\varepsilon}$ and since $\mathcal{D}(\mathbf{u}_N^0)$ is uniformly bounded in $\omega_{z,\varepsilon}$, we have

$$\varepsilon^2 \int_{\partial\omega} [\sigma(U_N, P_N)(y)\mathbf{n}] \cdot \mathbf{u}_N^0(z + \varepsilon y) = \varepsilon^2 \left(\int_{\partial\omega} \sigma(U_N, P_N)(y)\mathbf{n} \right) \cdot \mathbf{u}_N^0(z) + O(\varepsilon^3).$$

Taking into account that $\mathbf{n}$ is oriented inside ω , we use the jump condition of the single layer potential (see [77, Lemma 5.6.5]) to obtain that

$$-\sigma(U_N, P_N)\mathbf{n} + \sigma(\mathbf{W}, Q)\mathbf{n} = -\boldsymbol{\eta}_N,$$

where $\boldsymbol{\eta}_N \in \mathbf{H}^{-1/2}(\partial\omega)/\mathbb{R}\mathbf{n}$ is defined by (IV.2.3) and where $(\mathbf{W}, Q) \in \mathbf{H}^1(\omega) \times L^2(\omega)$ solves

$$\begin{cases} -\nu \Delta \mathbf{W} + \nabla Q &= \mathbf{0} & \text{in } \omega \\ \operatorname{div} \mathbf{W} &= 0 & \text{in } \omega \\ \mathbf{W} &= \mathbf{U}_N & \text{on } \partial\omega. \end{cases}$$

Since $-\operatorname{div}(\sigma(\mathbf{W}, Q)) = -\nu \Delta \mathbf{W} + \nabla Q = \mathbf{0}$ in ω , we have

$$\int_{\partial\omega} \sigma(\mathbf{W}, Q)\mathbf{n} = 0$$

and then

$$\int_{\partial\omega_{z,\varepsilon}} \left[\sigma(\mathbf{U}_N, P_N) \right] \left(\frac{x-z}{\varepsilon} \right) \mathbf{n} \cdot \mathbf{u}_N^0 = \varepsilon^2 \left(\int_{\partial\omega} \boldsymbol{\eta}_N \right) \cdot \mathbf{u}_N^0(z) + O(\varepsilon^3). \quad (\text{IV.4.9})$$

Gathering (IV.4.5), (IV.4.8) and (IV.4.9), we obtain

$$A_N = \varepsilon \left(\int_{\partial\omega} \boldsymbol{\eta}_N \right) \mathbf{u}_N^0(z) + o(\varepsilon). \quad (\text{IV.4.10})$$

IV.4.4 Asymptotic expansion of A_D

We recall that

$$A_D = \frac{1}{2} \nu \int_{\Omega_{z,\varepsilon}} \mathcal{D}(\mathbf{u}_D^\varepsilon - \mathbf{u}_D^0) : \mathcal{D}(\mathbf{u}_D^\varepsilon - \mathbf{u}_D^0) + \nu \int_{\Omega_{z,\varepsilon}} \mathcal{D}(\mathbf{u}_D^\varepsilon - \mathbf{u}_D^0) : \mathcal{D}(\mathbf{u}_D^0) - \frac{1}{2} \nu \int_{\omega_{z,\varepsilon}} |\mathcal{D}(\mathbf{u}_D^0)|^2.$$

Proceeding as in the previous section IV.4.3, we prove that

$$-\frac{1}{2} \nu \int_{\omega_{z,\varepsilon}} |\mathcal{D}(\mathbf{u}_D^0)|^2 = o(\varepsilon).$$

Moreover, using Green's formula, we have

$$\begin{aligned} \nu \int_{\Omega_{z,\varepsilon}} \mathcal{D}(\mathbf{u}_D^\varepsilon - \mathbf{u}_D^0) : \mathcal{D}(\mathbf{u}_D^0) &= 2 \int_{\partial\omega_{z,\varepsilon}} (\sigma(\mathbf{u}_D^0, p_D^0) \mathbf{n}) \cdot (\mathbf{u}_D^\varepsilon - \mathbf{u}_D^0) \\ &= -2 \int_{\partial\omega_{z,\varepsilon}} (\sigma(\mathbf{u}_D^0, p_D^0) \mathbf{n}) \cdot \mathbf{u}_D^0 = -\nu \int_{\omega_{z,\varepsilon}} |\mathcal{D}(\mathbf{u}_D^0)|^2 = o(\varepsilon). \end{aligned}$$

Now, let us study $\frac{1}{2} \nu \int_{\Omega_{z,\varepsilon}} \mathcal{D}(\mathbf{u}_D^\varepsilon - \mathbf{u}_D^0) : \mathcal{D}(\mathbf{u}_D^\varepsilon - \mathbf{u}_D^0)$. We define

$$\mathbf{r}_D^\varepsilon(x) := \mathbf{u}_D^\varepsilon(x) - \mathbf{u}_D^0(x) - \mathbf{U}_D \left(\frac{x-z}{\varepsilon} \right) \quad \text{and} \quad s_D^\varepsilon(x) := p_D^\varepsilon(x) - p_D^0(x) - \frac{1}{\varepsilon} P_D \left(\frac{x-z}{\varepsilon} \right),$$

where $(\mathbf{U}_D, P_D) \in \mathbf{W}_0^{1,2}(\mathbb{R}^3 \setminus \bar{\omega}) \times L^2(\mathbb{R}^3 \setminus \bar{\omega})$ solves (IV.3.1) with $\mathfrak{h} = D$. Using Green's formula

$$\begin{aligned} \nu \int_{\Omega_{z,\varepsilon}} |\mathcal{D}(\mathbf{u}_D^\varepsilon - \mathbf{u}_D^0)|^2 &= 2 \int_{\partial\omega_{z,\varepsilon}} [\sigma(\mathbf{u}_D^\varepsilon - \mathbf{u}_D^0, p_D^\varepsilon - p_D^0) \mathbf{n}] \cdot (\mathbf{u}_D^\varepsilon - \mathbf{u}_D^0) \\ &= -2 \int_{\partial\omega_{z,\varepsilon}} [\sigma(\mathbf{r}_D^\varepsilon, s_D^\varepsilon) \mathbf{n}] \cdot \mathbf{u}_D^0 - \frac{2}{\varepsilon} \int_{\partial\omega_{z,\varepsilon}} \left[\sigma(\mathbf{U}_D, P_D) \left(\frac{x-z}{\varepsilon} \right) \mathbf{n} \right] \cdot \mathbf{u}_D^0. \end{aligned}$$

Proceeding as in the previous section IV.4.3 (see Inequality (IV.4.8)), we use an inequality similar to (IV.4.7), the asymptotic expansion of $\mathbf{u}_D^\varepsilon$ given by Proposition IV.3.1 and Lemma IV.4.1 to obtain

$$\left| \int_{\partial\omega_{z,\varepsilon}} [\sigma(\mathbf{r}_D^\varepsilon, s_D^\varepsilon) \mathbf{n}] \cdot \mathbf{u}_D^0 \right| \leq c \varepsilon^{1/2} \|\mathbf{r}_D^\varepsilon\|_{1,\Omega_{z,\varepsilon}} = O(\varepsilon^{3/2}).$$

Moreover, using the change of variables $x = z + \varepsilon y$, we have

$$\int_{\partial\omega_{z,\varepsilon}} \left[\sigma(\mathbf{U}_D, P_D) \left(\frac{x-z}{\varepsilon} \right) \mathbf{n} \right] \cdot \mathbf{u}_D^0 = \varepsilon^2 \int_{\partial\omega} [\sigma(\mathbf{U}_D, P_D)(y) \mathbf{n}] \cdot \mathbf{u}_D^0(z + \varepsilon y).$$

Since $\mathbf{u}_D^0(z + \varepsilon y) = \mathbf{u}_D^0(z) + \varepsilon \nabla \mathbf{u}_D^0(\xi_y)y$, $\xi_y \in \omega_{z,\varepsilon}$ and since $\nabla \mathbf{u}_D^0$ is uniformly bounded in $\omega_{z,\varepsilon}$, we have

$$\varepsilon^2 \int_{\partial\omega} [\sigma(\mathbf{U}_D, P_D)(y) \mathbf{n}] \cdot \mathbf{u}_D^0(z + \varepsilon y) ds(y) = \varepsilon^2 \left(\int_{\partial\omega} \sigma(\mathbf{U}_D, P_D)(y) \mathbf{n} \right) \mathbf{u}_D^0(z) + O(\varepsilon^3).$$

Taking into account that $\mathbf{n}$ is oriented inside ω , we use the jump condition of the single layer potential (see [77, Lemma 5.6.5]) to obtain that

$$-\sigma(\mathbf{U}_D, P_D) + \sigma(\mathbf{W}_D, Q_D) = -\boldsymbol{\eta}_D,$$

where $\boldsymbol{\eta}_D \in \mathbf{H}^{-1/2}(\partial\omega)/\mathbb{R}\mathbf{n}$ is defined by (IV.2.2) and where $(\mathbf{W}_D, Q_D) \in \mathbf{H}^1(\omega) \times L^2(\omega)$ solves

$$\begin{cases} -\nu \Delta \mathbf{W}_D + \nabla Q_D &= \mathbf{0} & \text{in } \omega \\ \operatorname{div} \mathbf{W}_D &= 0 & \text{in } \omega \\ \mathbf{W}_D &= \mathbf{U}_D & \text{on } \partial\omega. \end{cases}$$

Since $\operatorname{div}(\sigma(\mathbf{W}_D, Q_D)) = -\nu \Delta \mathbf{W}_D + \nabla Q_D = 0$ in ω , we have

$$\int_{\partial\omega} \sigma(\mathbf{W}_D, Q_D) \mathbf{n} = 0$$

and then

$$\int_{\partial\omega_{z,\varepsilon}} \left[\sigma(\mathbf{U}_D, P_D) \left(\frac{x-z}{\varepsilon} \right) \mathbf{n} \right] \cdot \mathbf{u}_D^0 = \varepsilon^2 \left(\int_{\partial\omega} \boldsymbol{\eta}_D \right) \mathbf{u}_D^0(z) + O(\varepsilon^3).$$

Hence, we obtain

$$\nu \int_{\Omega_{z,\varepsilon}} |\mathcal{D}(\mathbf{u}_D^\varepsilon - \mathbf{u}_D^0)|^2 = -2\varepsilon \left(\int_{\partial\omega} \boldsymbol{\eta}_D \right) \mathbf{u}_D^0(z) + o(\varepsilon).$$

Therefore

$$A_D = -\varepsilon \left(\int_{\partial\omega} \boldsymbol{\eta}_D \right) \mathbf{u}_D^0(z) + o(\varepsilon). \quad (\text{IV.4.11})$$

IV.4.5 Conclusion of the proof: asymptotic expansion of $\mathcal{J}_{KN}$

Using (IV.4.1), (IV.4.10) and (IV.4.11), we conclude the proof of Theorem IV.2.1:

$$\mathcal{J}_{KV}(\Omega_{z,\varepsilon}) - \mathcal{J}_{KV}(\Omega) = -\varepsilon \left(\int_{\partial\omega} \boldsymbol{\eta}_D \right) \mathbf{u}_D^0(z) + \varepsilon \left(\int_{\partial\omega} \boldsymbol{\eta}_N \right) \mathbf{u}_N^0(z) + o(\varepsilon).$$

Chapitre V

Cas de petites inclusions : détection numérique à l'aide de la dérivée topologique

Dans ce chapitre, nous reconstruisons numériquement un (ou des) petit(s) objet(s) immergé(s) dans un fluide stationnaire lorsque le mouvement du fluide est supposé régi par les équations de Stokes incompressibles. Nous utilisons la dérivée topologique de la fonctionnelle de Kohn-Vogelius présentée dans le chapitre précédent (voir Corollaire IV.2.2) pour minimiser celle-ci.

L'utilisation de la dérivée topologique a pour objectif de fournir le nombre d'inclusions ainsi que leur emplacement approximatif. En utilisant un algorithme de type gradient topologique, nous détectons ici de petits objets immergés dans un fluide à partir d'une mesure effectuée sur une partie du bord extérieur. Des résultats satisfaisants sont obtenus mais nous soulignons également les limites de cette méthode dans le cadre de notre étude. En particulier, il s'avère très difficile de détecter des inclusions éloignées du domaine sur lequel nous effectuons la mesure.

Ce chapitre se décompose en cinq parties. Après avoir exposé le cadre des simulations numériques que nous réalisons, nous effectuons de premiers tests montrant que la reconstruction est efficace. Nous détectons des objets éloignés ou proches les uns des autres et également des objets possédant des angles (à savoir des cubes). Nous explorons ensuite l'efficacité et les limites de cette méthode. Nous commençons par étudier l'influence de la distance de l'objet avec le domaine de mesure. Il apparaît alors qu'il est très difficile de détecter des objets éloignés du domaine sur lequel les mesures sont effectuées. Nous explorons ensuite l'influence de la taille et la forme de l'objet sur la détection de celui-ci. Les simulations montrent que si la taille caractéristique de l'obstacle devient trop importante, la méthode devient inefficace. Ceci est raisonnable car l'étude théorique du chapitre précédent est basée sur cette hypothèse de petitesse des inclusions. De plus, la forme des obstacles semble influencer la détection : des domaines avec des singularités géométriques semblent être plus facilement détectés. Pour terminer, l'influence du domaine de mesure O est également testé et nous montrons que, naturellement, si le domaine de mesure est petit, la détection n'est pas efficace compte tenu du fait qu'il est difficile de reconstruire des objets éloignés du domaine de mesure.

V.1 Framework of the numerical simulations

The use of the topological derivative aims to give us the number of inclusions and their rough location. Then it can provide initial shapes for an optimization method based on the boundary variation method for which we have to know the number of connected objects we have to reconstruct.

Let Ω be a bounded Lipschitz open set of $\mathbb{R}^3$ containing a Newtonian and incompressible fluid with coefficient of kinematic viscosity $\nu > 0$. Let $\omega_\varepsilon \subset\subset \Omega$ a small bounded Lipschitz domain such that $\Omega \setminus \overline{\omega_\varepsilon}$ is connected (ω_ε can have several connected components). Let $\mathbf{f} \in \mathbf{H}^{1/2}(\partial\Omega)$ such that $\mathbf{f} \neq \mathbf{0}$

satisfying the compatibility condition

$$\int_{\partial\Omega} \mathbf{f} \cdot \mathbf{n} = 0. \quad (\text{V.1.1})$$

In order to determine the location of the objects, we make a measurement $\mathbf{g} \in \mathbf{H}^{-1/2}(O)$ on a part O of the exterior boundary $\partial\Omega$ with $O \subsetneq \partial\Omega$. We propose a resolution of the inverse problem of reconstructing ω_ε based on the minimization of the following Kohn-Vogelius functional

$$\mathcal{J}_{KV}(\Omega \setminus \overline{\omega_\varepsilon}) := \frac{1}{2} \int_{\Omega \setminus \overline{\omega_\varepsilon}} \nu |\mathcal{D}(\mathbf{u}_D^\varepsilon) - \mathcal{D}(\mathbf{u}_N^\varepsilon)|^2,$$

where $\mathcal{D}(\mathbf{u}) := (\nabla \mathbf{u} + {}^t \nabla \mathbf{u})$. Then, we consider two problems:

$$(\mathcal{P}_D^\varepsilon) \left\{ \begin{array}{l} \text{Find } (\mathbf{u}_D^\varepsilon, p_D^\varepsilon) \in \mathbf{H}^1(\Omega \setminus \overline{\omega_\varepsilon}) \times L_0^2(\Omega \setminus \overline{\omega_\varepsilon}) \text{ such that} \\ -\nu \Delta \mathbf{u}_D^\varepsilon + \nabla p_D^\varepsilon = \mathbf{0} \quad \text{in } \Omega \setminus \overline{\omega_\varepsilon} \\ \operatorname{div} \mathbf{u}_D^\varepsilon = 0 \quad \text{in } \Omega \setminus \overline{\omega_\varepsilon} \\ \mathbf{u}_D^\varepsilon = \mathbf{f} \quad \text{on } \partial\Omega \\ \mathbf{u}_D^\varepsilon = \mathbf{0} \quad \text{on } \partial\omega_\varepsilon \end{array} \right. \quad (\text{V.1.2})$$

and

$$(\mathcal{P}_N^\varepsilon) \left\{ \begin{array}{l} \text{Find } (\mathbf{u}_N^\varepsilon, p_N^\varepsilon) \in \mathbf{H}^1(\Omega \setminus \overline{\omega_\varepsilon}) \times L^2(\Omega \setminus \overline{\omega_\varepsilon}) \text{ such that} \\ -\nu \Delta \mathbf{u}_N^\varepsilon + \nabla p_N^\varepsilon = \mathbf{0} \quad \text{in } \Omega \setminus \overline{\omega_\varepsilon} \\ \operatorname{div} \mathbf{u}_N^\varepsilon = 0 \quad \text{in } \Omega \setminus \overline{\omega_\varepsilon} \\ \sigma(\mathbf{u}_N^\varepsilon, p_N^\varepsilon) \mathbf{n} = \mathbf{g} \quad \text{on } O \\ \mathbf{u}_N^\varepsilon = \mathbf{f} \quad \text{on } \partial\Omega \setminus \overline{O} \\ \mathbf{u}_N^\varepsilon = \mathbf{0} \quad \text{on } \partial\omega_\varepsilon, \end{array} \right. \quad (\text{V.1.3})$$

where $\sigma(\mathbf{u}, p)$ represents the stress tensor defined by

$$\sigma(\mathbf{u}, p) := \nu (\nabla \mathbf{u} + {}^t \nabla \mathbf{u}) - p \mathbf{I}.$$

The numerical simulations presented here are made using the finite elements library FREEFEM++ (see [70]). We use a $P1b$ - $P1$ finite elements discretization to solve the Stokes equations (V.1.2) and (V.1.3). The framework is the following: we assume the kinematic viscosity ν is equal to 1 (we obtain exactly the same results when we work with the water kinematic viscosity $\nu = 10^{-6}$), the exterior boundary is assumed to be the parallelepiped rectangle $[-0.5, 0.5] \times [-0.25, 0.25] \times [-0.25, 0.25]$. Except when mentioned, the measurement is assumed to be made on all the faces except on the one given by $x = -0.5$. We consider the exterior Dirichlet boundary condition

$$\mathbf{f} := \begin{pmatrix} -(x^2 + xy + 4 \cos(3x) + 8y + 2xz + \sin(z^2)) \\ -(-\frac{1}{2}y^2 - 2xy + x^2 + 12y \sin(3x) + y \cos(3z) + z^3) \\ -(x^2 + xy + y^3 - z^2 - \frac{1}{3} \sin(3z) + \cos(xy)) \end{pmatrix}.$$

Notice that $\mathbf{f}$ is such that the compatibility condition (V.1.1) is satisfied (since $\operatorname{div} \mathbf{f} = 0$). In order to have a suitable pair (*measure* $\mathbf{g}$, *domain* ω^*), we use a synthetic data: we fix a shape ω^* (more precisely a finite number of obstacles $\omega_1^*, \dots, \omega_m^*$), solve the Stokes problem (IV.1.3) in $\Omega \setminus \overline{\omega^*}$ using another finite elements method (here a $P2$ - $P1$ finite elements discretization) and extract the measurement $\mathbf{g}$ by computing $\sigma(\mathbf{u}, p) \mathbf{n}$ on O .

Remark V.1.1. *The fact that the results with $\nu = 10^{-6}$ are the same than those obtained with $\nu = 1$ comes to the fact that the topological derivative of $\mathcal{J}_{KV}$ depends only on the velocity and not on the pressure. Then, since*

$$\left\{ \begin{array}{l} -\nu \Delta \mathbf{u} + \nabla p = \mathbf{0} \quad \text{in } \Omega \setminus \overline{\omega} \\ \operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega \setminus \overline{\omega} \\ \mathbf{u} = \mathbf{f} \quad \text{on } \partial\Omega \\ \mathbf{u} = \mathbf{0} \quad \text{on } \partial\omega \end{array} \right. \iff \left\{ \begin{array}{l} -\Delta \mathbf{u} + \nabla \left(\frac{p}{\nu}\right) = \mathbf{0} \quad \text{in } \Omega \setminus \overline{\omega} \\ \operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega \setminus \overline{\omega} \\ \mathbf{u} = \mathbf{f} \quad \text{on } \partial\Omega \\ \mathbf{u} = \mathbf{0} \quad \text{on } \partial\omega \end{array} \right.$$

(idem for the mixed problem), the velocity does not change and so does the topological derivative.

In the practical simulations that we present, we add spherical objects. Indeed, in this very particular case, we have an explicit formula for the topological gradient of the functional $\mathcal{J}_{KV}$ (see Corollary IV.2.2) given by

$$\delta\mathcal{J}_{KV}(P) = 6\pi\nu\varepsilon \left(|\mathbf{u}_D^0(z)|^2 - |\mathbf{u}_N^0(z)|^2 \right), \quad (\text{V.1.4})$$

where z is the coordinates of P and where $\mathbf{u}_D^0 \in \mathbf{H}^1(\Omega)$ and $\mathbf{u}_N^0 \in \mathbf{H}^1(\Omega)$ solve respectively Problems (V.1.2) and (V.1.3) with $\omega_\varepsilon = \emptyset$. In order to determine the radius of these spheres, we use a thresholding method. For an iteration k , it consists in determining the minimum argument P^* of the topological gradient $\delta\mathcal{J}_{KV}$ in $\Omega \setminus \left(\bigcup_{j=1}^k \overline{\omega_j} \right)$ and in defining the set $\mathcal{P}$ of the points $P \in \Omega \setminus \left(\bigcup_{j=1}^k \overline{\omega_j} \right)$ such that

$$\delta\mathcal{J}_{KV}(P) = \delta\mathcal{J}_{KV}(P^*) + 0.25 * |\delta\mathcal{J}_{KV}(P^*)|.$$

Then we fix a minimum radius $r_{\min} := 0.01$ and we define the radius of the k^{th} sphere by

$$r_k := \max \left(r_{\min}, \min_{P \in \mathcal{P}} (|x_P - x_{P^*}|, |y_P - y_{P^*}|, |z_P - z_{P^*}|) \right). \quad (\text{V.1.5})$$

We use the classical topological gradient algorithm (see for example [16, 40, 66, 69]) that we recall here for reader's convenience:

Algorithm 2

1. fix an initial shape $\omega_0 = \emptyset$, a maximum number of iterations M and set $i = 1$ and $k = 0$,
2. solve Problems (V.1.2) and (V.1.3) in $\Omega \setminus \left(\bigcup_{j=0}^k \overline{\omega_j} \right)$,
3. compute the topological gradient $\delta\mathcal{J}_{KV}$ in $\Omega \setminus \left(\bigcup_{j=0}^k \overline{\omega_j} \right)$ using Formula (V.1.4),
4. seek $P_{k+1}^* := \operatorname{argmin} \left(\delta\mathcal{J}_{KV}(P), P \in \Omega \setminus \left(\bigcup_{j=0}^k \overline{\omega_j} \right) \right)$,
5. if $\|P_{k+1}^* - P_{j_0}\| < r_{k+1} + r_{j_0} + 0.01$ for $j_0 \in \{1, \dots, k\}$, where r_{j_0} is the radius of ω_{j_0} and r_{k+1} is defined by (V.1.5), then $r_{j_0} = 1.1 * r_{j_0}$, get back to the step 2. and $i \leftarrow i + 1$ while $i \leq M$,
6. set $\omega_{k+1} = B(P_{k+1}^*, r_{k+1})$, where r_{k+1} is defined by (V.1.5),
7. while $i \leq M$, get back to the step 2, $i \leftarrow i + 1$ and $k \leftarrow k + 1$.

