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SHAPE OPTIMIZATION FOR HEAT EXCHANGERS WITH A THIN LAYER

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Abstract. This paper focuses on a shape optimization method applied to fluid-to-fluid heat exchangers. We consider the framework of two fluids separated by a solid thin layer (the wall of the pipes) and we perform an asymptotic expansion in order to obtain an approximated model without thin layer. Due to this approximation, the multi-physics problem is reduced to a weak-coupled problem, between the steady-state Navier-Stokes equations for the two fluids dynamics and the convection-diffusion equation for the heat. The aim is then to optimize the shape of the heat exchanger in order to maximize the heat exchange and minimize the pressure drop. Thus, we characterize the shape derivative for the objective functional and perform numerical simulations in two dimensions.

Keywords: Asymptotic analysis, Shape optimization, Heat exchangers.
AMS classification: 35C20, 49Q10, 76B75.

§1. Introduction

A heat exchanger is a device that allows the heat exchange between two or more fluids without mixing of fluids. There is a growing interest in heat exchangers due to energy consumption and aiming for efficiency. Engineers propose new designs based on physics, intuition and experience for different applications of the heat exchangers, trying to improve the performance.

The shape and topology optimization community has been interested in this application in the last few years, since the work [10] based on the SIMP method [3]. One of the main difficulties in solving a shape and topology optimization problem concerning heat exchangers is how to deal with the non-mixing constraint. Recently, [6] proposed a level-set approach with which is more natural to deal with the distance constraint. We highlight that in the literature, the separation between the fluids is assumed to be large enough.

Our work takes into account that a thin layer separates the two fluids (i.e. the pipe). This would require a very fine mesh of this solid region to numerically solve the problem, which is too expensive. Hence, in order to avoid that difficulty, we perform an asymptotic analysis to obtain effective transmission conditions between the two fluids which takes into account the diffusion in the solid, without meshing it. Another advantage is that we do not have to deal with the distance constraint as in the works mentioned above. However the obtained system, which can contain non standard transmission conditions, can be harder to solve.

To find the optimal design, we rely on the Hadamard's boundary variation method (see [2, 8] for more details) as in [6]. In this conference paper, we present the method in a first simplified model: more elaborated ones are the topic of a forthcoming work.

Paper organization. The paper is organized as follows. Section 2 is devoted to introduce the considered problem. In section 3 are obtained the effective boundary conditions, using an asymptotic expansion, and we present the approximate optimization problem on which we focus. In section 4 are established the shape derivatives. In section 5 are presented numerical simulations. Finally, in section 6, some conclusions and perspectives of future work are given.

§2. Formulation of the optimal heat exchanger design problem

The domains. In this work, we consider $\Omega_{cold} := (-1,1)\times(-1,0)$ and $\Omega_{hot} := (-1,1)\times(0,1)$ be two disjoint open bounded domains of $\mathbb{R}^2$, such that the whole domain Ω is formed by their union. In what follows, Ω_{hot} represents the hot fluid domain and Ω_{cold} the cold fluid domain. The interface between them is denoted by $\Gamma := \partial\Omega_{cold} \cap \partial\Omega_{hot} = (-1,1)\times\{0\}$. Here and in the following, the subscript $_f$ stands for $_{cold}$ or $_{hot}$. The boundaries of Ω_f are respectively composed by four disjoint regions: $\partial\Omega_f =: \Gamma_{f,\,in} \cup \Gamma_{f,\,out} \cup \Gamma_{f,\,wall} \cup \Gamma$, where $\Gamma_{f,\,in}$ is the input of the fluid with a given velocity (Dirichlet boundary condition), $\Gamma_{f,\,out}$ contains outlet-pressure condition (Neumann boundary condition), and the classical non-slip condition (homogeneous Dirichlet boundary condition) is imposed on $\Gamma_{f,\,wall} \cup \Gamma$. The notations are summarized in Fig. 1.

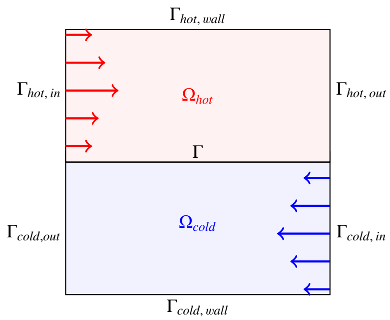


Figure 1: Illustration of the domain without thin layer

Let $\epsilon > 0$. We define the open bounded domains

$$D^\epsilon := (-1,1)\times\left(-1-\frac{\epsilon}{2}, 1+\frac{\epsilon}{2}\right), \qquad \Omega_s^\epsilon := \left\{x \in D^\epsilon;\ d(x,\Gamma) < \frac{\epsilon}{2}\right\}$$

and

$$\begin{aligned} \Omega^\epsilon_{cold} &:= \{x \in D^\epsilon \setminus \Omega^\epsilon_s; x_2 < 0\}, & \Gamma^\epsilon_{cold} &:= \left\{x \in D^\epsilon;\ d(x,\Gamma) = \frac{\epsilon}{2};\ x_2 < 0\right\}, \\ \Omega^\epsilon_{hot} &:= \{x \in D^\epsilon \setminus \Omega^\epsilon_s; x_2 > 0\}, & \Gamma^\epsilon_{hot} &:= \left\{x \in D^\epsilon;\ d(x,\Gamma) = \frac{\epsilon}{2};\ x_2 > 0\right\}. \end{aligned}$$

We denote $\Gamma^\epsilon := \Gamma^\epsilon_{cold} \cup \Gamma^\epsilon_{hot}$ and we define $\Gamma^\epsilon_{f,\,in}$, $\Gamma^\epsilon_{f,\,out}$ and $\Gamma^\epsilon_{f,\,wall}$ by translating $\Gamma_{f,\,in}$, $\Gamma_{f,\,out}$ and $\Gamma_{f,\,wall}$ by $\pm\frac{\epsilon}{2}$. The boundaries of Ω^ϵ_f are then composed by $\Gamma^\epsilon_{f,\,in}, \Gamma^\epsilon_{f,\,out}, \Gamma^\epsilon_{f,\,wall}$ and Γ^ϵ_f. The boundaries of Ω^ϵ_s are Γ^ϵ and $\Gamma^\epsilon_{s,wall} := \partial\Omega^\epsilon_s \setminus \Gamma^\epsilon$. Finally we define Ω^ϵ as the union of Ω^ϵ_{cold}, Ω^ϵ_{hot} and Ω^ϵ_s. Accordingly, $\partial\Omega^\epsilon = \Gamma^\epsilon_{in} \cup \Gamma^\epsilon_{out} \cup \Gamma^\epsilon \cup \Gamma^\epsilon_{wall}$, where $\Gamma^\epsilon_{in} := \Gamma^\epsilon_{cold,\,in} \cup \Gamma^\epsilon_{hot,\,in}$, $\Gamma^\epsilon_{out} := \Gamma^\epsilon_{cold,\,out} \cup \Gamma^\epsilon_{hot,\,out}$ and $\Gamma^\epsilon_{wall} := \Gamma^\epsilon_{cold,\,wall} \cup \Gamma^\epsilon_{hot,\,wall} \cup \Gamma^\epsilon_{s,\,wall}$ (see Fig. 2).

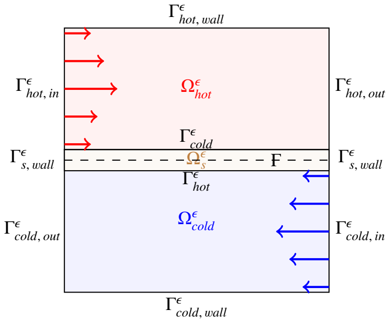


Figure 2: Illustration of the domain of the exact problem with a thin layer

The physical models. We model the fluid flow with the stationary Navier-Stokes equations: u^ϵ_f stands for the velocity of each fluid and p^ϵ_f for the pressure. The physical properties of the fluids, that is, the viscosity ν_f and the density ρ_f, are assumed to be constants. Let $u_{f,\,in} \in \mathrm{H}^{\frac{1}{2}}(\Gamma_{f,in})^d$ and let us consider $u^\epsilon_{f,\,in}(x_1, x_2) := u_{f,\,in}(x_1, x_2 \pm \frac{\epsilon}{2})$. Then we have the following system

$$\begin{cases} -\nu_f \Delta u^\epsilon_f + \rho_f (\nabla u^\epsilon_f) u^\epsilon_f + \nabla p^\epsilon_f & = 0 \quad \text{in } \Omega^\epsilon_f, \\ \operatorname{div}(u^\epsilon_f) & = 0 \quad \text{in } \Omega^\epsilon_f, \\ u^\epsilon_f & = u^\epsilon_{f,in} \quad \text{on } \Gamma^\epsilon_{f,in}, \\ \sigma(u^\epsilon_f, p^\epsilon_f) n & = 0 \quad \text{on } \Gamma^\epsilon_{f,out}, \\ u^\epsilon_f & = 0 \quad \text{on } \Gamma^\epsilon_{f,\,wall} \cup \Gamma^\epsilon_f, \end{cases} \tag{1}$$

where the fluid stress tensor is defined by

$$\sigma(u, p) := 2\nu\varepsilon(u) - pI, \qquad \text{with } \varepsilon(u) := \frac{1}{2}\left(\nabla u + \nabla u^T\right).$$