We add to this algorithm a stop test (in addition of the maximum number of iterations). In every iteration, we compute the functional $\mathcal{J}_{KV}$. This non-negative functional has to decrease at each iteration. Thus, we stop our implementation when it is not the case, *i.e.* when

$$\mathcal{J}_{KV} \left(\Omega \setminus \left(\bigcup_{j=0}^{k+1} \overline{\omega_j} \right) \right) > \mathcal{J}_{KV} \left(\Omega \setminus \left(\bigcup_{j=0}^k \overline{\omega_j} \right) \right).$$

Notice that with this algorithm, we add only one object at each iteration. This method can be slower than the one proposed by Carpio *et al.* in [34]: by seeking local minima of $\delta\mathcal{J}_{KV}$ and not a global minimum, they can add several objects simultaneously. However, in our case, it seems to be more appropriate because otherwise objects can be added wrongly. Moreover, notice that step 5 comes to the assumption that the objects are well separated. Finally, since we assumed that the obstacles are *far away from the exterior boundary*, we have to take away the added objects on it. Then, if the minimum of the topological gradient is on the exterior boundary, we push the added inclusion inside with a depth 0.005.

Concerning the mesh, we impose a fixed number of discretization points for the exterior boundary $\partial\Omega$, that is 20 points for $[-1, 1]$ and 10 points for $[-0.5, 0.5]$ (in order to have a uniform mesh). Concerning the added objects, we fix the characteristic size of a cell, that is $\frac{r}{2}$, where r is the radius of the sphere, in order to have approximatively 100 triangles to mesh the surface of each added sphere.

V.2 First simulations

First we want to detect three spheres ω_1^* , ω_2^* and ω_3^* centered respectively in $(0.425, 0, 0.225)$, $(-0.430, -0.225, 0.1)$ and $(-0.15, 0.222, -0.222)$ (*i.e.* near from the exterior boundary) with shared radius $r^* = 0.013$. The detection is quite efficient (see Figure V.1). Indeed we detect three objects

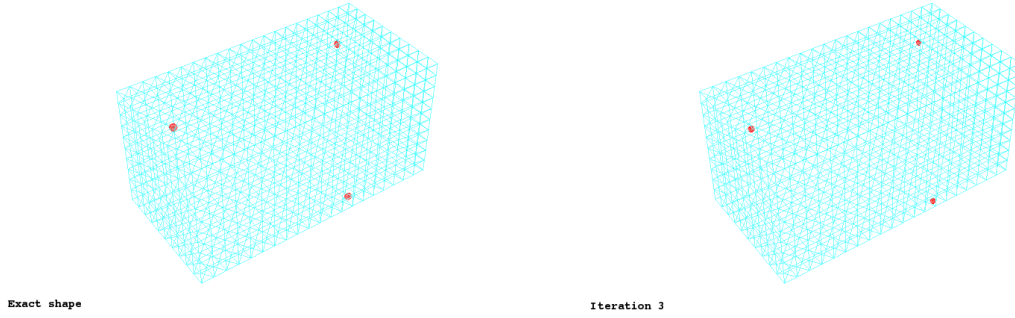


Figure V.1 – Detection of ω_1^* , ω_2^* and ω_3^*

with shared radius $r = 0.01$ that we summarized in Table V.1 Here, we stop the algorithm because of

Tableau V.1 – Detection of ω_1^* , ω_2^* and ω_3^*

actual objects	$(0.425, 0, 0.225)$	$(-0.430, -0.225, 0.1)$	$(-0.15, 0.222, -0.222)$
approximate objects	$(0.435, 0, 0.235)$	$(-0.430, -0.235, 0.102)$	$(-0.149, 0.235, -0.235)$

the functional increases as we can see in Figure V.2.

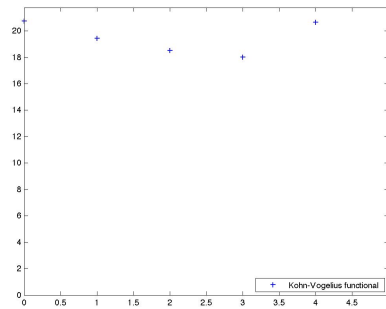


Figure V.2 – Evolution of the functional $\mathcal{J}_{KV}$ during the detection of ω_1^* , ω_2^* and ω_3^*

In this first simulation, the objects are very far away from each other. But *what happens when the obstacles are closed from each other?* Figure V.3 shows that the detection of closed objects is quite efficient too. Indeed, we want to detect three spheres ω_4^* , ω_5^* and ω_6^* centered respectively in $(0, 0.237, -0.237)$, $(0.04, 0.237, -0.237)$ and $(0.08, 0.237, -0.237)$ with shared radius $r^* = 0.01$. We obtain three spheres with shared radius $r = 0.01$ that we summarized in Table V.2

Now the question we asked is: *can we detect other shapes than spheres?* Thus, we want to detect objects with different shapes: a sphere ω_7^* centered in $(0.2, -0.237, 0.237)$ with radius $r^* = 0.01$ and a

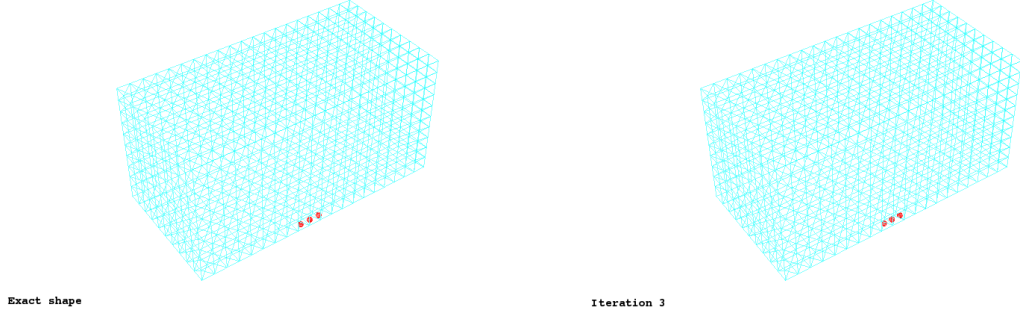


Figure V.3 – Detection of ω_4^* , ω_5^* and ω_6^*

Tableau V.2 – Detection of ω_4^* , ω_5^* and ω_6^*

actual objects	$(0,0.237,-0.237)$	$(0.04,0.237,-0.237)$	$(0.08,0.237,-0.237)$
approximate objects	$(0.0088,0.235,-0.235)$	$(0.044,0.235,-0.235)$	$(0.079,0.235,-0.235)$

cube ω_8^* given by $[0.08, 0.1] \times [0.227, 0.247] \times [-0.247, -0.227]$. We obtain Figure V.4: a sphere centered

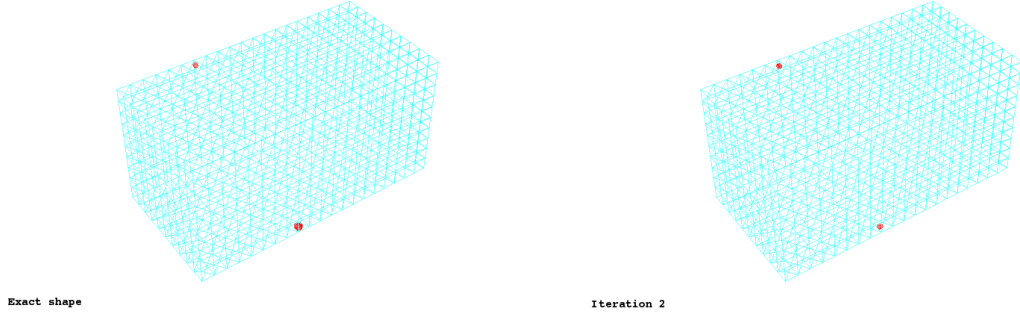


Figure V.4 – Detection of ω_7^* and ω_8^*

in $(0.096, 0.235, -0.235)$ and one centered in $(0.202, -0.235, 0.235)$ with shared radius $r = 0.01$. Here again, we stop our implementation because of the functional increases.

In conclusion of these first simulations, as we expected, this method permits to give us the number of objects we have to determine and their rough location. Moreover, it is efficient for different types of shapes, including shapes with corners.

V.3 Influence to the distance to the location of measurements

Now, we want to know how far we can detect an inclusion by trying to recover a sphere increasingly pushed away from the boundary: precisely, the sphere with radius $r^* = 0.01$ centered in $(0, y^*, 0.1)$ with $y^* = 0.232, 0.225, 0.2, 0.1, 0$. In the three first tests, the inclusion is approximatively detected. However, the method fails when $y^* = 0.1$ and $y^* = 0$ as we can see in Figure V.5 where we represent

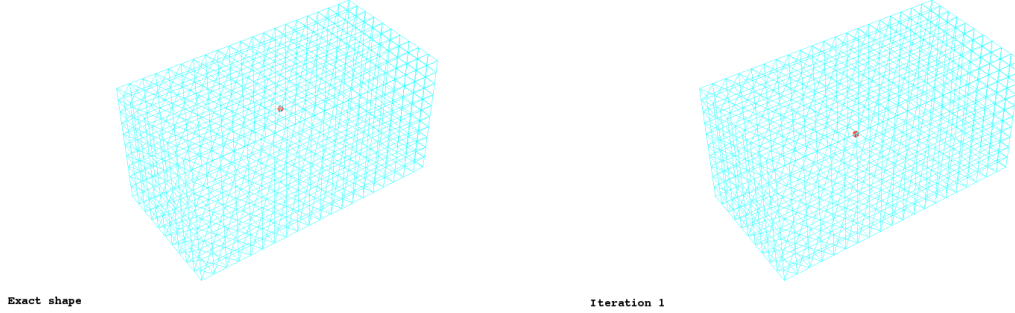


Figure V.5 – Detection of an object which is far away from the exterior boundary

the case $y^* = 0$: we obtain a sphere centered in $(0.185, 0.235, 0.235)$. In these two last cases, the functional increases when we add the first object (which means that we add it wrongly).

In order to be more precise, let us define d the distance between the object and the exterior boundary given by $y = 0.25$ (*i.e.* $d = 0.25 - y^* + r^*$). Let us define the non-dimensional distance $\eta := \frac{d}{2r^*}$. We want to study the error between $P^* := (x^*, y^*, z^*)$ and $P(\eta) := (x(\eta), y(\eta), z(\eta))$ which are the respective coordinates of the center of the real object and its approximation. Then, we define the two following errors

$$\text{Err}_1(\eta) := \frac{\|P^* - P(\eta)\|}{d} \quad \text{and} \quad \text{Err}_2(\eta) := \frac{\|P^* - P(\eta)\|}{2r^*},$$

where $\|\cdot\|$ represents the Euclidean norm. Notice that the first error Err_1 is the classical relative error. In Figure V.6, we represent Err_1 and Err_2 versus the non-dimensional distance η . The result

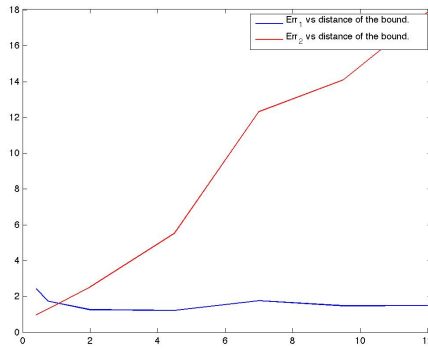


Figure V.6 – The Euclidean norm of the error between the exact and the approximate coordinates versus the non-dimensional distance η

obtained is similar to the one obtained by Ben Abda *et al.* in [23, Fig. 4.2.(b)] and shows that the error dramatically increases when the object is far away of the exterior boundary.

Finally, Figure V.7 shows that the more the obstacle is far away from the exterior boundary, the more the initial value of the functional $\mathcal{J}_{KV}$ is small and the initial value of the minimum of the topological gradient $\delta\mathcal{J}_{KV}(P_1^*)$ is big: they seem to tend to 0. This means that the sensitivity of the functional and its topological gradient decreases when the distance d of the object with the exterior boundary increases. This explains the encountered difficulties.

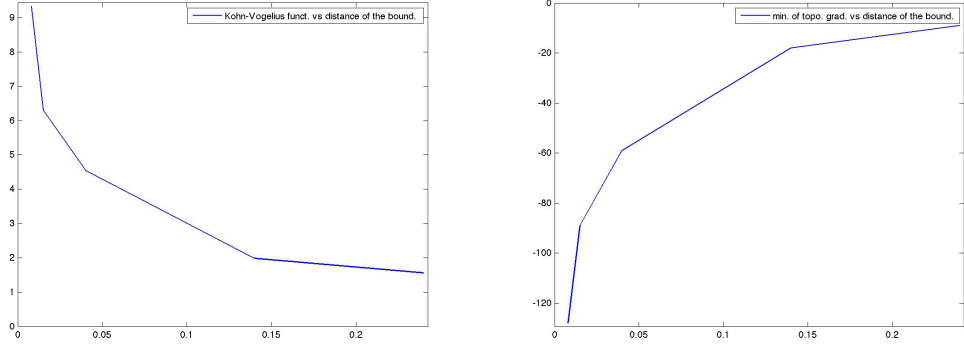


Figure V.7 – The initial Kohn-Vogelius functional and minimum of the topological gradient versus the distance of the obstacle from the exterior boundary d

Hence, these simulations show that the detection of obstacles far away from the exterior boundary is difficult. This phenomenon can be explained by the regularizing behavior of the Stokes equations. We emphasized this difficulty of detection pointing out that the functional $\mathcal{J}_{KV}$ and its topological gradient are less sensitive to the addition of obstacles when they are far away from the exterior boundary. This lack of sensitivity with respect to the object far away from the measurements is maybe due to the static (here stationary) physical context. Indeed, the sensitivity is better in a dynamical or waves context (see [34]). This constraint may seem very restrictive in particular for practical use. However, detecting objects closed of the exterior boundary has many applications as for example the location of mines mentioned in the work of Brühl *et al.* in [32].

Remark V.3.1. *An intuitive solution may be to cumulate several experiments (with different choices of $\mathbf{f}$) in order to improve the detection. We then sum the topological gradients for each experiment. However, this manipulation does not lead to efficient results and even poor results.*

V.4 Influence of the size and the shape of the objects

Now, we want to study the influence of the volume of the objects on the efficiency of the detection. Thus, we want to detect a sphere with radius $r^* = 0.015, 0.02, 0.03, 0.04, 0.05$ and which is, for each test, taken away from the boundary of $d = 0.003$. The fourth first tests are efficient and permit to detect the object decreasing the Kohn-Vogelius functional $\mathcal{J}_{KV}$. Moreover, once the object detected, the radius of the sphere increases. We recapitulate the radius r obtained in each case in Table V.3. However, for $r = 0.05$, the object is badly positioned as we can see in Figure V.8 and the functional

Tableau V.3 – Radius of the approximate obstacles versus real radius of the object.

r^*	0.015	0.02	0.03	0.04
r	0.012	0.025	0.035	0.052

increases immediately: the real sphere is centered in $(0, 0.197, 0.1)$ and we obtain a sphere centered in $(0.395, 0.235, 0.235)$.

Then, we want to know the influence of the size of the objects when several objects have to be detected. We make the simulations detecting two spheres with radius r_1 and r_2 when $r_1 = 2r_2$,

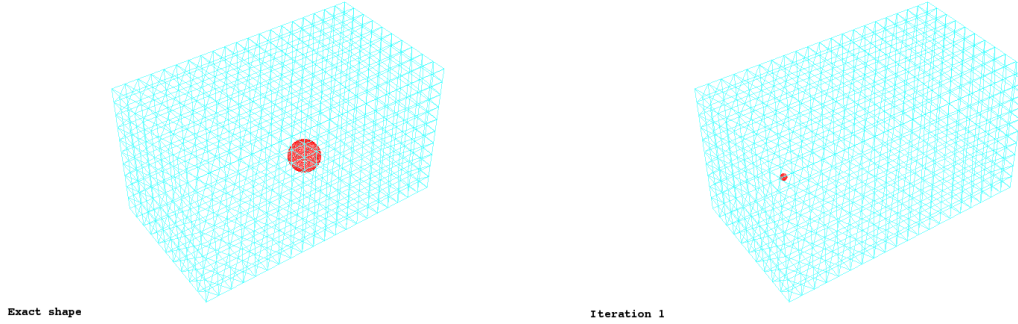


Figure V.8 – Detection of a big obstacle

$r_1 = 3r_2$, $r_1 = 4r_2$ and $r_1 = 5r_2$. The simulations shows that the first object detected is the biggest sphere (even if we interchange the position of the objects). However, when an object is too big, we have the same conclusion than above: we add an obstacle badly positioned and the functional increases.

Finally, we wonder if the shape of the objects has an influence on the detection. We have seen in Section V.2 that cubes can be detected. But *are the cubes more difficult or easier to detect than spheres?* In the simulation represented in Figure V.4, the first object detected is the cube. Notice that this is unchanged by interchanging the position of the objects. Then, we want to detect a *big* sphere and a *small* cube: the sphere ω_9^* centered in $(0.05, -0.222, 0.222)$ with radius $r^* = 0.025$ and the cube ω_{10}^* given by $[0.08, 0.1] \times [0.227, 0.247] \times [-0.247, -0.227]$. Notice that they are equally far away of the boundary. The results are presented in Figure V.9. The first object detected is again the

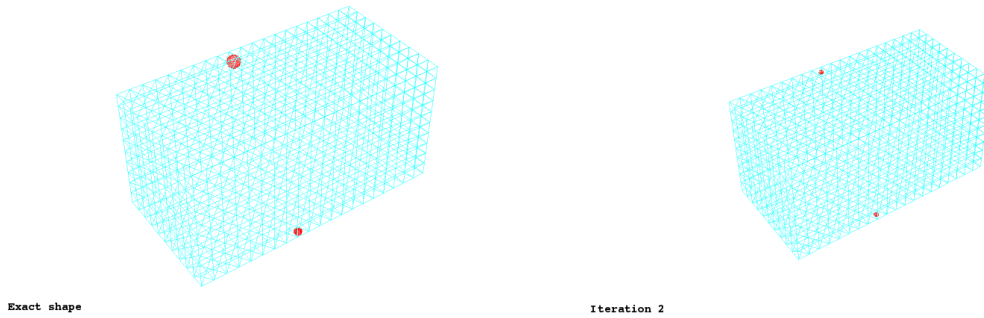


Figure V.9 – Detection of ω_9^* and ω_{10}^*

cube, even if it is the smaller. Finally, when we have to detect a *big* cube and a *small* sphere, the simulations shows that only the cube is well-detected.

In conclusion, the volume of the objects seems to have a major importance in the detection of them. It seems that we can only detect small obstacles. This is reasonable since the theoretical construction started with this assumption. Moreover, the shape of the obstacles seems to be important too: domains with geometrical singularities seems to be more easily imaged with the topological gradient approach.

V.5 Influence of the size of the domain where measurements are made

At this point, the measurements are made on all the faces of the parallelepiped rectangle except on the face $x = -0.5$. Now, we want to study the efficiency of the detection if the domain of measurement $O \subset \partial\Omega$ is reduced. To do this, we want to detect the objects ω_1^* , ω_2^* and ω_3^* presented in Section V.2 (see Figure V.1). One can note that here we have to refine the mesh to obtain efficient results: we use 32 discretization points for $[-1, 1]$ and 16 for $[-0.5, 0.5]$ and the characterized size of a cell of a sphere of radius r is now $\frac{r}{2.3}$.

We have seen that when $\partial\Omega \setminus \overline{O}$ is the face $x = -0.5$, the three objects are detected. However, if $\partial\Omega \setminus \overline{O}$ is the faces $x = -0.5$ and $y = -0.25$ (*i.e.* the faces near ω_2^*), then we only detect two objects: one centered in $(-0.165, 0.235, -0.235)$ and one centered in $(0.431, 0.05, 0.235)$. Hence, we do not detect ω_2^* (see Figure V.10 on the left). Moreover, if $\partial\Omega \setminus \overline{O}$ is the faces $x = 0.5$ and $z = 0.25$ (*i.e.* the faces near ω_1^*), then we detect only two objects again: ω_1^* is not located in this case. Finally, if we make the measurement only on the face $y = 0.25$, then we only detect an object centered in $(-0.159, 0.235, -0.217)$, *i.e.* the object ω_3^* which is near this face (see Figure V.10 on the right).

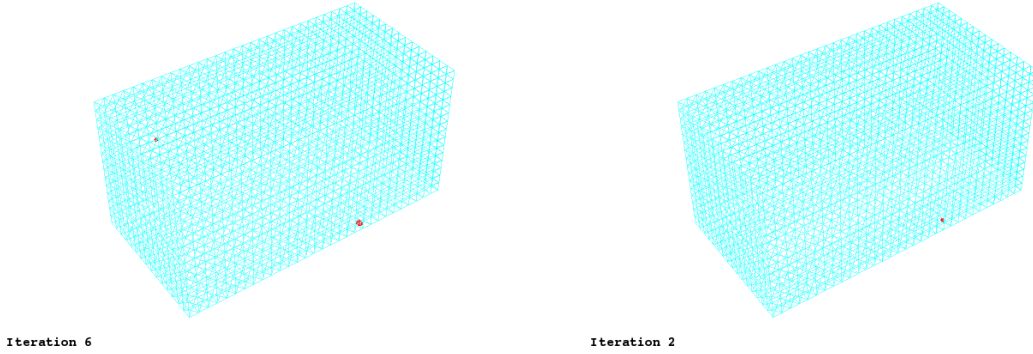


Figure V.10 – Detection of ω_1^* , ω_2^* and ω_3^* for different measurement domains

In conclusion, as we expected, it seems that we can only detect objects which are near from a measurement part of the exterior boundary.