In the following, we introduce the notations

$$\begin{aligned} u^\epsilon &:= u^\epsilon_{cold} \mathbb{1}_{\Omega^\epsilon_{cold}} + u^\epsilon_{hot} \mathbb{1}_{\Omega^\epsilon_{hot}}, \qquad & p^\epsilon &:= p^\epsilon_{cold} \mathbb{1}_{\Omega^\epsilon_{cold}} + p^\epsilon_{hot} \mathbb{1}_{\Omega^\epsilon_{hot}}, \\ \nu^\epsilon &:= \nu_{cold} \mathbb{1}_{\Omega^\epsilon_{cold}} + \nu_{hot} \mathbb{1}_{\Omega^\epsilon_{hot}}, \qquad & \rho^\epsilon &:= \rho_{cold} \mathbb{1}_{\Omega^\epsilon_{cold}} + \rho_{hot} \mathbb{1}_{\Omega^\epsilon_{hot}}. \end{aligned}$$

Concerning the temperature, we model it as the solution T^ϵ of the stationary convection-diffusion equation in Ω^ϵ, where T^ϵ_{hot}, T^ϵ_{cold} and T^ϵ_s are the restriction of the temperature to Ω^ϵ_{hot}, Ω^ϵ_{cold} and Ω^ϵ_s respectively. The physical parameters are the thermal conductivity κ_f and the thermal capacity c_f are assumed to be constant. We then define

$$T^\epsilon := T^\epsilon_{cold} \mathbb{1}_{\Omega^\epsilon_{cold}} + T^\epsilon_{hot} \mathbb{1}_{\Omega^\epsilon_{hot}} + T^\epsilon_s \mathbb{1}_{\Omega^\epsilon_s}, \; k^\epsilon := k_{cold} \mathbb{1}_{\Omega^\epsilon_{cold}} + k_{hot} \mathbb{1}_{\Omega^\epsilon_{hot}} + k_s \mathbb{1}_{\Omega^\epsilon_s}, \; c^\epsilon := c_{cold} \mathbb{1}_{\Omega^\epsilon_{cold}} + c_{hot} \mathbb{1}_{\Omega^\epsilon_{hot}}.$$

On the inlet Γ^ϵ_{in} we impose a given temperature (colder on $\Gamma^\epsilon_{cold,in}$ and hotter on $\Gamma^\epsilon_{hot,in}$), and on Γ^ϵ_{wall} and Γ^ϵ_{out} we assume adiabatic conditions. On the interface Γ^ϵ we suppose continuity of the temperature and the flux, this is, the jumps are zero. Let $T_{in} \in \mathrm{H}^{\frac{1}{2}}(\Gamma_{in})$, then we define $T^{\epsilon}_{in}(x_1, x_2) := T_{in}(x_1, x_2 \pm \frac{\epsilon}{2})$. The convection-diffusion equation is stated as

$$\left\{\begin{array}{rll} -\operatorname{div}(k_{cold}\nabla T^{\epsilon}_{cold}) + \rho_{cold}c_{cold}u^{\epsilon}_{cold}\cdot\nabla T^{\epsilon}_{cold} & = 0 & \text{in } \Omega^{\epsilon}_{cold},\\ -\operatorname{div}(k_{hot}\nabla T^{\epsilon}_{hot}) + \rho_{hot}c_{hot}u^{\epsilon}_{hot}\cdot\nabla T^{\epsilon}_{hot} & = 0 & \text{in } \Omega^{\epsilon}_{hot},\\ -\operatorname{div}(k_s\nabla T^{\epsilon}_s) & = 0 & \text{in } \Omega^{\epsilon}_s,\\ T^{\epsilon} & = T^{\epsilon}_{in} & \text{on } \Gamma^{\epsilon}_{in},\\ \dfrac{\partial T^{\epsilon}}{\partial n} & = 0 & \text{on } \Gamma^{\epsilon}_{out}\cup\Gamma^{\epsilon}_{wall},\\ [T^{\epsilon}] & = 0 & \text{on } \Gamma^{\epsilon},\\ \left[k\dfrac{\partial T^{\epsilon}}{\partial n}\right] & = 0 & \text{on } \Gamma^{\epsilon}. \end{array}\right. \tag{2}$$

Remark 1. In this work, we assume that the thermal expansion/contraction effects due to temperature are small enough. Thus the velocity does not depend on the temperature and then the problem is weakly coupled. This represents a first step for forthcoming work.

Functional framework and optimization problem. We introduce the following functional spaces for the Navier-Stokes and convection-diffusion equations respectively,

$$\begin{array}{rcl} V(\Omega^{\epsilon}_f) & := & \{w^{\epsilon}\in \mathrm{H}^1(\Omega^{\epsilon}_f)^d;\ w^{\epsilon}=0 \ \text{ on } \Gamma^{\epsilon}_{f,in}\cup\Gamma^{\epsilon}_{f,wall}\cup\Gamma^{\epsilon}_f\},\\ \mathcal{H}^1(\Omega^{\epsilon}) & := & \{S^{\epsilon}\in \mathrm{H}^1(\Omega^{\epsilon});\ S^{\epsilon}=0 \ \text{ on } \Gamma^{\epsilon}_{in}\}. \end{array}$$

Hence, the variational formulations are the following. For the Navier-Stokes equations (1),

$$\left\{\begin{array}{l} \text{Find } (u^{\epsilon}_f, p^{\epsilon}_f)\in\left(u^{\epsilon}_{f,in}+V(\Omega^{\epsilon}_f)\right)\times \mathrm{L}^2(\Omega^{\epsilon}_f) \text{ such that } \forall w^{\epsilon}\in V^{\epsilon}(\Omega^{\epsilon}_f), r^{\epsilon}\in \mathrm{L}^2(\Omega^{\epsilon}_f),\\ \displaystyle\int_{\Omega^{\epsilon}_f}\left(2\nu_f\varepsilon(u^{\epsilon}_f):\varepsilon(w^{\epsilon})+\rho_f(\nabla u^{\epsilon}_f)u^{\epsilon}_f\cdot w^{\epsilon}-p^{\epsilon}_f\operatorname{div}(w^{\epsilon})-r^{\epsilon}\operatorname{div}(u^{\epsilon}_f)\right)dx=0. \end{array}\right. \tag{3}$$

For the convection-diffusion equation (2),

$$\left\{\begin{array}{l} \text{Find } T^{\epsilon}\in T^{\epsilon}_{in}+\mathcal{H}^1(\Omega^{\epsilon}) \text{ such that } \forall S^{\epsilon}\in\mathcal{H}^1(\Omega^{\epsilon}),\\ \displaystyle\int_{\Omega^{\epsilon}_s}k_s\nabla T^{\epsilon}\cdot\nabla S^{\epsilon}dx+\int_{\Omega^{\epsilon}_{hot}\cup\Omega^{\epsilon}_{cold}}(k^{\epsilon}\nabla T^{\epsilon}\cdot\nabla S^{\epsilon}+\rho^{\epsilon}c^{\epsilon}S^{\epsilon}u^{\epsilon}\cdot\nabla T^{\epsilon})dx=0. \end{array}\right. \tag{4}$$

Our original aim is to minimize a certain functional with respect to the shape of the thin layer which is the variable. The others boundaries are assumed to be fixed. For our applications, we want to maximize the heat exchanged and minimize the pressure drop. The negative heat exchanged W ($-W$ is the heat exchanged) can be defined as

$$W(\Omega^{\epsilon}, u^{\epsilon}, T^{\epsilon}) = \int_{\Omega^{\epsilon}_{hot}}\rho^{\epsilon}c^{\epsilon}u^{\epsilon}\cdot\nabla T^{\epsilon}dx - \int_{\Omega^{\epsilon}_{cold}}\rho^{\epsilon}c^{\epsilon}u^{\epsilon}\cdot\nabla T^{\epsilon}dx. \tag{5}$$

For the drop pressure we consider the difference between the pressure on the output and input, that is $DP(\Omega^{\epsilon}_{cold}, p^{\epsilon}) = \int_{\Gamma^{\epsilon}_{cold,in}} p^{\epsilon}ds - \int_{\Gamma^{\epsilon}_{cold,out}} p^{\epsilon}ds$. This functional is directly linked with the energy dissipation of the fluid given by

$$E(\Omega^{\epsilon}, u^{\epsilon}) = \int_{\Omega^{\epsilon}_{hot}\cup\Omega^{\epsilon}_{cold}} 2\nu^{\epsilon}|\varepsilon(u^{\epsilon})|^2dx. \tag{6}$$