Conclusion générale et perspectives

Conclusion générale

Ce travail de thèse a porté sur la résolution d'un problème inverse de détection à partir d'une mesure dans le cas d'un objet immergé dans un fluide. Ce problème a été étudié à l'aide de deux méthodes d'optimisation de forme : l'optimisation géométrique et l'optimisation topologique. Nous avons en particulier étudié sa stabilité en démontrant qu'il est sévèrement mal posé à l'aide d'une analyse d'ordre deux de la sensibilité par rapport à la forme.

Nous avons étudié le cas d'un (ou plusieurs) objet(s) ω immergé(s) dans un fluide visqueux, incompressible et stationnaire s'écoulant dans un domaine plus grand Ω . La mesure permettant la détection a alors été supposée effectuée sur une partie O du bord extérieur $\partial\Omega$. Le mouvement du fluide a principalement été supposé régi par les équations de Stokes avec des conditions de Dirichlet homogènes sur le bord de l'objet $\partial\omega$. Cependant, dans le cadre de l'optimisation géométrique de forme, nous avons également analysé les conditions de Neumann en soulignant les difficultés engendrées par celles-ci ainsi que le problème non linéaire de Navier-Stokes stationnaire.

Ce problème inverse a été ici étudié comme un problème de minimisation de fonctionnelle de forme. Nous nous sommes intéressés à deux types de fonctionnelles : une de type moindre carrés et une de type Kohn-Vogelius. Cette minimisation s'est effectuée à l'aide de deux différents algorithmes d'optimisation de descente selon la méthode adoptée : un algorithme de gradient de forme et un de gradient topologique.

Concernant l'approche d'Hadamard utilisant le gradient de forme, nous avons, dans un premier temps, montré l'existence des dérivées de forme d'ordre un et deux par rapport à l'objet ω . Nous avons alors établi un cadre mathématique rigoureux permettant de démontrer l'idée intuitive que, dans ce type de problème de détection, la régularité du bord extérieur $\partial\Omega$ n'a pas d'importance (il peut être supposé seulement Lipschitz). Nous avons également souligné les modifications des démonstrations entraînées par les conditions de Neumann sur le bord de l'objet $\partial\omega$.

Pour analyser ce problème de détection, nous avons ensuite démontré un résultat d'identifiabilité permettant d'assurer que notre problème de minimisation est bien posé au sens où il admet un unique minimum. Nous avons ensuite caractérisé le gradient de forme des deux types de fonctionnelles considérées en montrant les différences entre elles. En particulier, nous soulignons là encore les difficultés significatives engendrées si des conditions de Neumann sont imposées sur le bord intérieur $\partial\omega$. Nous avons ensuite effectué une analyse d'ordre deux et caractérisé la Hessienne de forme de ces fonctionnelles. Nous avons alors montré à l'aide de résultats de régularité locale des solutions que ce problème est sévèrement mal posé : la fonctionnelle est dégénérée pour certaines directions de déformation du domaine, à savoir les *hautes fréquences*. Nous avons finalement illustré cette dégénérescence en calculant explicitement la Hessienne de forme dans le cas de deux cercles concentriques après avoir résolu les équations de Stokes dans cet anneau.

Ces résultats théoriques ont motivés nos simulations numériques. En effet, nous avons dû tenir compte de notre résultat d'instabilité nous indiquant que le gradient de forme n'a pas une sensibilité uniforme par rapport aux directions de perturbation. Ainsi, afin de résoudre numériquement ce problème inverse, nous avons adopté une méthode de régularisation par paramétrisation permettant d'écarter les *hautes fréquences*. Nous avons alors proposé une *méthode adaptative* permettant de reconstruire des objets non réguliers, nécessitant l'utilisation de ces fréquences et obtenu ainsi des résultats satisfaisants. Cette méthode consiste à augmenter progressivement le nombre de paramètres utilisés dans la paramétrisation de l'objet. Nous avons alors évité l'apparition d'oscillations du bord dues aux *hautes fréquences*. Nous avons finalement reconstruit plusieurs objets à l'aide de cette méthode.

Cependant, afin de localiser plusieurs objets avec la méthode d'optimisation géométrique de forme, le nombre d'obstacles présents doit être connu. En effet, cette méthode ne permet pas de modifier la topologie des domaines. Nous avons alors changé de point de vue pour outrepasser cette limite en utilisant une méthode d'optimisation topologique de forme : sous des hypothèses de petitesse des inclusions, nous avons démontré un développement asymptotique de la solution des problèmes de Stokes considérés lorsque l'on ajoute un petit obstacle à l'intérieur. Nous en avons alors déduit une caractérisation de la dérivée topologique de la fonctionnelle de Kohn-Vogelius considérée. Nous avons ici étudié le cas de conditions de Dirichlet sur le bord intérieur $\partial\omega$ et des conditions mixtes sur le bord extérieur $\partial\Omega$ du fait de notre approche de type Kohn-Vogelius.

L'utilisation de cette notion de gradient topologique a ainsi permis de déterminer le nombre d'objets présents et leur position approximative. En utilisant un algorithme de type gradient topologique, nous avons détecté de petits objets immergés dans un fluide à partir de la mesure effectuée sur la partie O du bord extérieur. Des résultats satisfaisants ont été obtenus mais nous avons également souligné les limites de cette méthode dans le cadre de notre étude. En particulier, il s'avère très difficile de détecter des inclusions éloignées du domaine sur lequel nous effectuons la mesure. Une intuition pour expliquer ce manque de sensibilité par rapport aux objets éloignés du domaine de mesure a été le contexte physique statique (ici stationnaire) de notre étude. En effet, cette sensibilité est meilleure dans un contexte dynamique ou ondulatoire.

Perspectives

Nous avons vu que nos deux méthodes diffèrent dans leur aspect pratique. En effet, nos simulations numériques ont montré que l'approche à l'aide du gradient de forme permet de reconstruire des objets volumineux alors que l'approche concernant le gradient topologique est efficace pour la détection de petites inclusions (situées très près du bord). Cependant, un contexte physique dynamique ou ondulatoire devrait relaxer ces limites. Ainsi, il serait intéressant de coupler ces deux méthodes dans le cadre des équations de Helmholtz par exemple. Le gradient topologique permettrait donc de détecter le nombre d'objets et leur position approximative et nous pourrions alors utiliser un algorithme de gradient de forme (avec la *méthode adaptative* proposée dans cette thèse) afin d'affiner la détection (en particulier la forme de l'objet). Cela fournirait ainsi un algorithme complet et précis de détection d'objets avec une initialisation de la méthode de gradient de forme à l'aide du résultat donné par l'algorithme de gradient topologique.

Dans cette thèse, nous avons démontré que le problème inverse d'un objet immergé dans un fluide est instable. Une perspective intéressante est d'étudier la stabilité d'un autre problème, à savoir la minimisation de la traînée (dans le cas de Stokes, de Navier-Stokes, dans un domaine borné, dans un domaine extérieur, etc.). Plus précisément, nous pourrions étudier la minimisation de la fonctionnelle de forme

$$\mathcal{J}(\omega) := \frac{1}{2} \int_{\Omega \setminus \overline{\omega}} \nu |\nabla \mathbf{u} + {}^t \nabla \mathbf{u}|^2,$$

où ω et Ω sont deux ouverts tels que $\omega \subset\subset \Omega$ et où $(\mathbf{u}, p)$ est, par exemple, solution de

$$\begin{cases} -\nu\Delta\mathbf{u} + \nabla p = \mathbf{0} & \text{dans } \Omega \setminus \overline{\omega} \\ \operatorname{div} \mathbf{u} = 0 & \text{dans } \Omega \setminus \overline{\omega} \\ \mathbf{u} = \mathbf{g} & \text{sur } \partial\Omega \\ \mathbf{u} = \mathbf{0} & \text{sur } \partial\omega. \end{cases}$$

Un résultat de stabilité est intuitivement attendu mais il faudra s'attendre à des difficultés de non-équivalence des normes rentrant en jeu comme dans le papier [45] de Dambrine. En effet, la coercivité que nous allons sans doute démontrer aura lieu dans la norme $\|\cdot\|_{\mathbf{H}^{1/2}}$ qui est plus faible que la norme de différentiabilité $\|\cdot\|_{\mathbf{W}^{3,\infty}}$. Ainsi, les outils basiques classiques n'assureront pas que la coercivité entraîne qu'un point critique est un minimum local strict de la fonctionnelle de forme $\mathcal{J}$ considérée. L'existence d'un minimum local strict pourra alors être montré à l'aide d'une borne supérieure précise des variations de la dérivée seconde de forme par rapport aux normes de coercivité et de différentiabilité (voir [45, Théorème 1]).

L'étude menée dans cette thèse ouvre également la porte à des analyses de problème plus complexes. La plus naturelle est sans doute le cas instationnaire d'un objet mobile. Dans ce cas, les équations étudiées seront par exemple

$$\begin{cases} \partial_t \mathbf{u} - \nu\Delta\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{0} & \text{dans } \Omega \setminus \overline{\omega(t)} \times (0, T) \\ \operatorname{div} \mathbf{u} = 0 & \text{dans } \Omega \setminus \overline{\omega(t)} \times (0, T) \\ \mathbf{u} = \mathbf{g} & \text{sur } \partial\Omega \times (0, T) \\ \mathbf{u} = \mathbf{h}'(t) + r(t) (x - \mathbf{h}(t))^\perp & \text{sur } \partial\omega(t) \times (0, T) \\ \mathbf{u}(0) = \mathbf{u}_0 & \text{dans } \Omega \setminus \overline{\omega(t)}, \end{cases}$$

où $\mathbf{h}(t)$ est la position du centre de gravité de l'objet $\omega(t)$, où $r(t)$ est sa vitesse angulaire et où $x^\perp := \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}$ pour tout vecteur $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ de $\mathbb{R}^2$ (voir [95, 96] par exemple). Le premier résultat à démontrer, avant même l'existence des dérivées de forme, sera un résultat d'identifiabilité. Nous pourrons alors, dans un premier temps, simplifier le problème en étudiant le cas quasi-stationnaire, c'est-à-dire le cas d'un objet mobile dans un fluide stationnaire (voir par exemple le récent papier [43] de Conca *et al.*). En effet, ce résultat semble difficile à obtenir même si le cas $r = 0$ (*i.e.* le cas d'une translation) simplifie le problème.

Nous pourrons également envisager de prendre en compte la rugosité du matériau à l'aide de conditions au bord non standard, comme par exemple des conditions de glissement de Navier (voir par exemple [29, 30]) :

$$\begin{cases} \mathbf{u} \cdot \mathbf{n} = \mathbf{u}^S \cdot \mathbf{n} & \text{sur } \partial\omega \\ [\sigma(\mathbf{u}, p)\mathbf{n}]_{tan} + \beta\mathbf{u} = \beta\mathbf{u}^S & \text{sur } \partial\omega, \end{cases}$$

où $\mathbf{u}^S$ est le *champ eulérien* du solide et β est le *coefficient de friction* de la paroi $\partial\omega$. Ici aussi, il faudra dans un premier temps étudier le résultat d'identifiabilité. Une autre difficulté pourra concerner la dérivation par rapport à la forme de ces conditions de bord sur l'objet. En effet, nous avons vu que les conditions de type Neumann conduisent à des complications majeures et nous pouvons donc nous attendre à de sévères difficultés pour celles-ci.

Enfin, nous pourrons compléter le problème en cherchant non seulement la position et la forme de l'objet mais aussi sa température. Il faudra alors considérer des équations de type Boussinesq données

par :

$$\left\{ \begin{array}{ll} -\nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p &= \theta \mathbf{g} & \text{dans } \Omega \setminus \overline{\omega} \\ \operatorname{div} \mathbf{u} &= 0 & \text{dans } \Omega \setminus \overline{\omega} \\ -\eta \Delta \theta + \mathbf{u} \cdot \nabla \theta &= 0 & \text{dans } \Omega \setminus \overline{\omega} \\ \mathbf{u} &= \mathbf{g} & \text{sur } \partial \Omega \\ \mathbf{u} &= \mathbf{0} & \text{sur } \partial \omega \\ \theta &= b & \text{sur } \partial \Omega \\ \theta &= c & \text{sur } \partial \omega, \end{array} \right.$$

où θ représente la température. Là encore, le résultat d'identifiabilité n'est pas trivial et nous ne pourrions en particulier pas nous appuyer sur le résultat de continuation unique de Fabre *et al.* (voir [55]). Il faudra ainsi commencer par démontrer ce même résultat dans le cas des équations de Boussinesq à l'aide d'estimations de Carleman. Des premiers résultats allant dans ce sens ont été obtenus par Doubova *et al.* dans [49, 50].

Annexe : Résultats sur les équations de Stokes et Navier-Stokes

A.1 Résultats concernant les équations de Stokes

Dans cette section, nous rappelons des résultats classiques concernant les équations de Stokes avec des conditions de bord particulières : nous étudions ici le cas des conditions de bord de type Neumann ou mixtes peu détaillées dans la littérature. Pour une étude précise de ces équations, en particulier pour des conditions de bord de type Dirichlet et pour la régularité des solutions, nous renvoyons par exemple à [31], [56] ou encore [64].

On introduit les notations classiques suivantes : pour Ω un ouvert de $\mathbb{R}^d$ ($d = 2$ ou 3), on définit

$$\mathbf{L}_0^2(\Omega) := \left\{ \mathbf{v} \in \mathbf{L}^2(\Omega); \int_{\Omega} \mathbf{v} = \mathbf{0} \right\} \quad \text{et} \quad \mathbf{V}^1(\Omega) := \{ \mathbf{u} \in \mathbf{H}^1(\Omega); \operatorname{div} \mathbf{u} = 0 \}.$$

Pour $(\mathbf{u}, p) \in \mathbf{H}^1(\Omega) \times L^2(\Omega)$, on introduit également tenseur des contraintes défini par

$$\sigma(\mathbf{u}, p) := \nu (\nabla \mathbf{u} + {}^t \nabla \mathbf{u}) - p \mathbf{I},$$

où $\mathbf{I}$ est la matrice identité de taille $d \times d$ et ν est la viscosité cinématique (supposée constante) associée au fluide considéré. Enfin, pour énoncer les résultats de régularité locale, nous introduisons les espaces suivants : pour $k, m \in \mathbb{N}$, $k < m$ et pour deux ouverts Ω_1 et Ω_2 tels que $\Omega_2 \subset \Omega_1$, on note $X^{k,m}(\Omega_1, \Omega_2)$ l'espace des fonctions de $H^k(\Omega_1)$ telles que leur restriction à Ω_2 appartient à $H^m(\Omega_2)$. De manière similaire, $X^{*,m}(\Omega_1, \Omega_2)$ est l'espace des fonctions de $[H^1(\Omega_1)]'$ telles que leur restriction à Ω_2 appartient à $H^m(\Omega_2)$.

Dans toute cette section C représente une constante générique positive dépendant seulement de la géométrie du domaine pouvant changer d'une ligne à l'autre.

Remarque A.1.1. Notons que les résultats démontrés dans cette section concernant le système de Stokes

$$\begin{cases} -\nu \Delta \mathbf{u} + \nabla p &= \mathbf{f} & \text{dans } \Omega \\ \operatorname{div} \mathbf{u} &= g & \text{dans } \Omega \\ -\nu \partial_{\mathbf{n}} \mathbf{u} + p \mathbf{n} &= \mathbf{f}_b & \text{sur } \partial \Omega \end{cases}$$

sont aussi vrais pour le système de Stokes

$$\begin{cases} -\operatorname{div} (\sigma(\mathbf{u}, p)) &= \mathbf{f} & \text{dans } \Omega \\ \operatorname{div} \mathbf{u} &= g & \text{dans } \Omega \\ \sigma(\mathbf{u}, p) \mathbf{n} &= \mathbf{f}_b & \text{sur } \partial \Omega, \end{cases}$$

et inversement. On rappelle également que, si $\operatorname{div} \mathbf{u} = 0$ dans Ω alors

$$-\nu \Delta \mathbf{u} + \nabla p = -\operatorname{div} (\sigma(\mathbf{u}, p)) \quad \text{dans } \Omega.$$

A.1.1 With Neumann boundary conditions

Theorem A.1.2 (Existence and uniqueness of the solution). *Let Ω be a bounded connected Lipschitz open set of $\mathbb{R}^d$ ($d \in \mathbb{N}^*$) and $\nu > 0$. Let $\mathbf{f} = (f_1, \dots, f_N) \in [\mathbf{H}^1(\Omega)]'$, $g \in L^2(\Omega)$ and $\mathbf{f}_b = (f_{b1}, \dots, f_{bN}) \in \mathbf{H}^{-1/2}(\partial\Omega)$ such that $\langle f_i, 1 \rangle_\Omega = \langle f_{bi}, 1 \rangle_{\partial\Omega}$ for all $i = 1, \dots, N$. Then, the problem: find $(\mathbf{u}, p)$ in $[\mathbf{H}^1(\Omega) \cap \mathbf{L}_0^2(\Omega)] \times L^2(\Omega)$ such that*

$$\begin{cases} \int_{\Omega} \nu \nabla \mathbf{u} : \nabla \mathbf{v} - \int_{\Omega} p \operatorname{div} \mathbf{v} = \langle \mathbf{f}, \mathbf{v} \rangle_{\Omega} - \langle \mathbf{f}_b, \mathbf{v} \rangle_{\partial\Omega} & \forall \mathbf{v} \in \mathbf{H}^1(\Omega) \\ \int_{\Omega} \xi \operatorname{div} \mathbf{u} = \int_{\Omega} \xi g & \forall \xi \in L^2(\Omega), \end{cases} \quad (\text{A.1.1})$$

admits a unique solution and the following estimate holds:

$$\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} + \|p\|_{L^2(\Omega)} \leq C \left(\|\mathbf{f}\|_{[\mathbf{H}^1(\Omega)]'} + \|g\|_{L^2(\Omega)} + \|\mathbf{f}_b\|_{\mathbf{H}^{-1/2}(\partial\Omega)} \right).$$

Moreover, for $k \geq 0$, if the open set Ω has a $C^{k+1,1}$ boundary and if

$$(\mathbf{f}, g, \mathbf{f}_b) \in \mathbf{H}^k(\Omega) \times H^{k+1}(\Omega) \times \mathbf{H}^{k+\frac{1}{2}}(\partial\Omega),$$

then the pair $(\mathbf{u}, p)$ belongs to $\mathbf{H}^{k+2}(\Omega) \times H^{k+1}(\Omega)$, satisfies

$$\begin{cases} -\nu \Delta \mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega \\ \operatorname{div} \mathbf{u} = g & \text{in } \Omega \\ -\nu \partial_{\mathbf{n}} \mathbf{u} + p \mathbf{n} = \mathbf{f}_b & \text{on } \partial\Omega \end{cases}$$

and the following estimate holds:

$$\|\mathbf{u}\|_{\mathbf{H}^{k+2}(\Omega)} + \|p\|_{H^{k+1}(\Omega)} \leq C \left(\|\mathbf{f}\|_{\mathbf{H}^k(\Omega)} + \|g\|_{H^{k+1}(\Omega)} + \|\mathbf{f}_b\|_{\mathbf{H}^{k+\frac{1}{2}}(\partial\Omega)} \right).$$

Proof. Existence and uniqueness. For the proof of the first part of Theorem A.1.2, let us begin by studying the case of null divergence. According to Lax-Milgram's Theorem, there exists a unique $\mathbf{u} \in \mathbf{V}^1(\Omega) \cap \mathbf{L}_0^2(\Omega)$ such that

$$\int_{\Omega} \nu \nabla \mathbf{u} : \nabla \mathbf{v} = \langle \mathbf{f}, \mathbf{v} \rangle_{\Omega} - \langle \mathbf{f}_b, \mathbf{v} \rangle_{\partial\Omega} \quad \forall \mathbf{v} \in \mathbf{V}^1(\Omega) \cap \mathbf{L}_0^2(\Omega), \quad (\text{A.1.2})$$

and we have

$$\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} \leq C \left(\|\mathbf{f}\|_{[\mathbf{H}^1(\Omega)]'} + \|\mathbf{f}_b\|_{\mathbf{H}^{-1/2}(\partial\Omega)} \right). \quad (\text{A.1.3})$$

For $\mathbf{v} \in \mathbf{V}^1(\Omega)$, we define

$$\mathbf{c}(\mathbf{v}) := \frac{1}{|\Omega|} \int_{\Omega} \mathbf{v}$$

and we use $\mathbf{v} - \mathbf{c}(\mathbf{v}) \in \mathbf{V}^1(\Omega) \cap \mathbf{L}_0^2(\Omega)$ as a test function in (A.1.2). According to the compatibility condition, we check that Equality (A.1.2) is also true for all $\mathbf{v} \in \mathbf{V}^1(\Omega)$ and in particular for all $\mathbf{v} \in \mathbf{V}^1(\Omega) \cap \mathbf{H}_0^1(\Omega)$. Then, using De Rham's Theorem (see for example Lemma 2.7 in [14]), there exists $p \in L^2(\Omega)$, up to an additive constant, such that

$$\int_{\Omega} \nu \nabla \mathbf{u} : \nabla \mathbf{v} - \int_{\Omega} p \operatorname{div} \mathbf{v} = \left\langle \mathbf{f}|_{\mathbf{H}_0^1(\Omega)}, \mathbf{v} \right\rangle_{\mathbf{H}^{-1}(\Omega), \mathbf{H}_0^1(\Omega)} \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega). \quad (\text{A.1.4})$$

We define $\varphi_{\mathbf{N}}(\mathbf{x}) := {}^t(x_1, \dots, x_N)/N$, where N is the dimension and $(x_1, \dots, x_N)$ are the coordinates of $\mathbf{x}$: $\varphi_{\mathbf{N}}$ is such that $\operatorname{div} \varphi_{\mathbf{N}} = 1$. Let $\mathbf{v} \in \mathbf{H}^1(\Omega)$ and

$$c_b(\mathbf{v}) := \frac{1}{|\Omega|} \int_{\partial\Omega} \mathbf{v} \cdot \mathbf{n}.$$

Using Theorem 3.2 in [56] or Lemma 3.3 in [14], let us define $\mathbf{v}_2 \in \mathbf{V}^1(\Omega)$ in such a way that $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 + c_b(\mathbf{v})\boldsymbol{\varphi}_N$, where $\mathbf{v}_1 \in \mathbf{H}_0^1(\Omega)$ satisfies

$$\operatorname{div} \mathbf{v}_1 = \operatorname{div} (\mathbf{v} - c_b(\mathbf{v})\boldsymbol{\varphi}_N).$$

According to (A.1.2) (which remains valid for $\mathbf{v} \in \mathbf{V}^1(\Omega)$) and (A.1.4) and choosing the additive constant for p such that $\int_{\Omega} p = -\langle \mathbf{f}, \boldsymbol{\varphi}_N \rangle_{\Omega} + \langle \mathbf{f}_b, \boldsymbol{\varphi}_N \rangle_{\partial\Omega}$, we check that

$$\int_{\Omega} \nu \nabla \mathbf{u} : \nabla \mathbf{v} - \int_{\Omega} p \operatorname{div} \mathbf{v} = \langle \mathbf{f}, \mathbf{v} \rangle_{\Omega} - \langle \mathbf{f}_b, \mathbf{v} \rangle_{\partial\Omega}.$$