We consider this functional rather than DP for mathematical reason: firstly it is a volume integral and secondly, since the pressure does not have boundary conditions, the adjoint state associated to DP can be ill-posed. Therefore, the shape optimization problem that we aim to solve is

$$\begin{cases} \min & W(\Omega^\epsilon, u^\epsilon(\Omega^\epsilon), T^\epsilon(\Omega^\epsilon)) \\ \text{s.t.} & E(\Omega^\epsilon, u^\epsilon(\Omega^\epsilon)) \leq E_{max} \\ & d(\Omega^\epsilon_{f,cold}, \Omega^\epsilon_{f,hot}) \geq d_{min}, \end{cases} \tag{7}$$

where $d(\Omega^\epsilon_{f,cold}, \Omega^\epsilon_{f,hot}) := \inf_{x\in\Omega^\epsilon_{f,cold},\ y\in\Omega^\epsilon_{f,hot}} |x-y|$ and u^ϵ, T^ϵ are the solutions of (3) and (4).

§3. Derivation of the effective boundary conditions, convergence analysis and approximate optimization problem

In order to deal with the convection-diffusion equation (2), we aim to make an asymptotic expansion to obtain a new model without thin layer and with new transmission conditions. In this way, we will get rid of the meshing of Ω^ϵ_s associated to the small parameter, but the equations will change. More details about this technique can be found it in [11].

Derivation of the new transmission conditions. Since the Navier-Stokes equations (1) does not depend on Ω^ϵ_s, we only have to replace $\Omega^\epsilon_f, \Gamma^\epsilon_f, \Gamma^\epsilon_{f,in}, \Gamma^\epsilon_{f,out}, u^\epsilon_{f,in}$ by Ω_f, Γ_f, $\Gamma_{f,in}$, $\Gamma_{f,out}$, $u_{f,in}$, respectively. We define (u_f, p_f) the solution of the following problem:

$$\begin{cases} -\nu_f \Delta u_f + \rho_f (\nabla u_f) u_f + \nabla p_f &= 0 & \text{in } \Omega_f, \\ \operatorname{div}(u_f) &= 0 & \text{in } \Omega_f, \\ u_f &= u_{f,in} & \text{on } \Gamma_{f,in}, \\ \sigma(u_f, p_f) n &= 0 & \text{on } \Gamma_{f,out}, \\ u_f &= 0 & \text{on } \Gamma_{f,wall} \cup \Gamma. \end{cases} \tag{8}$$

Note that u_f, p_f is the translated solution of $u^\epsilon_f, p^\epsilon_f$ in Ω^ϵ_f to Ω_f. Then, we define,

$$\begin{array}{lll} u := u_{cold}\mathbb{1}_{\Omega_{cold}} + u_{hot}\mathbb{1}_{\Omega_{hot}}, & & p := p_{cold}\mathbb{1}_{\Omega_{cold}} + p_{hot}\mathbb{1}_{\Omega_{hot}}, \\ \nu := \nu_{cold}\mathbb{1}_{\Omega_{cold}} + \nu_{hot}\mathbb{1}_{\Omega_{hot}}, & & \rho := \rho_{cold}\mathbb{1}_{\Omega_{cold}} + \rho_{hot}\mathbb{1}_{\Omega_{hot}}, \\ k := k_{cold}\mathbb{1}_{\Omega_{cold}} + k_{hot}\mathbb{1}_{\Omega_{hot}}, & & c := c_{cold}\mathbb{1}_{\Omega_{cold}} + c_{hot}\mathbb{1}_{\Omega_{hot}}. \end{array}$$

Proposition 1. *We consider $T = T_{cold}\mathbb{1}_{\Omega_{cold}} + T_{hot}\mathbb{1}_{\Omega_{hot}}$ the solution of the problem*

$$\begin{cases} -\operatorname{div}(k_{cold}\nabla T_{cold}) + \rho_{cold}c_{cold}u_{cold}\cdot\nabla T_{cold} &= 0 & \text{in } \Omega_{cold}, \\ -\operatorname{div}(k_{hot}\nabla T_{hot}) + \rho_{hot}c_{hot}u_{hot}\cdot\nabla T_{hot} &= 0 & \text{in } \Omega_{hot}, \\ T &= T_{in} & \text{on } \Gamma_{in}, \\ k\dfrac{\partial T}{\partial n} &= 0 & \text{on } \Gamma_{out}\cup\Gamma_{wall}, \\ [T] &= 0 & \text{on } \Gamma, \\ \left[k\dfrac{\partial T}{\partial n}\right] &= 0 & \text{on } \Gamma. \end{cases} \tag{9}$$