Therefore, there is a unique pair $(\mathbf{u}, p) \in [\mathbf{V}^1(\Omega) \cap \mathbf{L}_0^2(\Omega)] \times L^2(\Omega)$ such that

$$\int_{\Omega} \nu \nabla \mathbf{u} : \nabla \mathbf{v} - \int_{\Omega} p \operatorname{div} \mathbf{v} = \langle \mathbf{f}, \mathbf{v} \rangle_{\Omega} - \langle \mathbf{f}_b, \mathbf{v} \rangle_{\partial\Omega} \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega). \quad (\text{A.1.5})$$

Let $\mathbf{v} := \tilde{\mathbf{v}} + c(p)\boldsymbol{\varphi}_N$, where

$$c(p) := \frac{1}{|\Omega|} \int_{\Omega} p$$

and $\tilde{\mathbf{v}} \in \mathbf{H}_0^1(\Omega)$ is such that $\operatorname{div} \tilde{\mathbf{v}} = p - c(p)$ and $\|\tilde{\mathbf{v}}\|_{\mathbf{H}_0^1(\Omega)} \leq C\|p\|_{L^2(\Omega)}$ (see Theorem 3.2 in [56] or Lemma 3.3 in [14]). Using $\mathbf{v}$ in (A.1.5), and according to (A.1.3), we obtain

$$\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} + \|p\|_{L^2(\Omega)} \leq C \left(\|\mathbf{f}\|_{[\mathbf{H}^1(\Omega)]'} + \|\mathbf{f}_b\|_{\mathbf{H}^{-1/2}(\partial\Omega)} \right).$$

The first part of the theorem is proved for $g = 0$. The case $g \neq 0$ is obtained by a lifting argument. Let us define

$$\mathbf{u}_g := \tilde{\mathbf{u}}_g + c(g)\boldsymbol{\varphi}_N - \frac{1}{|\Omega|} \int_{\Omega} (\tilde{\mathbf{u}}_g + c(g)\boldsymbol{\varphi}_N) \in \mathbf{H}^1(\Omega),$$

where

$$c(g) := \frac{1}{|\Omega|} \int_{\Omega} g$$

and where, according to Theorem 3.2 in [56] (or Lemma 3.3 in [14]), $\tilde{\mathbf{u}}_g \in \mathbf{H}_0^1(\Omega)$ is such that $\operatorname{div} \tilde{\mathbf{u}}_g = g - c(g)$ and $\|\tilde{\mathbf{u}}_g\|_{\mathbf{H}_0^1(\Omega)} \leq C\|g\|_{L^2(\Omega)}$. Thus $\mathbf{u}_g$ is such that

$$\int_{\Omega} \mathbf{u}_g = \mathbf{0}, \quad \operatorname{div} \mathbf{u}_g = g \quad \text{in } \Omega \quad \text{and} \quad \|\mathbf{u}_g\|_{\mathbf{H}^1(\Omega)} \leq C\|g\|_{L^2(\Omega)}.$$

Thus, defining $\mathbf{u}_0 := \mathbf{u} - \mathbf{u}_g$ and

$$\tilde{\mathbf{f}} : \mathbf{v} \in \mathbf{H}^1(\Omega) \mapsto \langle \mathbf{f}, \mathbf{v} \rangle_{\Omega} - \int_{\Omega} \nu \nabla \mathbf{u}_g : \nabla \mathbf{v},$$

the problem (A.1.1) is equivalent to

$$\left\{ \begin{array}{l} \text{Find } (\mathbf{u}_0, p) \in [\mathbf{V}^1(\Omega) \cap \mathbf{L}_0^2(\Omega)] \times L^2(\Omega) \text{ such that} \\ \int_{\Omega} \nu \nabla \mathbf{u}_0 : \nabla \mathbf{v} - \int_{\Omega} p \operatorname{div} \mathbf{v} = \langle \tilde{\mathbf{f}}, \mathbf{v} \rangle_{\Omega} - \langle \mathbf{f}_b, \mathbf{v} \rangle_{\partial\Omega} \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega). \end{array} \right.$$

Regularity of the solution. For the second part of Theorem A.1.2 concerning the regularity of $(\mathbf{u}, p)$, we refer to Section 5.3 in [31] and [64] for the homogeneous case $g = 0$ and $\mathbf{f}_b = \mathbf{0}$. To obtain the result for $g \neq 0$ and $\mathbf{f}_b \neq \mathbf{0}$ let us define

$$\boldsymbol{\varphi}_g := \frac{1}{|\Omega|} \left(\int_{\Omega} g \right) \boldsymbol{\varphi}_N \quad \text{and} \quad \Psi(g) := 2\nu \mathbf{H} \boldsymbol{\varphi}_g \cdot \mathbf{n} - \nu \operatorname{div}_{\Gamma} (\boldsymbol{\varphi}_g)_{\tau}$$

where $(\varphi_g)_\tau$ is the tangential component of φ_g and H is the mean curvature of $\partial\Omega$. According to Theorem A in [13], there exists $\tilde{\mathbf{u}} \in \mathbf{H}^{k+2}(\Omega)$ such that

$$\operatorname{div} \tilde{\mathbf{u}} = g \text{ in } \Omega, \quad \tilde{\mathbf{u}} = \varphi_g \quad \text{and} \quad \nu \partial_{\mathbf{n}} \tilde{\mathbf{u}} = -(\mathbf{f}_b)_\tau + \Psi(g)\mathbf{n} + \nu g \mathbf{n} \text{ on } \partial\Omega,$$

and which satisfies the following estimate:

$$\|\tilde{\mathbf{u}}\|_{\mathbf{H}^{k+2}(\Omega)} \leq C \left(\|g\|_{H^{k+1}(\Omega)} + \|\mathbf{f}_b\|_{\mathbf{H}^{k+1/2}(\partial\Omega)} \right).$$

Moreover, with a continuous right inverse of the trace operator, we define a pressure function $\tilde{p}$ in $H^{k+1}(\Omega)$ such that $\tilde{p} = \mathbf{f}_b \cdot \mathbf{n} + \Psi(g) + g$ on $\partial\Omega$ and which depends continuously on $(g, \mathbf{f}_b)$ in $H^{k+1}(\Omega) \times \mathbf{H}^{k+1/2}(\partial\Omega)$. Then we have constructed a pair $(\tilde{\mathbf{u}}, \tilde{p})$ such that

$$-\nu \partial_{\mathbf{n}} \tilde{\mathbf{u}} + \tilde{p} \mathbf{n} = \mathbf{f}_b \text{ on } \partial\Omega,$$

and which obeys the following estimate:

$$\|\tilde{\mathbf{u}}\|_{\mathbf{H}^{k+2}(\Omega)} + \|\tilde{p}\|_{H^{k+1}(\Omega)} \leq C \left(\|g\|_{H^{k+1}(\Omega)} + \|\mathbf{f}_b\|_{\mathbf{H}^{k+1/2}(\partial\Omega)} \right).$$

Finally, we check that the vector field $\mathbf{F} := \mathbf{f} + \nu \Delta \tilde{\mathbf{u}} - \nabla \tilde{p} \in \mathbf{H}^k(\Omega)$ satisfies

$$\|\mathbf{F}\|_{\mathbf{H}^k(\Omega)} \leq C \left(\|\mathbf{f}\|_{\mathbf{H}^k(\Omega)} + \|g\|_{H^{k+1}(\Omega)} + \|\mathbf{f}_b\|_{\mathbf{H}^{k+1/2}(\partial\Omega)} \right), \quad (\text{A.1.6})$$

and, since regularity results in the homogeneous case (see [31] and [64]) ensures that the solution to

$$\begin{cases} \nu \Delta \mathbf{w} + \nabla q &= \mathbf{F} & \text{in } \Omega \\ \operatorname{div} \mathbf{w} &= 0 & \text{on } \Omega \\ -\nu \partial_{\mathbf{n}} \mathbf{w} + q \mathbf{n} &= \mathbf{0} & \text{on } \partial\Omega \end{cases}$$

belongs to $\mathbf{H}^{k+2}(\Omega) \times H^{k+1}(\Omega)$ and satisfies:

$$\|\mathbf{w}\|_{\mathbf{H}^{k+2}(\Omega)} + \|q\|_{H^{k+1}(\Omega)} \leq C \|\mathbf{F}\|_{\mathbf{H}^k(\Omega)}, \quad (\text{A.1.7})$$

the conclusion follows by remarking that $(\mathbf{u}, p) = (\mathbf{w} + \tilde{\mathbf{u}}, q + \tilde{p})$ and by combining (A.1.6) with (A.1.7). $\square$

Remark A.1.3. *As in the proof of Theorem A.1.2, we can obtain the following result from Theorem A in [13]: if Ω is a bounded connected open set of $\mathbb{R}^N$ with a C^{k+1} boundary and if*

$$(g, \mathbf{b}_0, \mathbf{b}_1) \in H^{k+1}(\Omega) \times \mathbf{H}^{k+3/2}(\partial\Omega) \times \mathbf{H}^{k+1/2}(\partial\Omega)$$

satisfies $\int_{\partial\Omega} \mathbf{b}_0 \cdot \mathbf{n} = \int_{\Omega} g$, then there exists a pair $(\mathbf{v}, \pi) \in \mathbf{H}^{k+2}(\Omega) \times H^{k+1}(\Omega)$ such that

$$\operatorname{div} \mathbf{v} = g \text{ in } \Omega, \quad \mathbf{v} = \mathbf{b}_0 \quad \text{and} \quad -\nu \partial_{\mathbf{n}} \mathbf{v} + \pi \mathbf{n} = \mathbf{b}_1 \text{ on } \partial\Omega$$

and

$$\|\mathbf{v}\|_{\mathbf{H}^{k+2}(\Omega)} + \|\pi\|_{H^{k+1}(\Omega)} \leq C \left(\|g\|_{H^{k+1}(\Omega)} + \|\mathbf{b}_0\|_{\mathbf{H}^{k+3/2}(\partial\Omega)} + \|\mathbf{b}_1\|_{\mathbf{H}^{k+1/2}(\partial\Omega)} \right).$$

A.1.2 Case where the boundary has two connected components

A.1.2.1 Neumann boundary conditions on the two connected components

We state here a local regularity result using the spaces $X^{k,m}$ defined at the beginning of Section A.1.

Theorem A.1.4 (Local regularity). *Let $k \in \mathbb{N}$, $\nu > 0$, Ω a bounded connected Lipschitz open set of $\mathbb{R}^d$ ($d \in \mathbb{N}^*$) and ω an open set with a $C^{k+1,1}$ boundary strictly included in Ω (that is to say there exists $\delta > 0$ such that ω is at least at distance δ of the boundary $\partial\Omega$) and such that $\Omega \setminus \bar{\omega}$ is connected. Let $\mathcal{C}$ and $\mathcal{C}'$ two smooth open subsets of $\Omega \setminus \bar{\omega}$ such that $\partial\omega \subset \partial\mathcal{C}$, $\partial\omega \subset \partial\mathcal{C}'$, $\bar{\mathcal{C}} \subset \mathcal{C}'$ and $\bar{\mathcal{C}'} \subset \Omega$. Let*

$$(\mathbf{f}, g, \mathbf{h}_{ext}, \mathbf{h}_{int}) \in \mathbf{X}^{*,k}(\Omega \setminus \bar{\omega}, \mathcal{C}') \times X^{0,k+1}(\Omega \setminus \bar{\omega}, \mathcal{C}') \times \mathbf{H}^{-1/2}(\partial\Omega) \times \mathbf{H}^{k+\frac{1}{2}}(\partial\omega),$$

such that $\int_{\Omega \setminus \bar{\omega}} f^i = \langle h_{ext}^i, 1 \rangle_{\partial\Omega} + \int_{\partial\omega} h_{int}^i$ for all $i = 1, \dots, N$, where f^i , h_{ext}^i and h_{int}^i are the respective components of $\mathbf{f}$, $\mathbf{h}_{ext}$ and $\mathbf{h}_{int}$. We consider $(\mathbf{u}, p)$ the solution in $\mathbf{H}^1(\Omega \setminus \bar{\omega}) \times L^2(\Omega \setminus \bar{\omega})$ of the following Stokes problem with Neumann boundary conditions

$$\begin{cases} -\nu\Delta\mathbf{u} + \nabla p &= \mathbf{f} & \text{in } \Omega \setminus \bar{\omega} \\ \operatorname{div} \mathbf{u} &= g & \text{in } \Omega \setminus \bar{\omega} \\ -\nu\partial_{\mathbf{n}}\mathbf{u} + p\mathbf{n} &= \mathbf{h}_{ext} & \text{on } \partial\Omega \\ -\nu\partial_{\mathbf{n}}\mathbf{u} + p\mathbf{n} &= \mathbf{h}_{int} & \text{on } \partial\omega. \end{cases} \quad (\text{A.1.8})$$

Then $(\mathbf{u}, p)$ belongs to $\mathbf{H}^{k+2}(\mathcal{C}) \times \mathbf{H}^{k+1}(\mathcal{C})$ and the following estimate holds:

$$\begin{aligned} & \|\mathbf{u}\|_{\mathbf{H}^{k+2}(\mathcal{C})} + \|p\|_{\mathbf{H}^{k+1}(\mathcal{C})} \\ & \leq C \left(\|\mathbf{f}\|_{\mathbf{X}^{*,k}(\Omega \setminus \bar{\omega}, \mathcal{C}')} + \|g\|_{X^{0,k+1}(\Omega \setminus \bar{\omega}, \mathcal{C}')} + \|\mathbf{h}_{int}\|_{\mathbf{H}^{k+\frac{1}{2}}(\partial\omega)} + \|\mathbf{h}_{ext}\|_{\mathbf{H}^{-1/2}(\partial\Omega)} \right). \end{aligned}$$

Proof. First, let us consider the case $k = 0$. We define $\mathcal{V} := \mathcal{C} \cup \bar{\omega}$, $\mathcal{V}' := \mathcal{C}' \cup \bar{\omega}$ and $\varphi \in C_c^\infty(\Omega)$ such that $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ in $\mathcal{V}$ and $\varphi \equiv 0$ in $\Omega \setminus \mathcal{V}'$. Let

$$(\mathbf{u}, p) \in [\mathbf{H}^1(\Omega \setminus \bar{\omega}) \cap \mathbf{L}_0^2(\Omega \setminus \bar{\omega})] \times L^2(\Omega \setminus \bar{\omega})$$

the solution of problem (A.1.8) given by Theorem A.1.2. Using (A.1.8), we check that

$$\begin{cases} -\nu\Delta(\varphi\mathbf{u}) + \nabla(\varphi p) &= \tilde{\mathbf{f}} & \text{in } \mathcal{C}' \\ \operatorname{div}(\varphi\mathbf{u}) &= \tilde{g} & \text{in } \mathcal{C}' \\ -\nu\partial_{\mathbf{n}}(\varphi\mathbf{u}) + (\varphi p)\mathbf{n} &= \mathbf{0} & \text{on } \partial\mathcal{V}' \\ -\nu\partial_{\mathbf{n}}(\varphi\mathbf{u}) + (\varphi p)\mathbf{n} &= \mathbf{h}_{int} & \text{on } \partial\omega, \end{cases}$$

where $\tilde{\mathbf{f}} := \varphi\mathbf{f} - \nu\mathbf{u}\Delta\varphi - 2\nu\nabla\mathbf{u}\nabla\varphi + p\nabla\varphi \in \mathbf{L}^2(\mathcal{C}')$ and $\tilde{g} := \varphi g + \mathbf{u} \cdot \nabla\varphi \in H^1(\mathcal{C}')$. From Theorem A.1.2, $\varphi\mathbf{u} \in \mathbf{H}^2(\mathcal{C}')$, $\varphi p \in H^1(\mathcal{C}')$ and, taking into account of the expression of $\tilde{\mathbf{f}}$ and $\tilde{g}$, we also have

$$\begin{aligned} & \|\varphi\mathbf{u}\|_{\mathbf{H}^2(\mathcal{C}')} + \|\varphi p\|_{H^1(\mathcal{C}')} \\ & \leq C \left(\|\varphi\mathbf{f}\|_{\mathbf{L}^2(\mathcal{C}')} + \|\varphi g\|_{H^1(\mathcal{C}')} + \|\mathbf{u}\|_{\mathbf{H}^1(\Omega \setminus \bar{\omega})} + \|p\|_{L^2(\Omega \setminus \bar{\omega})} + \|\mathbf{h}_{int}\|_{\mathbf{H}^{1/2}(\partial\omega)} \right). \quad (\text{A.1.9}) \end{aligned}$$

Using this inequality and the estimate on $\|\mathbf{u}\|_{\mathbf{H}^1(\Omega \setminus \bar{\omega})}$ and $\|p\|_{L^2(\Omega \setminus \bar{\omega})}$ given by Theorem A.1.2, we obtain the announced estimate for $k = 0$. Proceeding as above, we conclude by induction. $\square$

A.1.2.2 Mixed boundary conditions on the exterior boundary and Dirichlet boundary conditions on the interior boundary

First, let us introduce a notation: for Ω an open set of $\mathbb{R}^d$ an open subset $\omega \subset\subset \Omega$ and a part O of the exterior boundary $\partial\Omega$, we define

$$\mathbf{V}_O(\Omega \setminus \bar{\omega}) := \{ \mathbf{u} \in \mathbf{H}^1(\Omega \setminus \bar{\omega}); \operatorname{div} \mathbf{u} = 0 \text{ in } \Omega \setminus \bar{\omega}, \mathbf{u} = \mathbf{0} \text{ on } \partial\omega \cup (\partial\Omega \setminus \bar{O}) \}.$$

Theorem A.1.5 (Existence and uniqueness of the solution). *Let Ω be a bounded Lipschitz open set of $\mathbb{R}^d$ ($d \in \mathbb{N}^*$) and let $\omega \subset\subset \Omega$ be a Lipschitz open subset of Ω such that $\Omega \setminus \bar{\omega}$ is connected. Let O be an open subset of the exterior boundary $\partial\Omega$ and $\nu > 0$. Let*

$$(\mathbf{f}, g, \mathbf{h}_O, \mathbf{h}_{ext}, \mathbf{h}_{int}) \in [\mathbf{H}^1(\Omega \setminus \bar{\omega})]' \times L^2(\Omega \setminus \bar{\omega}) \times \mathbf{H}^{-1/2}(O) \times \mathbf{H}^{1/2}(\partial\Omega \setminus \bar{O}) \times \mathbf{H}^{1/2}(\partial\omega).$$

Then, the problem

$$\begin{cases} -\nu \Delta \mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega \setminus \bar{\omega} \\ \operatorname{div} \mathbf{u} = g & \text{in } \Omega \setminus \bar{\omega} \\ -\nu \partial_{\mathbf{n}} \mathbf{u} + p \mathbf{n} = \mathbf{h}_O & \text{on } O \\ \mathbf{u} = \mathbf{h}_{ext} & \text{on } \partial\Omega \setminus \bar{O} \\ \mathbf{u} = \mathbf{h}_{int} & \text{on } \partial\omega \end{cases} \quad (\text{A.1.10})$$

admits a unique solution $(\mathbf{u}, p) \in \mathbf{H}^1(\Omega \setminus \bar{\omega}) \times L^2(\Omega \setminus \bar{\omega})$ and the following estimate holds:

$$\begin{aligned} \|\mathbf{u}\|_{\mathbf{H}^1(\Omega \setminus \bar{\omega})} + \|p\|_{L^2(\Omega \setminus \bar{\omega})} &\leq C \left(\|\mathbf{f}\|_{[\mathbf{H}^1(\Omega \setminus \bar{\omega})]'} + \|g\|_{L^2(\Omega \setminus \bar{\omega})} \right. \\ &\quad \left. + \|\mathbf{h}_O\|_{\mathbf{H}^{-1/2}(O)} + \|\mathbf{h}_{ext}\|_{\mathbf{H}^{1/2}(\partial\Omega \setminus \bar{O})} + \|\mathbf{h}_{int}\|_{\mathbf{H}^{1/2}(\partial\omega)} \right). \end{aligned}$$

Proof. Step 1: existence and uniqueness. Let us begin by studying the case of null divergence. According to [14, Lemma 3.3], consider $\mathbf{H} \in \mathbf{H}^1(\Omega \setminus \bar{\omega})$ such that $\operatorname{div} \mathbf{H} = 0$, $\mathbf{H} = \mathbf{h}_{int}$ on $\partial\omega$, $\mathbf{H} = \mathbf{h}_{ext}$ on $\partial\Omega \setminus \bar{O}$ such that $\int_{\partial\Omega \cup \partial\omega} \mathbf{H} \cdot \mathbf{n} = 0$ and satisfying

$$\|\mathbf{H}\|_{\mathbf{H}^1(\Omega \setminus \bar{\omega})} \leq C \left(\|\mathbf{h}_{int}\|_{\mathbf{H}^{1/2}(\partial\omega)} + \|\mathbf{h}_{ext}\|_{\mathbf{H}^{1/2}(\partial\Omega \setminus \bar{O})} \right). \quad (\text{A.1.11})$$

Then the couple $(\mathbf{U} := \mathbf{u} - \mathbf{H}, p) \in \mathbf{H}^1(\Omega \setminus \bar{\omega}) \times L^2(\Omega \setminus \bar{\omega})$ satisfies

$$\begin{cases} -\nu \Delta \mathbf{U} + \nabla p = \mathbf{f} + \nu \Delta \mathbf{H} & \text{in } \Omega \setminus \bar{\omega} \\ \operatorname{div} \mathbf{U} = 0 & \text{in } \Omega \setminus \bar{\omega} \\ -\nu \partial_{\mathbf{n}} \mathbf{U} + p \mathbf{n} = \mathbf{h}_{ext} + \nu \partial_{\mathbf{n}} \mathbf{H} & \text{on } O \\ \mathbf{U} = \mathbf{0} & \text{on } \partial\Omega \setminus \bar{O} \\ \mathbf{U} = \mathbf{0} & \text{on } \partial\omega. \end{cases}$$