Then, it exists $C > 0$ independent of ϵ such that $\|S^\epsilon - T\|_{1,\Omega} \leq C\epsilon$, where $S^\epsilon := T^\epsilon_{cold}(\cdot, \cdot - \frac{\epsilon}{2})\mathbb{1}_{\Omega_{cold}} + T^\epsilon_{hot}(\cdot, \cdot + \frac{\epsilon}{2})\mathbb{1}_{\Omega_{hot}}$ and T^ϵ is the solution of (2).

Proof. We recall that we here consider a square geometry in the two dimensional case. We postulate the *Ansätze*, $T_f^\epsilon(x) = \sum_{j\geq 0} \epsilon^j T_f^j(x_1, x_2)$, and $T_s^\epsilon(x) = \sum_{j\geq 0} \epsilon^j T_s^j\left(x_1, \frac{x_2}{\epsilon}\right)$.

Defining $Y := \frac{x_2}{\epsilon} \in \left(\frac{-1}{2}, \frac{1}{2}\right)$ and inserting this expansion in $\operatorname{div}(k_s \nabla T_s^\epsilon) = 0$ in Ω_s^ϵ, we get

$$\sum_{j\geq 0} \epsilon^j \left(\frac{\partial}{\partial x_1} k_s \frac{\partial T_s^j}{\partial x_1} + \epsilon^{-2} \frac{\partial}{\partial Y} k_s \frac{\partial T_s^j}{\partial Y} \right) = 0, \quad \forall Y \in \left(\frac{-1}{2}, \frac{1}{2}\right).$$

Then, by formal identification of the powers of ϵ, we obtain in particular the first two terms,

$$\frac{\partial}{\partial Y}\left(k_s \frac{\partial T_s^0}{\partial Y}\right) = 0, \quad \frac{\partial}{\partial Y}\left(k_s \frac{\partial T_s^1}{\partial Y}\right) = 0, \quad \forall Y \in \left(\frac{-1}{2}, \frac{1}{2}\right).$$

Note that the problem solved by T_s^0, T_s^1 is one-dimensional, and x_1 is a parameter to that problem, so we can integrate with respect to Y, getting a first degree polynomial for T_s^0, T_s^1. In order to determine the coefficients of this polynomial we use the transmission conditions. The transmission conditions $[T] = 0$ on Γ^ϵ and $\left[k\frac{\partial T}{\partial n}\right] = 0$ on Γ^ϵ become

$$T_s^j\left(x_1, \frac{\pm 1}{2}\right) = T_f^j(x_1, x_2), \forall j \geq 0,\; k_s \frac{\partial T_s^0}{\partial Y}\left(x_1, \pm\frac{1}{2}\right) = 0,\; k_s \frac{\partial T_s^1}{\partial Y}\left(x_1, \pm\frac{1}{2}\right) = k_f \frac{\partial T_f^0}{\partial n}\left(x_1, \pm\frac{\epsilon}{2}\right),$$

where we choose the normal n pointing towards Ω_{hot}^ϵ. Using these equations leads to T_s^0, T_s^1 be affine with respect to Y. Furthermore, $[T^0]_\epsilon = 0$ and $\left[k\frac{\partial T^0}{\partial n}\right]_\epsilon (x_1) = 0$, with the notation $[S]_\epsilon(x_1) := S_{hot}\left(x_1, \frac{\epsilon}{2}\right) - S_{cold}\left(x_1, \frac{-\epsilon}{2}\right)$, for the jump between temperature on Γ_{hot}^ϵ and Γ_{cold}^ϵ respectively. To justify the convergence we proceed as in [11], observing that $T = S^0 := T^0(\cdot, \cdot - \frac{\epsilon}{2})\mathbb{1}_{\Omega_{cold}} + T^0(\cdot, \cdot + \frac{\epsilon}{2})\mathbb{1}_{\Omega_{hot}}$, due to S^0 satisfies Equation (9). □

Remark 2. Notice that, in order to extend the result in a more general framework than the square, we could use curvilinear coordinates as in [4].