According to Lax-Milgram's theorem, there exists a unique $\mathbf{U} \in \mathbf{V}_O(\Omega \setminus \bar{\omega})$ such that for all test functions $\mathbf{v} \in \mathbf{V}_O(\Omega \setminus \bar{\omega})$

$$\nu \int_{\Omega \setminus \bar{\omega}} \nabla \mathbf{U} : \nabla \mathbf{v} = \langle \mathbf{f}, \mathbf{v} \rangle_{\Omega \setminus \bar{\omega}} - \nu \int_{\Omega \setminus \bar{\omega}} \nabla \mathbf{H} : \nabla \mathbf{v} - \langle \mathbf{h}_O + \nu \partial_{\mathbf{n}} \mathbf{H}, \mathbf{v} \rangle_O \quad (\text{A.1.12})$$

and we have, using (A.1.11),

$$\|\mathbf{U}\|_{\mathbf{H}^1(\Omega \setminus \bar{\omega})} \leq C \left(\|\mathbf{f}\|_{[\mathbf{H}^1(\Omega \setminus \bar{\omega})]'} + \|\mathbf{h}_{int}\|_{\mathbf{H}^{1/2}(\partial\omega)} + \|\mathbf{h}_{ext}\|_{\mathbf{H}^{1/2}(\partial\Omega \setminus \bar{O})} + \|\mathbf{h}_O\|_{\mathbf{H}^{-1/2}(O)} \right). \quad (\text{A.1.13})$$

In particular (A.1.12) is true for all $\mathbf{v} \in \mathbf{V}_O(\Omega \setminus \bar{\omega}) \cap \mathbf{H}_0^1(\Omega \setminus \bar{\omega})$. Then using De Rham's theorem (see for example [13, Lemma 2.7]), there exists $p \in L^2(\Omega \setminus \bar{\omega})$, up to an additive constant, such that for all $\mathbf{v} \in \mathbf{H}_0^1(\Omega \setminus \bar{\omega})$

$$\nu \int_{\Omega \setminus \bar{\omega}} \nabla \mathbf{U} : \nabla \mathbf{v} - \int_{\Omega \setminus \bar{\omega}} p \operatorname{div} \mathbf{v} = \langle \mathbf{f}|_{\mathbf{H}_0^1(\Omega \setminus \bar{\omega})}, \mathbf{v} \rangle_{\mathbf{H}^{-1}(\Omega \setminus \bar{\omega}), \mathbf{H}_0^1(\Omega \setminus \bar{\omega})} - \nu \int_{\Omega \setminus \bar{\omega}} \nabla \mathbf{H} : \nabla \mathbf{v}. \quad (\text{A.1.14})$$

According to [14, Lemma 3.3] or [56, Theorem 3.2], we define $\varphi_N \in \mathbf{H}^1(\Omega \setminus \bar{\omega})$ such that $\operatorname{div} \varphi_N = 1$ in $\Omega \setminus \bar{\omega}$, $\varphi_N = \mathbf{0}$ on $\partial\Omega \setminus \bar{O}$ and $\varphi_N = \mathbf{0}$ on $\partial\omega$ with $\int_O \varphi_N \cdot \mathbf{n} \neq 0$. Let $\mathbf{v} \in \mathbf{H}^1(\Omega \setminus \bar{\omega})$ such that $\mathbf{v} = \mathbf{0}$ on $\partial\Omega \setminus \bar{O}$, $\mathbf{v} = \mathbf{0}$ on $\partial\omega$ and define

$$c_b(\mathbf{v}) = \frac{1}{\int_{\partial(\Omega \setminus \omega)} \varphi_N \cdot \mathbf{n}} \int_{\partial(\Omega \setminus \omega)} \mathbf{v} \cdot \mathbf{n}.$$

Using again [14, Lemma 3.3] or [56, Theorem 3.2], we define $\mathbf{v}_2 \in \mathbf{V}_O(\Omega \setminus \bar{\omega})$ in such a way that $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 + c_b(\mathbf{v})\varphi_N$, where $\mathbf{v}_1 \in \mathbf{H}_0^1(\Omega \setminus \bar{\omega})$ satisfies the following equality: $\operatorname{div} \mathbf{v}_1 = \operatorname{div}(\mathbf{v} - c_b(\mathbf{v})\varphi_N)$. Then, using (A.1.12) and (A.1.14), we obtain

$$\begin{aligned} \int_{\Omega \setminus \bar{\omega}} \nu \nabla \mathbf{U} : \nabla \mathbf{v} - \int_{\Omega \setminus \bar{\omega}} p \operatorname{div} \mathbf{v} &= \langle \mathbf{f}, \mathbf{v} \rangle_{\Omega \setminus \bar{\omega}} - \nu \int_{\Omega \setminus \bar{\omega}} \nabla \mathbf{H} : \nabla \mathbf{v} \\ &\quad - \langle \mathbf{h}_O + \nu \partial_{\mathbf{n}} \mathbf{H}, \mathbf{v} \rangle_O + \int_{\Omega \setminus \bar{\omega}} \nu \nabla \mathbf{U} : \nabla (c_b(\mathbf{v})\varphi_N) - \int_{\Omega \setminus \bar{\omega}} p \operatorname{div} (c_b(\mathbf{v})\varphi_N) \\ &\quad - \langle \mathbf{f}, c_b(\mathbf{v})\varphi_N \rangle_{\Omega \setminus \bar{\omega}} + \nu \int_{\Omega \setminus \bar{\omega}} \nabla \mathbf{H} : \nabla (c_b(\mathbf{v})\varphi_N) + \langle \mathbf{h}_O + \nu \partial_{\mathbf{n}} \mathbf{H}, c_b(\mathbf{v})\varphi_N \rangle_O. \end{aligned}$$

Therefore, choosing the additive constant for p such that

$$\int_{\Omega \setminus \bar{\omega}} p = \nu \int_{\Omega \setminus \bar{\omega}} \nabla \mathbf{U} : \nabla \varphi_N - \langle \mathbf{f}, c_b(\mathbf{v})\varphi_N \rangle_{\Omega \setminus \bar{\omega}} + \nu \int_{\Omega \setminus \bar{\omega}} \nabla \mathbf{H} : \nabla \varphi_N + \langle \mathbf{h}_O + \nu \partial_{\mathbf{n}} \mathbf{H}, c_b(\mathbf{v})\varphi_N \rangle_O,$$

we prove that there exists a unique pair $(\mathbf{U}, p) \in \mathbf{V}_O(\Omega \setminus \bar{\omega}) \times L^2(\Omega \setminus \bar{\omega})$ such that for all $\mathbf{v} \in \mathbf{H}^1(\Omega \setminus \bar{\omega})$ with $\mathbf{v} = \mathbf{0}$ on $\partial\Omega \setminus \bar{O}$ and $\mathbf{v} = \mathbf{0}$ on $\partial\omega$,

$$\int_{\Omega \setminus \bar{\omega}} \nu \nabla \mathbf{U} : \nabla \mathbf{v} - \int_{\Omega \setminus \bar{\omega}} p \operatorname{div} \mathbf{v} = \langle \mathbf{f}, \mathbf{v} \rangle_{\Omega \setminus \bar{\omega}} - \nu \int_{\Omega \setminus \bar{\omega}} \nabla \mathbf{H} : \nabla \mathbf{v} - \langle \mathbf{h}_O + \nu \partial_{\mathbf{n}} \mathbf{H}, \mathbf{v} \rangle_O. \quad (\text{A.1.15})$$

Step 2: estimate. Let $\mathbf{v} := \tilde{\mathbf{v}} + c(p)\varphi_N$, where

$$c(p) := \frac{1}{|\Omega \setminus \bar{\omega}|} \int_{\Omega \setminus \bar{\omega}} p$$

and $\tilde{\mathbf{v}} \in \mathbf{H}_0^1(\Omega \setminus \bar{\omega})$ is such that $\operatorname{div} \tilde{\mathbf{v}} = p - c(p)$ and $\|\tilde{\mathbf{v}}\|_{\mathbf{H}_0^1(\Omega \setminus \bar{\omega})} \leq C \|p\|_{L^2(\Omega \setminus \bar{\omega})}$ (see [14, Lemma 3.3]). Using $\mathbf{v}$ in (A.1.15), and according to (A.1.13), we obtain

$$\begin{aligned} \|\mathbf{U}\|_{\mathbf{H}^1(\Omega \setminus \bar{\omega})} + \|p\|_{L^2(\Omega \setminus \bar{\omega})} &\leq C \left(\|\mathbf{f}\|_{[\mathbf{H}^1(\Omega \setminus \bar{\omega})]'} + \|\mathbf{h}_{int}\|_{\mathbf{H}^{1/2}(\partial\omega)} + \|\mathbf{h}_{ext}\|_{\mathbf{H}^{1/2}(\partial\Omega \setminus \bar{O})} + \|\mathbf{h}_O\|_{\mathbf{H}^{-1/2}(O)} \right) \end{aligned}$$

and hence

$$\begin{aligned} \|\mathbf{u}\|_{\mathbf{H}^1(\Omega \setminus \bar{\omega})} + \|p\|_{L^2(\Omega \setminus \bar{\omega})} &\leq C \left(\|\mathbf{f}\|_{[\mathbf{H}^1(\Omega \setminus \bar{\omega})]'} + \|\mathbf{h}_{int}\|_{\mathbf{H}^{1/2}(\partial\omega)} + \|\mathbf{h}_{ext}\|_{\mathbf{H}^{1/2}(\partial\Omega \setminus \bar{O})} + \|\mathbf{h}_O\|_{\mathbf{H}^{-1/2}(O)} \right). \end{aligned}$$

Step 3: case $g \neq 0$. The first part of the theorem is proved for $g = 0$. The case $g \neq 0$ is obtained by a lifting argument. Let us define

$$\mathbf{u}_g := \widetilde{\mathbf{u}}_g + c(g)\varphi_N - \frac{1}{|\Omega \setminus \overline{\omega}|} \int_{\Omega \setminus \overline{\omega}} (\widetilde{\mathbf{u}}_g + c(g)\varphi_N) \in \mathbf{H}^1(\Omega \setminus \overline{\omega}),$$

where

$$c(g) := \frac{1}{|\Omega \setminus \overline{\omega}|} \int_{\Omega \setminus \overline{\omega}} g$$

and where, according to Theorem 3.2 in [56] (or Lemma 3.3 in [14]), $\widetilde{\mathbf{u}}_g \in \mathbf{H}_0^1(\Omega \setminus \overline{\omega})$ is such that $\operatorname{div} \widetilde{\mathbf{u}}_g = g - c(g)$ and $\|\widetilde{\mathbf{u}}_g\|_{\mathbf{H}_0^1(\Omega \setminus \overline{\omega})} \leq C\|g\|_{L^2(\Omega \setminus \overline{\omega})}$. Thus $\mathbf{u}_g$ is such that

$$\int_{\Omega \setminus \overline{\omega}} \mathbf{u}_g = \mathbf{0}, \quad \operatorname{div} \mathbf{u}_g = g \quad \text{in } \Omega \setminus \overline{\omega} \quad \text{and} \quad \|\mathbf{u}_g\|_{\mathbf{H}^1(\Omega \setminus \overline{\omega})} \leq C\|g\|_{L^2(\Omega \setminus \overline{\omega})}.$$

Thus, defining $\mathbf{u}_0 := \mathbf{u} - \mathbf{u}_g$ and

$$\widetilde{\mathbf{f}} : \mathbf{v} \in \mathbf{H}^1(\Omega \setminus \overline{\omega}) \mapsto \langle \mathbf{f}, \mathbf{v} \rangle_{\Omega \setminus \overline{\omega}} - \nu \int_{\Omega \setminus \overline{\omega}} \nabla \mathbf{H} : \nabla \mathbf{v} - \nu \int_{\Omega \setminus \overline{\omega}} \nabla \mathbf{u}_g : \nabla \mathbf{v},$$

the problem (A.1.10) is equivalent to

$$\left\{ \begin{array}{l} \text{Find } (\mathbf{u}_0, p) \in \mathbf{V}_O(\Omega \setminus \overline{\omega}) \times L^2(\Omega \setminus \overline{\omega}) \text{ such that} \\ \int_{\Omega \setminus \overline{\omega}} \nu \nabla \mathbf{u}_0 : \nabla \mathbf{v} - \int_{\Omega \setminus \overline{\omega}} p \operatorname{div} \mathbf{v} = \langle \widetilde{\mathbf{f}}, \mathbf{v} \rangle_{\Omega \setminus \overline{\omega}} - \langle \mathbf{h}_O + \nu \partial_n \mathbf{H}, \mathbf{v} \rangle_O \\ \forall \mathbf{v} \in \mathbf{H}^1(\Omega \setminus \overline{\omega}), \mathbf{v} = \mathbf{0} \text{ on } \partial\omega, \mathbf{v} = \mathbf{0} \text{ on } \partial\Omega \setminus \overline{O}. \end{array} \right.$$

Then we proceed in the same manner as the case $g = 0$. □

Let us now state and prove a local regularity result on the solution of the Stokes problem with mixed boundary conditions.

Theorem A.1.6 (Local regularity result). *Let $k \in \mathbb{N}$, $\nu > 0$, Ω a bounded Lipschitz open set of $\mathbb{R}^d$ ($d \in \mathbb{N}^*$) and ω an open set with a $C^{k+1,1}$ boundary such that $\omega \subset \subset \Omega$ and $\Omega \setminus \overline{\omega}$ is connected. Let O be an open subset of the exterior boundary $\partial\Omega$ and $\nu > 0$. Let $\mathcal{C}$ and $\mathcal{C}'$ two smooth open subsets of $\Omega \setminus \overline{\omega}$ such that $\partial\omega \subset \partial\mathcal{C}$, $\partial\omega \subset \partial\mathcal{C}'$, $\overline{\mathcal{C}} \setminus \partial\omega \subset \mathcal{C}'$ and $\overline{\mathcal{C}'} \subset \Omega$. Let*

$$(\mathbf{f}, g, \mathbf{h}_O, \mathbf{h}_{ext}, \mathbf{h}_{int}) \in \mathbf{X}^{*,k}(\Omega \setminus \overline{\omega}, \mathcal{C}') \times X^{0,k+1}(\Omega \setminus \overline{\omega}, \mathcal{C}') \times \mathbf{H}^{-1/2}(O) \times \mathbf{H}^{1/2}(\partial\Omega \setminus \overline{O}) \times \mathbf{H}^{k+\frac{1}{2}}(\partial\omega).$$

We consider $(\mathbf{u}, p) \in \mathbf{H}^1(\Omega \setminus \overline{\omega}) \times L^2(\Omega \setminus \overline{\omega})$ the solution of the following Stokes problem

$$\left\{ \begin{array}{ll} -\nu \Delta \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega \setminus \overline{\omega} \\ \operatorname{div} \mathbf{u} &= g \quad \text{in } \Omega \setminus \overline{\omega} \\ -\nu \partial_n \mathbf{u} + p \mathbf{n} &= \mathbf{h}_O \quad \text{on } O \\ \mathbf{u} &= \mathbf{h}_{ext} \quad \text{on } \partial\Omega \setminus \overline{O} \\ \mathbf{u} &= \mathbf{h}_{int} \quad \text{on } \partial\omega. \end{array} \right. \quad (\text{A.1.16})$$

Then $(\mathbf{u}, p)$ belongs to $\mathbf{H}^{k+2}(\mathcal{C}) \times \mathbf{H}^{k+1}(\mathcal{C})$ and the following estimate holds:

$$\begin{aligned} \|\mathbf{u}\|_{\mathbf{H}^{k+2}(\mathcal{C})} + \|p\|_{\mathbf{H}^{k+1}(\mathcal{C})} &\leq C \left(\|\mathbf{f}\|_{\mathbf{X}^{*,k}(\Omega \setminus \overline{\omega}, \mathcal{C}')} + \|g\|_{X^{0,k+1}(\Omega \setminus \overline{\omega}, \mathcal{C}')} \right. \\ &\quad \left. + \|\mathbf{h}_O\|_{\mathbf{H}^{-1/2}(O)} + \|\mathbf{h}_{ext}\|_{\mathbf{H}^{1/2}(\partial\Omega \setminus \overline{O})} + \|\mathbf{h}_{int}\|_{\mathbf{H}^{k+\frac{1}{2}}(\partial\omega)} \right). \end{aligned}$$

A.2 Résultats concernant les équations de Navier-Stokes et Navier-Stokes linéarisées

Proof. First, let us consider the case $k = 0$. We define $\mathcal{V} = \mathcal{C} \cup \bar{\omega}$, $\mathcal{V}' = \mathcal{C}' \cup \bar{\omega}$ and $\varphi \in C_c^\infty(\Omega)$ such that $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ in $\mathcal{V}$ and $\varphi = 0$ in $\Omega \setminus \mathcal{V}'$. Let

$$(\mathbf{u}, p) \in \mathbf{H}^1(\Omega \setminus \bar{\omega}) \times L^2(\Omega \setminus \bar{\omega})$$

the solution of problem (A.1.16) given by Theorem A.1.5. Using (A.1.16) we check

$$\begin{cases} -\nu \Delta(\varphi \mathbf{u}) + \nabla(\varphi p) &= \tilde{\mathbf{f}} & \text{in } \mathcal{C}' \\ \operatorname{div}(\varphi \mathbf{u}) &= \tilde{g} & \text{in } \mathcal{C}' \\ \varphi \mathbf{u} &= \mathbf{0} & \text{on } \partial \mathcal{V}' \\ \varphi \mathbf{u} &= \mathbf{h}_{int} & \text{on } \partial \omega, \end{cases}$$

where $\tilde{\mathbf{f}} := \varphi \mathbf{f} - \nu \mathbf{u} \Delta \varphi - 2\nu \nabla \mathbf{u} \nabla \varphi + p \nabla \varphi$ belongs to $\mathbf{L}^2(\mathcal{C}')$ and $\tilde{g} := \varphi g + \mathbf{u} \cdot \nabla \varphi$ belongs to $H^1(\mathcal{C}')$. From the regularity of the solutions of the Stokes equations with Dirichlet boundary conditions (see for example [31] or [56]), $\varphi \mathbf{u} \in \mathbf{H}^2(\mathcal{C}')$, $\varphi p \in H^1(\mathcal{C}')$ and from the expression of $\tilde{\mathbf{f}}$ and $\tilde{g}$ we obtain

$$\begin{aligned} \|\varphi \mathbf{u}\|_{\mathbf{H}^2(\mathcal{C}')} + \|\varphi p\|_{H^1(\mathcal{C}')} &\leq C \left(\|\varphi \mathbf{f}\|_{\mathbf{L}^2(\mathcal{C}')} + \|\varphi g\|_{H^1(\mathcal{C}')} \right. \\ &\quad \left. + \|\mathbf{u}\|_{\mathbf{H}^1(\Omega \setminus \bar{\omega})} + \|p\|_{L^2(\Omega \setminus \bar{\omega})} + \|\mathbf{h}_{int}\|_{\mathbf{H}^{\frac{1}{2}}(\partial \omega)} \right). \end{aligned}$$

Using this inequality and the estimate on $\|\mathbf{u}\|_{\mathbf{H}^1(\Omega \setminus \bar{\omega})}$ and $\|p\|_{L^2(\Omega \setminus \bar{\omega})}$ given by Theorem A.1.5, we obtain the announced estimate for $k = 0$. We then proceed by induction for the cases $k \geq 1$. $\square$

A.2 Some results concerning the Navier-Stokes and the linearized Navier-Stokes equations

A.2.1 Some results concerning the Navier-Stokes equations

We denote by $b(\cdot, \cdot, \cdot)$ the trilinear continuous form on $\mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega)$ defined by

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \sum_{i,j=1}^N \int_{\Omega} \mathbf{u}_i \frac{\partial \mathbf{v}_j}{\partial x_i} \mathbf{w}_j.$$

We first recall a classical existence theorem picked out in [61, Theorem IV.2.3]. We also refer for example to [44, 56, 97] for classical results concerning the Navier-Stokes equations and to [15] for the L^p theory.

Theorem A.2.1 (Existence of a solution. Girault, Raviart, [61]). *Let $d = 2, 3$, Ω a bounded connected Lipschitz open set of $\mathbb{R}^d$, Γ_i , $i = 1, \dots, I$, the connected components of $\partial \Omega$ and $\nu > 0$. Let $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$ and $\mathbf{g} \in \mathbf{H}^{1/2}(\partial \Omega)$ satisfying the condition*

$$\int_{\Gamma_i} \mathbf{g} \cdot \mathbf{n} = 0 \quad \forall i = 1, \dots, I. \quad (\text{A.2.1})$$

Then, the Navier-Stokes problem

$$\begin{cases} -\nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p &= \mathbf{f} & \text{in } \Omega \\ \operatorname{div} \mathbf{u} &= 0 & \text{in } \Omega \\ \mathbf{u} &= \mathbf{g} & \text{on } \partial \Omega \end{cases} \quad (\text{A.2.2})$$

admits at least a solution $(\mathbf{u}, p) \in \mathbf{H}^1(\Omega) \times L_0^2(\Omega)$.

In order to prove a regularity result, we first recall a classical corollary of the Sobolev imbeddings and we deduce a practical result.

Theorem A.2.2. *Assume that Ω is a bounded Lipschitz open subset of $\mathbb{R}^d$. Let $s_1, s_2 \in \mathbb{R}^+$ and $s \in [-1, +\infty[$ such that $s_1 \geq s$, $s_2 \geq s$ and either*

$$s_1 + s_2 - s \geq \frac{d}{2}, \quad s_i - s > 0 \quad i = 1, 2$$

or

$$s_1 + s_2 - s > \frac{d}{2}, \quad s_i - s \geq 0 \quad i = 1, 2.$$

Then $(u, v) \mapsto u \cdot v$ is a continuous bilinear map from $H^{s_1}(\Omega) \times H^{s_2}(\Omega)$ into $H^s(\Omega)$.

Proof. The usual result is given for $m_1 = s_1$, $m_2 = s_2$ and $m = s + 1$ integers (see [61, Corollary I.1.1]) but we deduce this result with an interpolation argument (see for example [24, Theorem 6.4.5]). $\square$

It leads the following result:

Corollary A.2.3. *Let Ω a bounded Lipschitz open subset of $\mathbb{R}^d$ ($d = 1, 2, 3$) and $s \in \mathbb{R}^+$. The mapping*

$$(u, v) \in H^{1+s}(\Omega) \times H^s(\Omega) \mapsto u \cdot v \in H^{s-1/2}(\Omega)$$

is bilinear continuous. In particular, the mapping

$$\mathbf{u} \in \mathbf{H}^{1+s} \mapsto (\mathbf{u} \cdot \nabla) \mathbf{u} \in \mathbf{H}^{s-1/2}(\Omega)$$

is continuous.

Corollary A.2.4 (Regularity of the solution). *Let Ω a bounded connected open set of $\mathbb{R}^d$ ($d = 2, 3$) with a $C^{k,1}$ boundary ($k \in \mathbb{N}$) and $\nu > 0$. Let $\mathbf{f} \in \mathbf{H}^{k-1}(\Omega)$ and $\mathbf{g} \in \mathbf{H}^{k+1/2}(\partial\Omega)$ satisfying the condition (A.2.1). Then Problem (A.2.2) admits a solution $(\mathbf{u}, p) \in \mathbf{H}^{k+1}(\Omega) \times (H^k(\Omega) \cap L_0^2(\Omega))$.*

Proof. We will use the regularity of the solutions of the Stokes equations and proceed by induction. We know using Theorem A.2.1 that the result is true for $k = 0$.

Let us assume that the result is true for one $k \in \mathbb{N}$. Then $\mathbf{u} \in \mathbf{H}^{k+1}(\Omega)$ and we have

$$\begin{cases} -\nu \Delta \mathbf{u} + \nabla p &= \mathbf{f} - (\mathbf{u} \cdot \nabla) \mathbf{u} & \text{in } \Omega \\ \operatorname{div} \mathbf{u} &= 0 & \text{in } \Omega \\ \mathbf{u} &= \mathbf{g} & \text{on } \partial\Omega. \end{cases}$$

Let us prove that $(\mathbf{u} \cdot \nabla) \mathbf{u} \in \mathbf{H}^k(\Omega)$. Since $\mathbf{u} \in \mathbf{H}^{k+1}(\Omega)$, then, according to Corollary A.2.3, $(\mathbf{u} \cdot \nabla) \mathbf{u}$ belongs to $\mathbf{H}^{k-1/2}(\Omega)$. Therefore, using the regularity of the solutions of the Stokes equations (see for example [14, Theorem 4.8]), $\mathbf{u} \in \mathbf{H}^{k+3/2}(\Omega)$. The same argument gives $(\mathbf{u} \cdot \nabla) \mathbf{u} \in \mathbf{H}^k(\Omega)$ and $(\mathbf{u}, p) \in \mathbf{H}^{k+2}(\Omega) \times H^{k+1}(\Omega)$ which concludes the proof. $\square$

Let us recall a uniqueness result (see [56, Theorem VIII.2.1]).

Theorem A.2.5 (Unicity of the solution. Galdi, [56]). *We consider the problem given in Theorem A.2.1. If $\nu > c_b(\Omega) \|\nabla \mathbf{u}\|_{L^2(\Omega)}$ where*

$$c_b(\Omega) := \begin{cases} \frac{|\Omega|^{1/2}}{2} & \text{if } d = 2 \\ \frac{2\sqrt{2}|\Omega|^{1/6}}{3} & \text{if } d = 3, \end{cases} \quad (\text{A.2.3})$$

then the solution given by Corollary A.2.4 is unique.

A.2 Résultats concernant les équations de Navier-Stokes et Navier-Stokes linéarisées

Let us prove the following local regularity result:

Theorem A.2.6 (Local regularity for the Navier-Stokes equations). *Let $k \in \mathbb{N}^*$, $\nu > 0$, Ω a bounded connected Lipschitz open set of $\mathbb{R}^d$ ($d = 2, 3$) and $\omega \subset\subset \Omega$ an open set with a $C^{k,1}$ boundary such that $\Omega \setminus \bar{\omega}$ is connected. Let $\mathcal{C}$ and $\mathcal{C}'$ two smooth open subsets of $\Omega \setminus \bar{\omega}$ such that $\partial\omega \subset \partial\mathcal{C}$, $\partial\omega \subset \partial\mathcal{C}'$, $\bar{\mathcal{C}} \setminus \partial\omega \subset \mathcal{C}'$ and $\bar{\mathcal{C}'} \subset \Omega$. Let*

$$(\mathbf{f}, \mathbf{h}_{ext}, \mathbf{h}_{int}) \in \mathbf{X}^{-1,k-1}(\Omega \setminus \bar{\omega}, \mathcal{C}') \times \mathbf{H}^{1/2}(\partial\Omega) \times \mathbf{H}^{k+1/2}(\partial\omega),$$

such that

$$\int_{\partial\Omega} \mathbf{h}_{ext} \cdot \mathbf{n} + \int_{\partial\omega} \mathbf{h}_{int} \cdot \mathbf{n} = 0.$$

Then every solution $(\mathbf{u}, p) \in \mathbf{H}^1(\Omega \setminus \bar{\omega}) \times L_0^2(\Omega \setminus \bar{\omega})$ of the Navier-Stokes problem

$$\begin{cases} -\nu\Delta\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p &= \mathbf{f} & \text{in } \Omega \setminus \bar{\omega} \\ \operatorname{div} \mathbf{u} &= 0 & \text{in } \Omega \setminus \bar{\omega} \\ \mathbf{u} &= \mathbf{h}_{ext} & \text{on } \partial\Omega \\ \mathbf{u} &= \mathbf{h}_{int} & \text{on } \partial\omega \end{cases}$$

belongs to $\mathbf{H}^{k+1}(\mathcal{C}) \times \mathbf{H}^k(\mathcal{C})$.

Proof. We will use the local regularity of the solutions of the Stokes equations (see for example [56, Theorem IV.5.1]).

First let us consider the case $k = 1$. We then proceed by induction for $k > 1$. We have

$$\begin{cases} -\nu\Delta\mathbf{u} + \nabla p &= \mathbf{f} - (\mathbf{u} \cdot \nabla)\mathbf{u} & \text{in } \Omega \\ \operatorname{div} \mathbf{u} &= 0 & \text{in } \Omega \\ \mathbf{u} &= \mathbf{g} & \text{on } \partial\Omega. \end{cases}$$

Let us prove that $(\mathbf{u} \cdot \nabla)\mathbf{u} \in \mathbf{X}^{-1,0}(\Omega \setminus \bar{\omega}, \mathcal{C}')$. Since $\mathbf{u} \in \mathbf{H}^1(\Omega \setminus \bar{\omega})$, then, according to Corollary A.2.3, $(\mathbf{u} \cdot \nabla)\mathbf{u} \in \mathbf{H}^{-1/2}(\Omega \setminus \bar{\omega})$. Therefore, using the local regularity of the solutions of the Stokes equations, $\mathbf{u} \in \mathbf{X}^{1,3/2}(\Omega \setminus \bar{\omega}, \mathcal{V})$, where $\mathcal{V}$ is a smooth open subset of $\Omega \setminus \bar{\omega}$ such that $\mathcal{C} \subset\subset \mathcal{V} \subset\subset \mathcal{C}'$. Therefore $\mathbf{u} \in \mathbf{X}^{1,3/2}(\Omega \setminus \bar{\omega}, \mathcal{V})$. The same argument gives $(\mathbf{u} \cdot \nabla)\mathbf{u} \in \mathbf{X}^{-1/2,0}(\Omega \setminus \bar{\omega}, \mathcal{V})$ and $(\mathbf{u}, p)$ belongs to $\mathbf{X}^{1,2}(\Omega \setminus \bar{\omega}, \mathcal{C}) \times \mathbf{X}^{0,1}(\Omega \setminus \bar{\omega}, \mathcal{C})$. $\square$

A.2.2 Some results concerning the linearized Navier-Stokes equations

Theorem A.2.7 (Existence and uniqueness of the solution of the linearized Navier-Stokes equations). *Let Ω a bounded connected Lipschitz open set of $\mathbb{R}^d$ ($d = 2, 3$). Let us consider $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$, $g \in L^2(\Omega)$ and $\mathbf{h} \in \mathbf{H}^{1/2}(\partial\Omega)$ such that*

$$\int_{\Omega} g = \int_{\partial\Omega} \mathbf{h} \cdot \mathbf{n}.$$

Let $\mathbf{v} \in \mathbf{H}^1(\Omega)$ such that $\nu > c_b(\Omega)\|\mathbf{v}\|_{\mathbf{H}^1(\Omega)}$, where $c_b(\Omega)$ is defined by (A.2.3). Then the problem

$$\begin{cases} -\nu\Delta\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{u} + \nabla p &= \mathbf{f} & \text{in } \Omega \\ \operatorname{div} \mathbf{u} &= g & \text{in } \Omega \\ \mathbf{u} &= \mathbf{h} & \text{on } \partial\Omega \end{cases} \quad (\text{A.2.4})$$

admits a unique solution $(\mathbf{u}, p) \in \mathbf{H}^1(\Omega) \times L_0^2(\Omega)$ and there exists a constant C (which depends on d , Ω , ν , and $\mathbf{v}$) such that

$$\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} + \|p\|_{L^2(\Omega)} \leq C \left(\|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} + \|g\|_{L^2(\Omega)} + \|\mathbf{h}\|_{\mathbf{H}^{1/2}(\partial\Omega)} \right). \quad (\text{A.2.5})$$

Proof. Using a lifting argument (see [14, Lemma 3.3]), existence and uniqueness of the solution of Problem (A.2.4) reduce to existence and uniqueness of the solution of the homogeneous problem. We then use Lax-Milgram's Theorem and finally De Rham's Theorem (see for example [14, Lemma 2.7]) to obtain the corresponding pressure p . $\square$

Theorem A.2.8 (Local regularity for the linearized Navier-Stokes equations). *Let $k \in \mathbb{N}^*$, $\nu > 0$, Ω a bounded connected Lipschitz open set of $\mathbb{R}^d$ ($d = 2, 3$) and $\omega \subset\subset \Omega$ an open set with a $C^{k,1}$ boundary such that $\Omega \setminus \bar{\omega}$ is connected. Let $\mathcal{C}$ and $\mathcal{C}'$ two smooth open subsets of $\Omega \setminus \bar{\omega}$ such that $\partial\omega \subset \partial\mathcal{C}$, $\partial\omega \subset \partial\mathcal{C}'$, $\bar{\mathcal{C}} \setminus \partial\omega \subset \mathcal{C}'$ and $\bar{\mathcal{C}}' \subset \Omega$. Let*

$$(\mathbf{f}, g, \mathbf{h}_{ext}, \mathbf{h}_{int}) \in \mathbf{X}^{-1,k-1}(\Omega \setminus \bar{\omega}, \mathcal{C}') \times X^{0,k}(\Omega \setminus \bar{\omega}, \mathcal{C}') \times \mathbf{H}^{1/2}(\partial\Omega) \times \mathbf{H}^{k+1/2}(\partial\omega),$$

such that

$$\int_{\Omega \setminus \bar{\omega}} g = \int_{\partial\Omega} \mathbf{h}_{ext} \cdot \mathbf{n} + \int_{\partial\omega} \mathbf{h}_{int} \cdot \mathbf{n}.$$

Let $\mathbf{v} \in \mathbf{H}^1(\Omega \setminus \bar{\omega})$ such that $\mathbf{v}|_{\mathcal{C}'} \in \mathbf{H}^{k+1}(\mathcal{C}')$. Every solution $(\mathbf{u}, p) \in \mathbf{H}^1(\Omega \setminus \bar{\omega}) \times L_0^2(\Omega \setminus \bar{\omega})$ of the linearized Navier-Stokes problem

$$\begin{cases} -\nu \Delta \mathbf{u} + (\mathbf{v} \cdot \nabla) \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{v} + \nabla p = \mathbf{f} & \text{in } \Omega \setminus \bar{\omega} \\ \operatorname{div} \mathbf{u} = g & \text{in } \Omega \setminus \bar{\omega} \\ \mathbf{u} = \mathbf{h}_{ext} & \text{on } \partial\Omega \\ \mathbf{u} = \mathbf{h}_{int} & \text{on } \partial\omega \end{cases} \quad (\text{A.2.6})$$

belongs to $\mathbf{H}^{k+1}(\mathcal{C}) \times \mathbf{H}^k(\mathcal{C})$. Moreover, if $\nu > c_b(\Omega \setminus \bar{\omega}) \|\mathbf{v}\|_{\mathbf{H}^1(\Omega \setminus \bar{\omega})}$ (where $c_b(\Omega \setminus \bar{\omega})$ is defined analogously to (A.2.3)), then the solution is unique and there exists a constant C (which depends on d , Ω , ν , and $\mathbf{v}$) such that

$$\begin{aligned} \|\mathbf{u}\|_{\mathbf{H}^{k+1}(\mathcal{C})} + \|p\|_{\mathbf{H}^k(\mathcal{C})} &\leq C \left(\|\mathbf{f}\|_{\mathbf{X}^{-1,k-1}(\Omega \setminus \bar{\omega}, \mathcal{C}')} + \|g\|_{X^{0,k}(\Omega \setminus \bar{\omega}, \mathcal{C}')} \right. \\ &\quad \left. + \|\mathbf{h}_{int}\|_{\mathbf{H}^{k+1/2}(\partial\omega)} + \|\mathbf{h}_{ext}\|_{\mathbf{H}^{1/2}(\partial\Omega)} \right). \end{aligned}$$

Proof. The proof of the regularity result is similar to the proof of Theorem A.2.6. Moreover, if $\nu > c_b(\Omega \setminus \bar{\omega}) \|\mathbf{v}\|_{\mathbf{H}^1(\Omega \setminus \bar{\omega})}$, then according to Theorem A.2.7, the solution is unique and we have the estimate (A.2.5). Thus using the local estimate on the solution of the Stokes equations (as in Theorem A.1.6 for example or [56, Theorem IV.5.1]), we obtain by induction the announced estimate. $\square$

Remark A.2.9. Theorems A.2.7 and A.2.8 are also true replacing the first equation of Systems (A.2.4) and (A.2.6) (i.e. $-\nu \Delta \mathbf{u} + (\mathbf{v} \cdot \nabla) \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{v} + \nabla p = \mathbf{f}$) by

$$-\nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{v} - (\mathbf{v} \cdot \nabla) \mathbf{u} + \nabla p = \mathbf{f}.$$

Remark A.2.10. Concerning Theorem A.2.8, we can prove the following sharper estimate

$$\|\varphi \mathbf{u}\|_{\mathbf{H}^{k+1}(\mathcal{C}')} + \|\varphi p\|_{\mathbf{H}^k(\mathcal{C}')} \leq C \left(\|\mathbf{f}\|_{\mathbf{X}^{-1,k-1}(\Omega \setminus \bar{\omega}, \mathcal{C}')} + \|\mathbf{h}_{int}\|_{\mathbf{H}^{k+1/2}(\partial\omega)} + \|\mathbf{h}_{ext}\|_{\mathbf{H}^{1/2}(\partial\Omega)} \right) \quad (\text{A.2.7})$$

where $\varphi \in C_c^\infty(\Omega)$ is a truncation function such that $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ in $\mathcal{C} \cup \bar{\omega}$ and $\varphi \equiv 0$ in $\Omega \setminus (\mathcal{C}' \cup \bar{\omega})$. The proof relies on the fact that $(\varphi \mathbf{u}, \varphi p)$ solves

$$\begin{cases} -\nu \Delta(\varphi \mathbf{u}) + \nabla(\varphi p) = \tilde{\mathbf{f}} & \text{in } \mathcal{C}' \\ \operatorname{div}(\varphi \mathbf{u}) = \varphi g + \mathbf{u} \cdot \nabla \varphi & \text{in } \mathcal{C}' \\ \varphi \mathbf{u} = \mathbf{0} & \text{on } \partial\mathcal{V}' \\ \varphi \mathbf{u} = \mathbf{h}_{int} & \text{on } \partial\omega, \end{cases}$$

where $\tilde{\mathbf{f}} := \varphi \mathbf{f} - \nu \mathbf{u} \Delta \varphi - 2\nu \nabla \mathbf{u} \nabla \varphi + p \nabla \varphi + (\mathbf{u} \otimes \nabla \varphi) \mathbf{v} - (\varphi \mathbf{u} \cdot \nabla) \mathbf{v} - (\mathbf{v} \cdot \nabla) \varphi \mathbf{u}$ (see for instance [56, Theorem IV.5.1] or Theorem A.1.6).

A.3 Some results concerning the exterior Stokes problem

First, we recall the definition of the weighted Sobolev spaces. We introduce the weight function $\rho(x) := (2 + |x|^2)^{1/2}$ and the following Sobolev spaces (for more details, see [5]):

Definition A.3.1. Let $1 < p < \infty$. For each real number α and each open set $\mathcal{O} \subset \mathbb{R}^d$, we set

$$\begin{aligned} \mathbf{L}_\alpha^p(\mathcal{O}) &:= \{u \in \mathcal{D}'(\mathcal{O}), \rho^\alpha u \in \mathbf{L}^p(\mathcal{O})\}, \\ \mathbf{W}_\alpha^{1,p}(\mathcal{O}) &:= \begin{cases} \{u \in \mathcal{D}'(\mathcal{O}), u \in \mathbf{L}_{\alpha-1}^p(\mathcal{O}), \nabla u \in \mathbf{L}_\alpha^p(\mathcal{O})\} & \text{if } \frac{d}{p} + \alpha \neq 1 \\ \{u \in \mathcal{D}'(\mathcal{O}), (\ln(\rho))^{-1}u \in \mathbf{L}_{\alpha-1}^p(\mathcal{O}), \nabla u \in \mathbf{L}_\alpha^p(\mathcal{O})\} & \text{if } \frac{d}{p} + \alpha = 1. \end{cases} \end{aligned}$$

Consider now the space $\overset{\circ}{\mathbf{W}}_\alpha^{1,p}(\mathcal{O}) := \overline{\mathcal{D}(\mathcal{O})}^{\|\cdot\|_{\mathbf{W}_\alpha^{1,p}(\mathcal{O})}}$. It is standard to check that

$$\overset{\circ}{\mathbf{W}}_\alpha^{1,p}(\mathcal{O}) = \{v \in \mathbf{W}_\alpha^{1,p}(\mathcal{O}), v|_{\partial\mathcal{O}} = 0\}.$$

The dual space of $\overset{\circ}{\mathbf{W}}_\alpha^{1,p}(\mathcal{O})$ is denoted by $\mathbf{W}_{-\alpha}^{-1,p'}(\mathcal{O})$, where p' is such that $\frac{1}{p} + \frac{1}{p'} = 1$ (it is a subspace of $\mathcal{D}'(\mathcal{O})$).

Notice that these spaces are reflexive Banach spaces with respect to the norms:

$$\begin{aligned} \|u\|_{\mathbf{L}_\alpha^p(\mathcal{O})} &:= \|\rho^\alpha u\|_{\mathbf{L}^p(\mathcal{O})}, \\ \|u\|_{\mathbf{W}_\alpha^{1,p}(\mathcal{O})} &:= \begin{cases} \left(\|u\|_{\mathbf{L}_{\alpha-1}^p(\mathcal{O})}^p + \|\nabla u\|_{\mathbf{L}_\alpha^p(\mathcal{O})}^p \right)^{1/p} & \text{if } \frac{d}{p} + \alpha \neq 1 \\ \left(\left\| \frac{u}{\ln(\rho)} \right\|_{\mathbf{L}_{\alpha-1}^p(\mathcal{O})}^p + \|\nabla u\|_{\mathbf{L}_\alpha^p(\mathcal{O})}^p \right)^{1/p} & \text{if } \frac{d}{p} + \alpha = 1. \end{cases} \end{aligned}$$

We also recall the following theorem extracted from [62, Theorem 3.4 and Remark 3.4] which guarantees the existence and uniqueness of the solution of an exterior Stokes problem with Dirichlet boundary conditions:

Theorem A.3.2 (Girault et al., [62]). Let $\mathcal{O}$ be a Lipschitz-continuous exterior domain. For each $\mathbf{f} \in \mathbf{W}_0^{-1,2}(\mathcal{O})$, $g \in \mathbf{L}^2(\mathcal{O})$ and for each $\boldsymbol{\varphi} \in \mathbf{H}^{1/2}(\partial\mathcal{O})$, the problem

$$\begin{cases} -\nu \Delta \mathbf{u} + \nabla \pi &= \mathbf{f} & \text{in } \mathcal{O} \\ \operatorname{div} \mathbf{u} &= g & \text{in } \mathcal{O} \\ \mathbf{u} &= \boldsymbol{\varphi} & \text{on } \partial\mathcal{O} \end{cases}$$

admits a unique solution $(\mathbf{u}, \pi) \in \mathbf{W}_0^{1,2}(\mathcal{O}) \times \mathbf{L}^2(\mathcal{O})$. Moreover, there exists a positive constant C depending on d , $\mathcal{O}$ and ν such that

$$\|\mathbf{u}\|_{\mathbf{W}_0^{1,2}(\mathcal{O})} + \|\pi\|_{\mathbf{L}^2(\mathcal{O})} \leq C \left(\|\mathbf{f}\|_{\mathbf{W}_0^{-1,2}(\mathcal{O})} + \|g\|_{\mathbf{L}^2(\mathcal{O})} + \|\boldsymbol{\varphi}\|_{\mathbf{H}^{1/2}(\mathcal{O})} \right).$$

Finally we recall that $E(y) = \frac{1}{8\pi\nu \|y\|} (\mathbf{I} + \mathbf{e}_r {}^t \mathbf{e}_r)$ with $\mathbf{e}_r := \frac{y}{\|y\|}$ is the Green function associated to the Stokes system (see (IV.2.1)). For an integrable vector $\boldsymbol{\varphi}$ on $\partial\omega$, we introduce the following integral operator (called simple layer hydrodynamic potential operator) for all $y \notin \partial\omega$:

$$S\boldsymbol{\varphi}(y) := \int_{\partial\omega} E(y-x) \boldsymbol{\varphi}(x) \mathrm{d}s(x).$$

Now we claim a theorem extracted from [48, §XI.B.5, Theorem 1, page 697–698]:

Theorem A.3.3. *Let ω an open bounded open set of $\mathbb{R}^d$ ($d = 2, 3$). Let $\mathbf{g} \in \mathbf{H}^{1/2}(\partial\omega)$ such that $\int_{\partial\omega} \mathbf{g} \cdot \mathbf{n} = 0$. We consider $(\mathbf{w}, s) \in \mathbf{W}_0^{1,2}(\mathbb{R}^3 \setminus \bar{\omega}) \times L^2(\mathbb{R}^3 \setminus \bar{\omega})$ the solution of the Stokes exterior problem*

$$\begin{cases} -\nu \Delta \mathbf{w} + \nabla s &= \mathbf{0} & \text{in } \mathbb{R}^3 \setminus \bar{\omega} \\ \operatorname{div} \mathbf{w} &= 0 & \text{in } \mathbb{R}^3 \setminus \bar{\omega} \\ \mathbf{w} &= \mathbf{g} & \text{on } \partial\omega. \end{cases}$$

Hence the velocity $\mathbf{w}$ can be written thanks to the representation formula with hydrodynamical potential

$$\mathbf{w}(y) = S\boldsymbol{\eta}(y) = \int_{\partial\omega} E(y-x)\boldsymbol{\eta}(x)ds(x), \quad y \in \mathbb{R}^3 \setminus \bar{\omega},$$

where $\boldsymbol{\eta} \in \mathbf{H}^{-1/2}(\partial\omega)^3/\mathcal{R}$ is the unique solution of

$$S\boldsymbol{\eta}(y) = \mathbf{g}(y) \quad \forall y \in \partial\omega.$$

Here $\mathcal{R}$ is the equivalence relation $t \sim t'$ if $t - t' = \lambda \mathbf{n}$, $\lambda \in \mathbb{R}$.