Formulation of the approximate optimal heat exchanger design problem. Since now, we will work with the approximated model (9) instead of the exact model (2). The Navier-Stokes equations (8) have following variational formulation

$$\begin{cases} \text{Find } (u_f, p_f) \in \left(u_{f,in} + V(\Omega_f)\right) \times \mathrm{L}^2(\Omega_f) \text{ such that } \forall w \in V(\Omega_f), r \in \mathrm{L}^2(\Omega_f), \\ \int_{\Omega_f} \left(2\nu_f \varepsilon(u_f) : \varepsilon(w) + \rho_f(\nabla u_f)u_f \cdot w - p_f \operatorname{div}(w) - r div(u_f)\right) dx = 0 \end{cases} \tag{10}$$

and the variational formulation associated to (9) is

$$\begin{cases} \text{Find } T \in T_{in} + \mathcal{H}_0^1(\Omega) \text{ such that } \forall S \in \mathcal{H}_0^1(\Omega), \\ \int_\Omega (k\nabla T \cdot \nabla S + \rho c S u \cdot \nabla T) dx = 0, \end{cases} \tag{11}$$

where $\mathcal{H}_0^1(\Omega) := \{S \in \mathrm{H}^1(\Omega);\; S = 0 \text{ on } \Gamma_{in}\}$.

Remark 3. If the viscosity ν_f is large enough, the problem (10) has a unique solution (see, e.g., [7] for details), and the problem (11) is also well posed (see, e.g., [5]).

The initial shape optimization problem (7) is then replaced by the following one that we will focus on:

$$\begin{cases} \min & W(\Omega, u(\Omega), T(\Omega)) \\ \text{s.t.} & E(\Omega, u(\Omega)) \leq E_{max}, \end{cases} \tag{12}$$

where u and T are the solutions of (10) and (11), respectively.

§4. Shape sensitivity analysis

The computation of the shape derivatives for the order zero model is a simplification of [5] which was derived in a fully Lagrangian setting. In the next proposition, we give the Eulerian derivative for the convection-diffusion equation. This proof is rather standard and has been studied in detail in the literature. Thus we do not detail the proof here and refer for exemple to [1] where the case of the diffusion equation (without the convective term) is detailed (see also [9] for the Navier-Stokes equations). The case of more elaborated transmission conditions, doing an asymptotic expansion of order two for example, is much more difficult, in particular because of the Laplace-Beltrami operator.

Proposition 2. *The shape derivative T' of T is the solution of the following problem*

$$\begin{cases} -\operatorname{div}(k\nabla T') + \rho c(u \cdot \nabla T' + u' \cdot \nabla T) & = 0 & \text{in } \Omega_{cold} \cup \Omega_{hot}, \\ T' & = 0 & \text{on } \Gamma_{in}, \\ k\dfrac{\partial T'}{\partial n} & = 0 & \text{on } \Gamma_{out}, \\ [T'] & = -\left[k^{-1}\right] k\frac{\partial T}{\partial n}(\theta \cdot n) & \text{on } \Gamma, \\ \left[k\dfrac{\partial T'}{\partial n}\right] & = -[k] \operatorname{div}_\tau((\theta \cdot n)\nabla_\tau T) & \text{on } \Gamma. \end{cases}$$

For the Lagrangian setting we require the concept of *transported functional*.

Definition 1. In our context, the *transported functional* (in Ω instead of Ω_θ) of a shape objective functional J, is the functional $\mathcal{J}$ such that for all $\theta \in \mathrm{W}_0^{1,\infty}(\Omega, \mathbb{R}^d)$ and all $(\bar{u}, \bar{p}, \bar{T}) \in \mathrm{H}^1(\Omega, \mathbb{R}^d) \times \mathrm{L}^2(\Omega) \times \mathrm{H}^1(\Omega)$

$$\mathcal{J}(\theta, \hat{u}, \hat{p}, \hat{T}) = J(\Gamma_\theta, \hat{u} \circ (I + \theta)^{-1}, \hat{p} \circ (I + \theta)^{-1}, \hat{T} \circ (I + \theta)^{-1}).$$

For the considered problem, $\mathcal{E}$ and $\mathcal{W}$ stand respectively for the transported functional of $E(\Omega, u)$ given in (6) and $W(\Omega, u, T)$ given in (5).