Moreover S is an isomorphism from $\mathbf{H}^{-1/2}(\partial\omega)/\mathcal{R}$ into $\left\{ \mathbf{g} \in \mathbf{H}^{1/2}(\partial\omega), \int_{\partial\omega} \mathbf{g} \cdot \mathbf{n} = 0 \right\}$.

Annexe : Résultats sur la dérivation par rapport au domaine

B.1 Quelques formalités

Nous énonçons ici des résultats généraux principalement extraits du livre de Henrot et Pierre [72, Chapitre 5].

Lemme B.1.1. *Soit $\boldsymbol{\theta} \in \mathbf{W}^{1,\infty}(\mathbb{R}^d)$ tel que $\|\boldsymbol{\theta}\|_{1,\infty} < 1$. Alors*

- (i) $\mathbf{I} + \boldsymbol{\theta}$ est inversible et $(\mathbf{I} + \boldsymbol{\theta})^{-1} \in \mathbf{W}^{1,\infty}(\mathbb{R}^d)$,
- (ii) l'application $\boldsymbol{\theta} \in \mathbf{W}^{1,\infty}(\mathbb{R}^d) \mapsto (\mathbf{I} + \boldsymbol{\theta})^{-1} \in \mathbf{W}^{1,\infty}(\mathbb{R}^d)$ est continue en $\mathbf{0}$,
- (iii) l'application $\boldsymbol{\theta} \in \mathbf{W}^{1,\infty}(\mathbb{R}^d) \mapsto (\mathbf{I} + \boldsymbol{\theta})^{-1} \in \mathbf{L}^\infty$ est différentiable en $\mathbf{0}$ de différentielle $-\mathbf{I}$.

Démonstration. (i) Soit $\boldsymbol{\theta} \in \mathbf{W}^{1,\infty}(\mathbb{R}^d)$ tel que $\|\boldsymbol{\theta}\|_{1,\infty} < 1$ et soit $x, y \in \mathbb{R}^d$ tels que

$$(\mathbf{I} + \boldsymbol{\theta})(x) = y \iff x = y - \boldsymbol{\theta}(x) =: f_y(x). \quad (\text{B.1.1})$$

Comme $x \mapsto -\boldsymbol{\theta}(x)$ est lipschitzienne de rapport strictement inférieur à 1, alors f_y l'est aussi (*i.e.* f_y est contractante). Ainsi d'après le théorème de point fixe de Banach, f_y admet un unique point fixe, *i.e.*, (B.1.1) admet une unique solution, *i.e.* $\mathbf{I} + \boldsymbol{\theta}$ est inversible.

Comme $(\mathbf{I} + \boldsymbol{\theta})^{-1} = \sum_{q \geq 0} (-1)^q \boldsymbol{\theta}^q$, il vient que $(\mathbf{I} + \boldsymbol{\theta})^{-1} \in \mathbf{W}^{1,\infty}(\mathbb{R}^d)$.

- (ii) On a $(\mathbf{I} + \boldsymbol{\theta})^{-1} - \mathbf{I} = -\boldsymbol{\theta} \sum_{k \geq 0} (-1)^k \boldsymbol{\theta}^k$. D'où

$$\|(\mathbf{I} + \boldsymbol{\theta})^{-1} - \mathbf{I}\|_{1,\infty} \leq \|\boldsymbol{\theta}\|_{1,\infty} \sum_{k \geq 0} \|\boldsymbol{\theta}\|_{1,\infty}^k = \|\boldsymbol{\theta}\|_{1,\infty} (1 - \|\boldsymbol{\theta}\|_{1,\infty})^{-1}.$$

Ainsi il vient que l'application $\boldsymbol{\theta} \in \mathbf{W}^{1,\infty}(\mathbb{R}^d) \mapsto (\mathbf{I} + \boldsymbol{\theta})^{-1} \in \mathbf{W}^{1,\infty}(\mathbb{R}^d)$ est continue en $\mathbf{0}$.

- (iii) On a $(\mathbf{I} + \boldsymbol{\theta})^{-1} - \mathbf{I} + \boldsymbol{\theta} = -\boldsymbol{\theta} \sum_{k \geq 1} (-1)^k \boldsymbol{\theta}^k = -\boldsymbol{\theta} \left(\sum_{k \geq 0} (-1)^k \boldsymbol{\theta}^k - \mathbf{I} \right)$. D'où

$$\|(\mathbf{I} + \boldsymbol{\theta})^{-1} - \mathbf{I} + \boldsymbol{\theta}\|_\infty \leq \|\boldsymbol{\theta}\|_\infty \|(\mathbf{I} + \boldsymbol{\theta})^{-1} - \mathbf{I}\|_\infty.$$

Ainsi il vient que l'application $\boldsymbol{\theta} \in \mathbf{W}^{1,\infty}(\mathbb{R}^d) \mapsto (\mathbf{I} + \boldsymbol{\theta})^{-1} \in \mathbf{L}^\infty$ est différentiable en $\mathbf{0}$ de différentielle $-\mathbf{I}$.

□

Lemme B.1.2. *L'application $\theta \in \mathbf{W}^{2,\infty}(\mathbb{R}^d) \mapsto (\mathbf{I} + \theta)^{-1} \in \mathbf{W}^{1,\infty}(\mathbb{R}^d)$ est différentiable en $\mathbf{0}$ de différentielle $-\mathbf{I}$.*

Démonstration. On procède comme dans la preuve du (iii) du Lemme B.1.1 précédent. On a, pour $\theta \in \mathbf{W}^{2,\infty}(\mathbb{R}^d)$ tel que $\|\theta\|_{2,\infty} < 1$, $(\mathbf{I} + \theta)^{-1} - \mathbf{I} + \theta = -\theta((\mathbf{I} + \theta)^{-1} - \mathbf{I})$. D'où

$$\|(\mathbf{I} + \theta)^{-1} - \mathbf{I} + \theta\|_{1,\infty} \leq \|\theta\|_{1,\infty} \|(\mathbf{I} + \theta)^{-1} - \mathbf{I}\|_{1,\infty}.$$

Ainsi il vient que l'application $\theta \in \mathbf{W}^{2,\infty}(\mathbb{R}^d) \mapsto (\mathbf{I} + \theta)^{-1} \in \mathbf{W}^{1,\infty}(\mathbb{R}^d)$ est différentiable en $\mathbf{0}$ de différentielle $-\mathbf{I}$. $\square$

Lemme B.1.3. *Soit $T > 0$. Soit une application $\phi : [0, T[\rightarrow \mathbf{W}^{1,\infty}(\mathbb{R}^d)$ dérivable en 0 avec $\phi(0) = \mathbf{I}$ et $\phi'(0) =: \mathbf{V}$. Alors pour t voisin de 0 , $\phi(t)$ inversible et*

- (i) *l'application $t \in [0, T[\mapsto \phi(t)^{-1} \in \mathbf{W}^{1,\infty}(\mathbb{R}^d)$ est continue en $\mathbf{0}$,*
- (ii) *l'application $t \in [0, T[\mapsto \phi(t)^{-1} \in \mathbf{L}^\infty(\mathbb{R}^d)$ est dérivable en $\mathbf{0}$ de dérivée $-\mathbf{V}$.*

Démonstration. Pour t voisin de 0 , $\phi(t)$ est voisin de $\mathbf{I}$. Posons $\tilde{\theta} = \phi(t) - \mathbf{I}$. Ainsi le Lemme B.1.1 appliqué à $\tilde{\theta}$ donne le résultat. $\square$

Lemme B.1.4. *Tout $\Psi \in \mathbf{W}^{1,\infty}(\mathbb{R}^d)$ tel que $\|\Psi\|_{1,\infty} < 1$ vérifie les deux propriétés suivantes :*

- (i) *Ψ est inversible et $\Psi^{-1} \in \mathbf{W}^{1,\infty}(\mathbb{R}^d)$,*
- (ii) *$j := \det(D\Psi^{-1})$ (c'est-à-dire le jacobien) est borné.*

Démonstration. Soit $\Psi \in \mathbf{W}^{1,\infty}(\mathbb{R}^d)$ tel que $\|\Psi\|_{1,\infty} < 1$.

- (i) On pose $\tilde{\Psi} := \Psi - \mathbf{I}$. On a alors $\tilde{\Psi} \in \mathbf{W}^{1,\infty}(\mathbb{R}^d)$ tel que $\|\tilde{\Psi}\|_{1,\infty} < 1$. Ainsi, en appliquant le Lemme B.1.1 (i) à $\theta = \tilde{\Psi}$, on a bien $\mathbf{I} + \tilde{\Psi} = \Psi$ inversible et $\Psi \in \mathbf{W}^{1,\infty}(\mathbb{R}^d)$.
- (ii) D'après le (i) démontré ci-dessus, on a $\Psi^{-1} \in \mathbf{W}^{1,\infty}(\mathbb{R}^d)$, donc en particulier $D(\Psi^{-1})$ est borné. Comme $\det$ est une application polynomiale, il vient que $\det(D(\Psi^{-1}))$ est borné.

$\square$

Lemme B.1.5. *Soit $k \geq 1$ entier.*

- (i) *Si $\theta \in \mathbf{W}^{1,\infty}(\mathbb{R}^d)$ et $f \in L^2(\Omega_\theta)$, alors $f \circ (\mathbf{I} + \theta) \in L^2(\Omega)$.*
- (ii) *Si $\theta \in \mathbf{W}^{k,\infty}$ et $f \in H^k(\Omega_\theta)$, alors $f \circ (\mathbf{I} + \theta) \in H^k(\Omega)$.*
- (iii) *Si $\theta \in \mathbf{W}^{k,\infty}$ et $f \in H_0^k(\Omega_\theta)$, alors $f \circ (\mathbf{I} + \theta) \in H_0^k(\Omega)$.*

Démonstration. La démonstration de (i) et (ii) se fait par changement de variables en utilisant le fait que le jacobien ainsi que $D^k\theta$ sont bornés.

Le résultat (iii) vient du fait que $(\mathbf{I} + \theta)(\partial\Omega) = \partial\Omega_\theta$. Ce dernier point est vérifié car $\mathbf{I} + \theta$ est un homéomorphisme donc est une application ouverte et fermée. $\square$

B.2 Des résultats de différentiabilité

Le résultat suivant, extrait de [72], est un résultat général de différentiabilité :

Corollaire B.2.1 (Lemme 5.3.3 dans [72]). *Soit $g \in W^{1,p}(\mathbb{R}^N)$ avec $1 \leq p < +\infty$. L'application*

$$\begin{aligned} G : \mathbf{W}^{1,\infty} &\longrightarrow L^p(\mathbb{R}^N) \\ \theta &\longmapsto g \circ (\mathbf{I} + \theta) \end{aligned}$$

est de classe C^1 sur un voisinage de θ , et on a

$$\forall \xi \in \mathbf{W}^{1,\infty}(\mathbb{R}^d) \quad G'(\theta) \cdot \xi = (\nabla g \circ (\mathbf{I} + \theta)) \cdot \xi.$$

Plus généralement, si l'application $\theta \in \mathbf{W}^{1,\infty} \mapsto \Psi(\theta) \in \mathbf{W}^{1,\infty}$ est continu en 0 avec $\Psi(0) = \mathbf{I}$ et si l'application $\theta \in \mathbf{W}^{1,\infty} \mapsto (g(\theta), \Psi(\theta)) \in \mathbf{L}^p \times \mathbf{L}^\infty$ est différentiable en 0 avec $g(0) \in \mathbf{W}^{1,p}$ et $g'(0) : \mathbf{W}^{1,\infty} \rightarrow \mathbf{W}^{1,p}$ continu, alors l'application

$$\mathcal{G} : \theta \in \mathbf{W}^{1,\infty} \mapsto g(\theta) \circ \Psi(\theta) \in \mathbf{L}^p$$

est différentiable en 0 et on a

$$\forall \xi \in \mathbf{W}^{1,\infty}(\mathbb{R}^d) \quad \mathcal{G}'(\theta) \cdot \xi = g'(0) \xi + \nabla g(0) \cdot \Psi'(\theta) \xi.$$

Le lemme suivant, de nouveau extrait de [72], donne un résultat de différentiabilité d'ordre plus élevé (en particulier d'ordre deux) :

Lemme B.2.2 (Lemme 5.3.9 dans [72]). *Soit $m \geq 0$ et $k \geq 1$ entiers.*

(i) *Soit $g \in H^{m+k}(\mathbb{R}^N)$. Alors l'application*

$$\theta \in \mathbf{W}^{max(m,1),\infty} \mapsto g \circ (\mathbf{I} + \theta) \in H^m(\mathbb{R}^N)$$

est de classe C^k sur un voisinage de 0.

(ii) *Soit $\xi \in \mathbf{W}^{max(m,1),\infty}$ et une application $t \in [0, T[\mapsto G(t) \in H^{m+k}(\mathbb{R}^N)$ de classe C^k . Alors l'application*

$$t \in [0, T[\mapsto G(t) \circ (\mathbf{I} + t\xi)^{-1} \in H^m(\mathbb{R}^N)$$

est de classe C^k sur un voisinage de 0.

(iii) *Soit $\xi \in \mathbf{W}^{max(m,1),\infty}$ et une application $t \in [0, T[\mapsto G(t) \in H^{m+k}(\mathbb{R}^N)$ k -fois dérivable en 0. Alors l'application*

$$t \in [0, T[\mapsto G(t) \circ (\mathbf{I} + t\xi)^{-1} \in H^m(\mathbb{R}^N)$$

est k -fois dérivable en 0.

B.3 Un résultat de dérivation sur un domaine variable

Le résultat suivant permet de dériver l'application $t \mapsto I(t) = \int_{\Omega_t} f(t, x)$ lorsque $f(t, \cdot)$ est seulement définie sur un domaine variable Ω_t .

Corollaire B.3.1 (Corollaire 5.2.5 dans [72]). *Soit une application $\phi : [0, T[\rightarrow \mathbf{W}^{1,\infty}(\mathbb{R}^d)$ dérivable en 0 telle que $\phi(0) = \mathbf{I}$ et $\phi'(0) =: \mathbf{V}$. Soit Ω un ouvert de $\mathbb{R}^N$ et une application $t \mapsto f(t) \in L^1(\Omega_t)$. Si l'application*

$$t \in [0, T[\mapsto F(t) := f(t, \phi(t, \cdot)) \in L^1(\Omega)$$

est dérivable en 0 et si $f(0) \in W^{1,1}(\Omega)$, alors

$$- t \in [0, T[\mapsto I(t) := \int_{\Omega_t} f(t) \text{ est dérivable en 0,}$$

$$- \text{ pour tout compact } K \subset \Omega, \text{ l'application } t \in [0, T[\mapsto f(t)|_K \in L^1(K) \text{ est dérivable en 0 avec}$$

$$f'(0) = F'(0) - \nabla f(0) \cdot \mathbf{V} \in L^1(\Omega),$$

$$- I'(0) = \int_{\Omega} f'(0) + \operatorname{div}(f(0) \mathbf{V}).$$

B.4 Des résultats de dérivation sur un bord variable

Commençons par rappeler quelques notions de géométrie différentielles :

Définition B.4.1. Pour $\Omega \subset \mathbb{R}^N$ un borné de bord Γ , $g \in C^1(\Gamma)$, $\mathbf{W} \in \mathbf{C}^1(\Gamma)$ et $\mathbf{u} \in \mathbf{W}^{2,1}(\Gamma)$, on définit sur Γ

- $\nabla_\Gamma g := \nabla \tilde{g} - (\nabla \tilde{g} \cdot \mathbf{n})\mathbf{n}$,
- $\operatorname{div}_\Gamma \mathbf{W} := \operatorname{div} \tilde{\mathbf{W}} - [\tilde{\mathbf{W}}'] \mathbf{n} \cdot \mathbf{n}$,
- $\mathbf{W}_\Gamma := \tilde{\mathbf{W}} - (\tilde{\mathbf{W}} \cdot \mathbf{n})\mathbf{n}$,
- $\Delta_\Gamma \mathbf{u} := \operatorname{div}_\Gamma (\nabla_\Gamma \mathbf{u})$,

où $\tilde{g} \in C^1(\mathbb{R}^N)$ et $\tilde{\mathbf{W}} \in \mathbf{C}^1(\mathbb{R}^N)$ sont des prolongements respectivement de g et de $\mathbf{W}$, où $\mathbf{n}$ est la normale unitaire extérieure à Γ et où $[\tilde{\mathbf{W}}']$ est la matrice jacobienne de $\tilde{\mathbf{W}}$.

Si de plus Ω est de classe C^2 , on définit $H := \operatorname{div}_\Gamma \mathbf{n} \in C^0(\Gamma)$, appelée courbure moyenne de Ω .

Remarque B.4.2. (i) Ces définitions sont indépendantes des prolongements.

(ii) On étend ces définitions à $g \in W^{1,1}(\Gamma)$ et $\mathbf{W} \in \mathbf{W}^{1,1}(\mathbb{R}^N)$.

Remarque B.4.3. Pour $\mathbf{u}, \mathbf{v} \in \mathbf{W}^{1,1}(\Gamma)$, on a

$$\begin{aligned} \nabla_\Gamma \mathbf{u} \cdot \nabla_\Gamma \mathbf{v} &= \nabla \mathbf{u} \cdot \nabla \mathbf{v} - \partial_{\mathbf{n}} \mathbf{u} \cdot \nabla \mathbf{v} \\ &= \nabla \mathbf{u} \cdot \nabla \mathbf{v} - \partial_{\mathbf{n}} \mathbf{u} \cdot \nabla \mathbf{v} + \partial_{\mathbf{n}} \mathbf{u} \cdot \partial_{\mathbf{n}} \mathbf{v} \mathbf{n} \\ &= \nabla \mathbf{u} \cdot \nabla \mathbf{v}. \end{aligned}$$

Et de même on a $\nabla_\Gamma \mathbf{u} \cdot \nabla_\Gamma \mathbf{v} = \nabla_\Gamma \mathbf{u} \cdot \nabla \mathbf{v}$.

Théorème B.4.4 (Proposition 5.4.9 dans [72]). Soit Ω un borné de bord Γ et de classe C^2 . Pour tout $f \in W^{1,1}(\Gamma)$ et tout $\mathbf{W} \in \mathbf{W}^{1,1}(\Gamma)$, on a

- $\operatorname{div}_\Gamma (f \mathbf{n}) = H f$,
- $\operatorname{div}_\Gamma (\mathbf{W}) = \operatorname{div}_\Gamma \mathbf{W}_\Gamma + H \mathbf{n} \cdot \mathbf{W}$,
- $\int_\Gamma \operatorname{div}_\Gamma \mathbf{W} = \int_\Gamma H \mathbf{n} \cdot \mathbf{W}$,
- $\int_\Gamma \mathbf{W} \cdot \nabla_\Gamma f = \int_\Gamma -f \operatorname{div}_\Gamma \mathbf{W} + H f \mathbf{W} \cdot \mathbf{n}$.

Théorème B.4.5 (Proposition 5.4.12 dans [72]). Soit Ω un ouvert borné de bord Γ et de classe C^2 . Soit $u : \bar{\Omega} \rightarrow \mathbb{R}$ de classe C^2 . Alors on a

$$\Delta u = \Delta_\Gamma u + H \partial_{\mathbf{n}} u + \frac{\partial^2 u}{\partial \mathbf{n}^2} \text{ sur } \Gamma.$$

Remarque B.4.6. Ce théorème s'étend par densité à $u \in H^3(\Omega)$.

Rappelons maintenant quelques formules d'intégration par parties pour des intégrales de bord.

Théorème B.4.7 (Théorème 5.4.13 dans [72]). Soit Ω un ouvert borné de bord Γ et de classe C^2 . Pour $f, g \in H^2(\Omega)$, on a

$$\int_\Gamma \frac{\partial f}{\partial x_i} g = - \int_\Gamma f \frac{\partial g}{\partial x_i} + \int_\Gamma \left(\frac{\partial}{\partial \mathbf{n}} (f g) + H f g \right) n_i.$$

Si $f \in H^2(\Omega)$ et $g \in H^3(\Omega)$, alors on a

$$\int_\Gamma \nabla f \cdot \nabla g = - \int_\Gamma f \Delta g + \int_\Gamma \left(\frac{\partial f}{\partial \mathbf{n}} \frac{\partial g}{\partial \mathbf{n}} + f \frac{\partial^2 g}{\partial \mathbf{n}^2} + H f \frac{\partial g}{\partial \mathbf{n}} \right),$$

et

$$\int_\Gamma \nabla_\Gamma f \cdot \nabla_\Gamma g = - \int_\Gamma f \Delta_\Gamma g.$$

B.5 Un dernier résultat concernant l'espace de traces

On rappelle ici un résultat concernant la trace des fonctions, en particulier lorsque les domaines dépendent d'un paramètre (voir [83, Chapter 4]) :

Théorème B.5.1 ([83] Section 4.1.3. page 214). *Soit Ω et ω deux ouverts bornés simplement connexes de $\mathbb{R}^d$ ($d \geq 2$) de classe $C^{0,1}$. Soit $p \in (1, +\infty)$, $\varepsilon \in (0, 1/2)$ et $\omega_\varepsilon := \varepsilon\omega$. Supposons que $\overline{\omega_\varepsilon} \subset \Omega$ et qu'il existe une constante $c > 0$ dépendant seulement de d, p, ω et Ω telle que $d(\omega_\varepsilon, \partial\Omega) > c\varepsilon$. Alors*

$$\langle \cdot \rangle_{p, \partial\omega_\varepsilon} \sim a(\varepsilon) \|\cdot\|_{L^p(\partial\omega_\varepsilon)} + [\cdot]_{p, \partial\omega_\varepsilon}$$

où

$$\begin{aligned} \langle f \rangle_{p, \partial\omega_\varepsilon} &:= \inf \left\{ \|u\|_{W^{1,p}(\Omega \setminus \overline{\omega_\varepsilon})}, u \in W^{1,p}(\Omega \setminus \overline{\omega_\varepsilon}), u|_{\partial\omega_\varepsilon} = f \right\}, \\ a(\varepsilon) &:= \begin{cases} \varepsilon^{\frac{1-N}{p}} \min(1, \varepsilon^{\frac{N}{p}-1}), & \text{pour } p < d \\ \varepsilon^{\frac{1-N}{p}} \min(1, |\log \varepsilon|^{\frac{1-p}{p}}), & \text{pour } p = d \\ \varepsilon^{\frac{1-N}{p}}, & \text{pour } p > d, \end{cases} \end{aligned}$$

et

$$\begin{aligned} [f]_{1, \partial\omega_\varepsilon} &:= |\partial\omega_\varepsilon|^{-1} \iint_{\partial\omega_\varepsilon \times \partial\omega_\varepsilon} |f(x) - f(y)| \, ds(x) ds(y) \\ [f]_{p, \partial\omega_\varepsilon} &:= \left(\iint_{\partial\omega_\varepsilon \times \partial\omega_\varepsilon} \frac{|f(x) - f(y)|^p}{|x - y|^{N+p-2}} ds(x) ds(y) \right)^{1/p} \quad \text{pour } p \in (1, +\infty). \end{aligned}$$

Références bibliographiques

- [1] M. Abdelwahed and M. Hassine. Topological optimization method for a geometric control problem in Stokes flow. *Appl. Numer. Math.*, 59(8):1823–1838, 2009.
- [2] R. A. Adams and J. J. F. Fournier. *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, second edition, 2003.
- [3] L. Afraites, M. Dambrine, K. Eppler, and D. Kateb. Detecting perfectly insulated obstacles by shape optimization techniques of order two. *Discrete Contin. Dyn. Syst. Ser. B*, 8(2):389–416 (electronic), 2007.
- [4] L. Afraites, M. Dambrine, and D. Kateb. On second order shape optimization methods for electrical impedance tomography. *SIAM J. Control Optim.*, 47(3):1556–1590, 2008.
- [5] F. Alliot and C. Amrouche. Weak solutions for the exterior Stokes problem in weighted Sobolev spaces. *Math. Methods Appl. Sci.*, 23(6):575–600, 2000.
- [6] C. Alvarez, C. Conca, L. Friz, O. Kavian, and J. H. Ortega. Identification of immersed obstacles via boundary measurements. *Inverse Problems*, 21(5):1531–1552, 2005.
- [7] C. J. S. Alves, R. Kress, and A. L. Silvestre. Integral equations for an inverse boundary value problem for the two-dimensional Stokes equations. *J. Inverse Ill-Posed Probl.*, 15(5):461–481, 2007.
- [8] H. Ammari, E. Beretta, E. Francini, H. Kang, and M. Lim. Reconstruction of small interface changes of an inclusion from modal measurements II: the elastic case. *J. Math. Pures Appl. (9)*, 94(3):322–339, 2010.
- [9] H. Ammari, R. Griesmaier, and M. Hanke. Identification of small inhomogeneities: asymptotic factorization. *Math. Comp.*, 76(259):1425–1448 (electronic), 2007.
- [10] H. Ammari and H. Kang. Reconstruction of elastic inclusions of small volume via dynamic measurements. *Appl. Math. Optim.*, 54(2):223–235, 2006.
- [11] H. Ammari and H. Kang. The method of small-volume expansions for medical imaging. In *Mathematical modeling in biomedical imaging. I*, volume 1983 of *Lecture Notes in Math.*, pages 99–132. Springer, Berlin, 2009.
- [12] H. Ammari, H. Kang, and H. Lee. Asymptotic expansions for eigenvalues of the Lamé system in the presence of small inclusions. *Comm. Partial Differential Equations*, 32(10-12):1715–1736, 2007.

Références bibliographiques

- [13] C. Amrouche and V. Girault. Problèmes généralisés de Stokes. *Portugal. Math.*, 49(4):463–503, 1992.
- [14] C. Amrouche and V. Girault. Decomposition of vector spaces and application to the Stokes problem in arbitrary dimension. *Czechoslovak Math. J.*, 44(119)(1):109–140, 1994.
- [15] C. Amrouche and M. Á. Rodríguez-Bellido. Stationary Stokes, Oseen and Navier-Stokes equations with singular data. *Arch. Ration. Mech. Anal.*, 199(2):597–651, 2011.
- [16] S. Amstutz. The topological asymptotic for the Navier-Stokes equations. *ESAIM Control Optim. Calc. Var.*, 11(3):401–425 (electronic), 2005.
- [17] S. Amstutz, M. Masmoudi, and B. Samet. The topological asymptotic for the Helmholtz equation. *SIAM J. Control Optim.*, 42(5):1523–1544 (electronic), 2003.
- [18] M. Badra, F. Caubet, and M. Dambrine. Detecting an obstacle immersed in a fluid by shape optimization methods. *Math. Models Methods Appl. Sci.*, 21(10):2069–2101, 2011.
- [19] A. Ballerini. Stable determination of an immersed body in a stationary Stokes fluid. *Inverse Problems*, 26(12):125015, 25, 2010.
- [20] A. Ballerini. Stable determination of a body immersed in a fluid: the nonlinear stationary case. *Appl. Anal.*, to appear.
- [21] J. A. Bello, E. Fernández Cara, and J. Simon. Optimal shape design for Navier-Stokes flow. In *System modelling and optimization (Zurich, 1991)*, volume 180 of *Lecture Notes in Control and Inform. Sci.*, pages 481–489. Springer, Berlin, 1992.
- [22] J. A. Bello, E. Fernández Cara, and J. Simon. The variation of the drag with respect to the domain in Navier-Stokes flow. In *Optimization, optimal control and partial differential equations (Iasi, 1992)*, volume 107 of *Internat. Ser. Numer. Math.*, pages 287–296. Birkhäuser, Basel, 1992.
- [23] A. Ben Abda, M. Hassine, M. Jaoua, and M. Masmoudi. Topological sensitivity analysis for the location of small cavities in Stokes flow. *SIAM J. Control Optim.*, 48(5):2871–2900, 2009/10.
- [24] J. Bergh and J. Löfström. *Interpolation spaces. An introduction*. Springer-Verlag, Berlin, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.
- [25] V. Bonnaillie-Noël, M. Dambrine, S. Tordeux, and G. Vial. Interactions between moderately close inclusions for the Laplace equation. *Math. Models Methods Appl. Sci.*, 19(10):1853–1882, 2009.
- [26] V. Bonnaillie-Noël, D. Brancherie, M. Dambrine, F. Hérau, S. Tordeux, and G. Vial. Multiscale expansion and numerical approximation for surface defects. *ESAIM: Proceedings*, 33:22–35, 2011.
- [27] M. Bonnet. Topological sensitivity for 3D elastodynamic and acoustic inverse scattering in the time domain. *Comput. Methods Appl. Mech. Engrg.*, 195(37-40):5239–5254, 2006.
- [28] M. Bonnet and B. B. Guzina. Sounding of finite solid bodies by way of topological derivative. *Internat. J. Numer. Methods Engrg.*, 61(13):2344–2373, 2004.
- [29] M. Bonnivard. Influence des perturbations géométriques de domaines sur les solutions d'équations aux dérivées partielles. *Thesis*, 2010. Université de Grenoble.
- [30] M. Bonnivard. On the stability of self-propelled bodies with respect to their shape motion. *Math. Models Methods Appl. Sci.*, 21(4):667–691, 2011.
- [31] F. Boyer and P. Fabrie. *Éléments d'analyse pour l'étude de quelques modèles d'écoulements de fluides visqueux incompressibles*, volume 52 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer-Verlag, Berlin, 2006.

- [32] M. Brühl, M. Hanke, and M. S. Vogelius. A direct impedance tomography algorithm for locating small inhomogeneities. *Numer. Math.*, 93(4):635–654, 2003.
- [33] D. Bucur and G. Buttazzo. *Variational methods in shape optimization problems*. Progress in Nonlinear Differential Equations and their Applications, 65. Birkhäuser Boston Inc., Boston, MA, 2005.
- [34] A. Carpio and M. L. Rapún. Topological derivatives for shape reconstruction. In *Inverse problems and imaging*, volume 1943 of *Lecture Notes in Math.*, pages 85–133. Springer, Berlin, 2008.
- [35] F. Caubet. Detecting an obstacle immersed in a fluid: the Stokes case. *Jaca: Proceedings*, 35:91–101, 2010.
- [36] F. Caubet. Instability of an inverse problem for the stationary Navier-Stokes equations. *HAL-00696174*, 2012.
- [37] F. Caubet and Dambrine. Localisation of small obstacles in Stokes flow. *HAL-00678037*, 2012. (submitted).
- [38] F. Caubet, M. Dambrine, D. Kateb, and C. Z. Timimoun. A Kohn-Vogelius formulation to detect an obstacle immersed in a fluid. *HAL-00678036*, 2012. (submitted).
- [39] J. C  a. Conception optimale ou identification de formes: calcul rapide de la d  riv  e directionnelle de la fonction co  t. *RAIRO Mod  l. Math. Anal. Num  r.*, 20(3):371–402, 1986.
- [40] J. C  a, S. Garreau, P. Guillaume, and M. Masmoudi. The shape and topological optimizations connection. *Comput. Methods Appl. Mech. Engrg.*, 188(4):713–726, 2000. IV WCCM, Part II (Buenos Aires, 1998).
- [41] C. Conca, P. Cumsille, J. Ortega, and L. Rosier. Detecting a moving obstacle in an ideal fluid by a boundary measurement. *C. R. Math. Acad. Sci. Paris*, 346(15-16):839–844, 2008.
- [42] C. Conca, M. Malik, and A. Munier. Detection of a moving rigid solid in a perfect fluid. *Inverse Problems*, 26(9):095010, 18, 2010.
- [43] C. Conca, E. L. Schwindt, and T. Takahashi. On the identifiability of a rigid body moving in a stationary viscous fluid. *Inverse Problems*, 28(1):015005, 22, 2012.
- [44] P. Constantin and C. Foias. *Navier-Stokes equations*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1988.
- [45] M. Dambrine. On variations of the shape Hessian and sufficient conditions for the stability of critical shapes. *RACSAM Rev. R. Acad. Cienc. Exactas F  s. Nat. Ser. A Mat.*, 96(1):95–121, 2002.
- [46] M. Dambrine and G. Vial. A multiscale correction method for local singular perturbations of the boundary. *M2AN Math. Model. Numer. Anal.*, 41(1):111–127, 2007.
- [47] R. Dautray and J.-L. Lions. *Analyse math  matique et calcul num  rique pour les sciences et les techniques. Vol. 4*. INSTN: Collection Enseignement. [INSTN: Teaching Collection]. Masson, Paris, 1988. M  thodes variationnelles. [Variational methods], With the collaboration of Michel Artola, Marc Authier, Michel Cessenat, Jean Michel Combes, Bertrand Mercier and Claude Wild, Reprinted from the 1984 edition.
- [48] R. Dautray and J.-L. Lions. *Analyse math  matique et calcul num  rique pour les sciences et les techniques. Vol. 6*. INSTN: Collection Enseignement. [INSTN: Teaching Collection]. Masson, Paris, 1988. M  thodes int  grales et num  riques. [Integral and numerical methods], With the collaboration of Michel Artola, Philippe B  nilan, Michel Bernadou, Michel Cessenat, Jean-Claude N  d  lec and Jacques Planchard, Reprinted from the 1984 edition.

Références bibliographiques

- [49] A. Doubova, E. Fernández-Cara, M. González-Burgos, and J. Ortega. Uniqueness and partial identification in a geometric inverse problem for the Boussinesq system. *C. R. Math. Acad. Sci. Paris*, 342(9):665–670, 2006.
- [50] A. Doubova, E. Fernández-Cara, M. González-Burgos, and J. H. Ortega. A geometric inverse problem for the Boussinesq system. *Discrete Contin. Dyn. Syst. Ser. B*, 6(6):1213–1238, 2006.
- [51] A. Doubova, E. Fernández-Cara, and J. H. Ortega. On the identification of a single body immersed in a Navier-Stokes fluid. *European J. Appl. Math.*, 18(1):57–80, 2007.
- [52] R. Dziri, M. Moubachir, and J.-P. Zolésio. Dynamical shape gradient for the Navier-Stokes system. *C. R. Math. Acad. Sci. Paris*, 338(2):183–186, 2004.
- [53] M. Engliš and J. Peetre. A Green’s function for the annulus. *Ann. Mat. Pura Appl. (4)*, 171:313–377, 1996.
- [54] K. Eppler and H. Harbrecht. A regularized Newton method in electrical impedance tomography using shape Hessian information. *Control Cybernet.*, 34(1):203–225, 2005.
- [55] C. Fabre and G. Lebeau. Prolongement unique des solutions de l’équation de Stokes. *Comm. Partial Differential Equations*, 21(3-4):573–596, 1996.
- [56] G. P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations.*, volume 38 of *Springer Tracts in Natural Philosophy*. Springer-Verlag, New York, 1994. Linearized steady problems.
- [57] Z. Gao, Y. Ma, and H. Zhuang. Shape optimization for Stokes flow. *Appl. Numer. Math.*, 58(6):827–844, 2008.
- [58] Z. M. Gao, Y. C. Ma, and H. W. Zhuang. Shape optimization for Navier-Stokes flow. *Inverse Probl. Sci. Eng.*, 16(5):583–616, 2008.
- [59] S. Garreau, P. Guillaume, and M. Masmoudi. The topological asymptotic for PDE systems: the elasticity case. *SIAM J. Control Optim.*, 39(6):1756–1778 (electronic), 2001.
- [60] S. Garreau, P. Guillaume, and M. Masmoudi. The topological asymptotic for PDE systems: the elasticity case. *SIAM J. Control Optim.*, 39(6):1756–1778 (electronic), 2001.
- [61] V. Girault and P.-A. Raviart. *Finite element methods for Navier-Stokes equations*, volume 5 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1986. Theory and algorithms.
- [62] V. Girault and A. Sequeira. A well-posed problem for the exterior Stokes equations in two and three dimensions. *Arch. Rational Mech. Anal.*, 114(4):313–333, 1991.
- [63] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 24 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1985.
- [64] G. Grubb and V. A. Solonnikov. Boundary value problems for the nonstationary Navier-Stokes equations treated by pseudo-differential methods. *Math. Scand.*, 69(2):217–290 (1992), 1991.
- [65] P. Guillaume and K. Sid Idris. The topological asymptotic expansion for the Dirichlet problem. *SIAM J. Control Optim.*, 41(4):1042–1072 (electronic), 2002.
- [66] P. Guillaume and K. Sid Idris. Topological sensitivity and shape optimization for the Stokes equations. *SIAM J. Control Optim.*, 43(1):1–31 (electronic), 2004.
- [67] J. Hadamard. Mémoire sur le problème d’analyse relatif à l’équilibre des plaques élastiques encastrées. *Paris : Imprimerie nationale*, 1909.

- [68] M. Hassine. Shape optimization for the Stokes equations using topological sensitivity analysis. *ARIMA*, 5:216–229, 2006.
- [69] M. Hassine and M. Masmoudi. The topological asymptotic expansion for the quasi-Stokes problem. *ESAIM Control Optim. Calc. Var.*, 10(4):478–504 (electronic), 2004.
- [70] F. Hecht. Finite Element Library FREEFEM++. <http://www.freefem.org/ff++/>.
- [71] H. Heck, G. Uhlmann, and J.-N. Wang. Reconstruction of obstacles immersed in an incompressible fluid. *Inverse Probl. Imaging*, 1(1):63–76, 2007.
- [72] A. Henrot and M. Pierre. *Variation et optimisation de formes*, volume 48 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer, Berlin, 2005. Une analyse géométrique. [A geometric analysis].
- [73] A. Henrot and Y. Privat. Une conduite cylindrique n’est pas optimale pour minimiser l’énergie dissipée par un fluide. *C. R. Math. Acad. Sci. Paris*, 346(19-20):1057–1061, 2008.
- [74] A. Henrot and Y. Privat. What is the optimal shape of a pipe? *Arch. Ration. Mech. Anal.*, 196(1):281–302, 2010.
- [75] F. Hettlich. Fréchet derivatives in inverse obstacle scattering. *Inverse Problems*, 11(2):371–382, 1995.
- [76] F. Hettlich and W. Rundell. The determination of a discontinuity in a conductivity from a single boundary measurement. *Inverse Problems*, 14(1):67–82, 1998.
- [77] G. C. Hsiao and W. L. Wendland. *Boundary integral equations*, volume 164 of *Applied Mathematical Sciences*. Springer-Verlag, Berlin, 2008.
- [78] D. Martin. Finite Element Library MÉLINA. <http://homepage.mac.com/danielmartin/melina/>.
- [79] M. Masmoudi, J. Pommier, and B. Samet. The topological asymptotic expansion for the Maxwell equations and some applications. *Inverse Problems*, 21(2):547–564, 2005.
- [80] V. Maz’ya and A. Movchan. Asymptotic treatment of perforated domains without homogenization. *Math. Nachr.*, 283(1):104–125, 2010.
- [81] V. Maz’ya, S. Nazarov, and B. Plamenevskij. *Asymptotic theory of elliptic boundary value problems in singularly perturbed domains. Vol. I*, volume 111 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 2000. Translated from the German by Georg Heinig and Christian Posthoff.
- [82] V. Maz’ya and T. Shaposhnikova. *Theory of multipliers in spaces of differentiable functions*, volume 23 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1985.
- [83] V. G. Maz’ya and S. V. Poborchii. *Differentiable functions on bad domains*. World Scientific Publishing Co. Inc., River Edge, NJ, 1997.
- [84] F. Murat and J. Simon. *Sur le contrôle par un domaine géométrique*. Rapport du L.A. 189, 1976. Université de Paris VI, France.
- [85] J. Nocedal and S. J. Wright. *Numerical optimization*. Springer Series in Operations Research and Financial Engineering. Springer, New York, second edition, 2006.

Références bibliographiques

- [86] O. A. Oleĭnik, A. S. Shamaev, and G. A. Yosifian. *Mathematical problems in elasticity and homogenization*, volume 26 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam, 1992.
- [87] J. Pommier and B. Samet. The topological asymptotic for the Helmholtz equation with Dirichlet condition on the boundary of an arbitrarily shaped hole. *SIAM J. Control Optim.*, 43(3):899–921 (electronic), 2004.
- [88] A. Schumacher. Topologieoptimisierung von Bauteilstrukturen unter Verwendung von Topologieoptimierungskriterien. *Thesis*, 1995. Universität-Gesamthochschule-Siegen.
- [89] J. R. Shewchuk. Mesh Generator TRIANGLE. <http://www.cs.cmu.edu/~quake/triangle.html>.
- [90] J. Simon. Differentiation with respect to the domain in boundary value problems. *Numer. Funct. Anal. Optim.*, 2(7-8):649–687 (1981), 1980.
- [91] J. Simon. Second variations for domain optimization problems. In *Control and estimation of distributed parameter systems (Vorau, 1988)*, volume 91 of *Internat. Ser. Numer. Math.*, pages 361–378. Birkhäuser, Basel, 1989.
- [92] J. Simon. Domain variation for drag in Stokes flow. In *Control theory of distributed parameter systems and applications (Shanghai, 1990)*, volume 159 of *Lecture Notes in Control and Inform. Sci.*, pages 28–42. Springer, Berlin, 1991.
- [93] J. Sokołowski and A. Żochowski. On the topological derivative in shape optimization. *SIAM J. Control Optim.*, 37(4):1251–1272 (electronic), 1999.
- [94] J. Sokołowski and J.-P. Zolésio. *Introduction to shape optimization*, volume 16 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1992. Shape sensitivity analysis.
- [95] T. Takahashi. Analysis of strong solutions for the equations modeling the motion of a rigid-fluid system in a bounded domain. *Adv. Differential Equations*, 8(12):1499–1532, 2003.
- [96] T. Takahashi and M. Tucsnak. Global strong solutions for the two-dimensional motion of an infinite cylinder in a viscous fluid. *J. Math. Fluid Mech.*, 6(1):53–77, 2004.
- [97] R. Temam. *Navier-Stokes equations*. AMS Chelsea Publishing, Providence, RI, 2001. Theory and numerical analysis, Reprint of the 1984 edition.

Résumé

Cette thèse s'inscrit dans le domaine des mathématiques appelé **optimisation de formes**. Plus précisément, nous étudions ici un **problème inverse de détection** à l'aide du **calcul de forme** et de l'**analyse asymptotique**. L'objectif est de localiser un objet immergé dans un fluide visqueux, incompressible et stationnaire. Les questions principales qui ont motivé ce travail sont les suivantes :

- *peut-on détecter un objet immergé dans un fluide à partir d'une mesure effectuée à la surface ?*
- *peut-on reconstruire numériquement cet objet, i.e. approcher sa position et sa forme, à partir de cette mesure ?*
- *peut-on connaître le nombre d'objets présents dans le fluide en utilisant cette mesure ?*

Les résultats obtenus sont décrits dans les cinq chapitres de cette thèse :

- le premier met en place un cadre mathématique pour démontrer l'**existence des dérivées de forme** d'ordre un et deux pour les problèmes de détection d'inclusions ;
- le deuxième analyse le problème de détection à l'aide de l'**optimisation géométrique de forme** : un résultat d'identifiabilité est montré, le gradient de forme de plusieurs types de fonctionnelles de forme est caractérisé et l'instabilité de ce problème inverse est enfin démontrée ;
- le chapitre 3 utilise nos résultats théoriques pour reconstruire numériquement des objets immergés dans un fluide à l'aide d'un **algorithme de gradient de forme** ;
- le chapitre 4 analyse la localisation de petites inclusions dans un fluide à l'aide de l'**optimisation topologique de forme** : le gradient topologique d'une fonctionnelle de forme de Kohn-Vogelius est caractérisé ;
- le dernier chapitre utilise cette dernière expression théorique pour déterminer numériquement le nombre et la localisation de petits obstacles immergés dans un fluide à l'aide d'un **algorithme de gradient topologique**.

Mots clés : optimisation de forme, problème inverse géométrique, sensibilité par rapport à la forme d'ordre deux, dérivées de forme, sensibilité topologique, dérivée topologique, EDP surdéterminée, équations de Stokes et Navier-Stokes stationnaires

Abstract

This dissertation takes place in the mathematic field called **shape optimization**. More precisely, we focus on a **detecting inverse problem** using **shape calculus** and **asymptotic analysis**. The aim is to localize an object immersed in a viscous, incompressible and stationary fluid. This work was motivated by the following main questions:

- *can we localize an obstacle immersed in a fluid from a boundary measurement?*
- *can we reconstruct numerically this object, i.e. be close to its localization and its shape, from this measure?*
- *can we know how many objects are included in the fluid using this measure?*

The results are described in the five chapters of the thesis:

- the first one gives a mathematical framework in order to prove the **existence of the shape derivatives** of order one and two in the frame of the detection of inclusions;
- the second one analyzes the detection problem using **geometric shape optimization**: an identifiability result is proved, the shape gradient of several shape functionals is characterized and the instability of this inverse problem is proved;
- the chapter 3 uses our theoretical results in order to reconstruct numerically some objets immersed in a fluid using a **shape gradient algorithm**;
- the fourth chapter analyzes the detection of small inclusions in a fluid using the **topological shape optimization**: the topological gradient of a Kohn-Vogelius shape functional is characterized;
- the last chapter uses this theoretical expression in order to determine numerically the number and the location of some small obstacles immersed in a fluid using a **topological gradient algorithm**.

Keywords : shape optimization, geometric inverse problem, second order shape sensitivity, shape derivatives, topological sensitivity, topological derivative, overdetermined PDE, stationary Stokes and Navier-Stokes equations