In order to obtain a suitable expression of the shape derivative, we introduce the following adjoint states. Firstly we consider $R \in \mathcal{H}_0^1(\Omega)$ solution of the following adjoint problem:

$$\int_\Omega (k\nabla R \cdot \nabla S + \rho c R u \cdot \nabla S) dx = \frac{\partial \mathcal{J}}{\partial T}(S), \qquad \forall S \in \mathcal{H}_0^1(\Omega). \tag{13}$$

Notice that this problem (13) is classically well-posed. Secondly we consider $(v_f, q_f) \in V(\Omega_f) \times \mathrm{L}^2(\Omega_f)$ solution of the following adjoint problem:

$$\int_{\Omega_f} \Big(2\nu_f \varepsilon(v_f) : \varepsilon(w_f) + \rho_f((\nabla u_f)^T v_f - (\nabla v_f) u_f) \cdot w_f - q_f \operatorname{div}(w_f) - r_f div(v_f)\Big) dx$$
$$= - \int_{\Omega_f} \rho_f c_f R_f \nabla T_f \cdot w_f dx + \frac{\partial \mathcal{J}}{\partial(u_f, p_f)}(w_f, r_f), \qquad \forall w_f \in V(\Omega_f),\ \forall r_f \in \mathrm{L}^2(\Omega_f). \quad (14)$$

If ν_f is large enough, it is well-known that this problem (14) has a unique solution (v_f, q_f).

Remark 4. To solve Problem (12), we consider the adjoint problem (13) with $\mathcal{J} = \mathcal{W}$ and the adjoint problem (14) with both $\mathcal{J} = \mathcal{W}$ and $\mathcal{J} = \mathcal{E}$, that is five adjoint problems ((v^E_{cold}, q^E_{cold}), (v^E_{hot}, q^E_{hot}) associated to E, and R^W, (v^W_{cold}, q^W_{cold}) and (v^W_{hot}, q^W_{hot}) associated to W).

Using the previous notations, the shape derivative can be written in the volume or surface version, as it is established in the following proposition. The proof is classical (see, e.g., [5]) and based on a change of variable to move from the perturbed domain $\Omega_\theta = (I + \theta)\Omega$ to Ω.

Proposition 3. *Let J be an objective shape function depending on u and T. Let $\mathcal{J}$ be the transported objective function. Then J is differentiable with respect to $\theta \in \mathrm{W}^{1,\infty}(\Omega, \mathbb{R}^d)$ and the volume shape derivative is given by*

$$\frac{d}{d\theta}(J(\Gamma_\theta, u(\Gamma_\theta), T(\Gamma_\theta))(\theta) = \frac{\partial \mathcal{J}}{\partial \theta}(\theta) - \int_\Omega (\sigma(u,p) : \nabla v + \rho(\nabla u)u \cdot v) \operatorname{div}(\theta) dx$$
$$+ \int_\Omega \sigma(u,p) : (\nabla v \nabla \theta) + \sigma(v,q) : (\nabla u \nabla \theta) + \rho(\nabla u \nabla \theta) u \cdot v dx$$
$$- \int_\Omega \operatorname{div}(\theta)(k\nabla T \cdot \nabla R + \rho c R u \cdot \nabla T) dx + \int_\Omega k((\nabla\theta + \nabla\theta^t)\nabla T) \cdot \nabla R + \rho c R u \cdot (\nabla \theta^t \nabla T) dx.$$

Moreover, the surface shape derivative is given by

$$\frac{d}{d\theta}(J(\Gamma_\theta, u(\Gamma_\theta), T(\Gamma_\theta))(\theta) = \overline{\frac{\partial \mathcal{J}}{\partial \theta}}(\theta) + \int_\Gamma \left(-[\sigma(v,q) : \nabla u] + [k \nabla T \cdot \nabla R] - 2\left[k \frac{\partial T}{\partial n} \frac{\partial R}{\partial n}\right]\right)(\theta \cdot n) ds.$$

where $\overline{\frac{\partial \mathcal{J}}{\partial \theta}}$ denotes the normal component of $\frac{\partial \mathcal{J}}{\partial \theta}$.

In order to consider the optimization problem (12), we specify here the derivative of the associated functional:

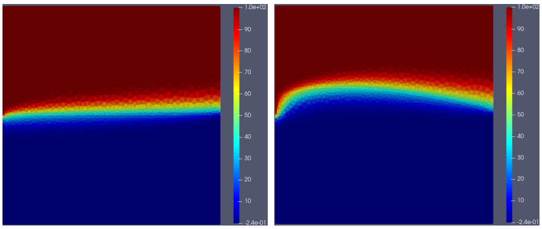
$$\frac{\partial \mathcal{E}}{\partial \theta}(\theta) = \int_\Omega |\epsilon(u)|^2 - 2\nu \epsilon(u) : \nabla u \nabla \theta dx,$$
$$\overline{\frac{\partial \mathcal{E}}{\partial \theta}}(\theta) = - \int_\Gamma |\epsilon(u)|^2 (\theta \cdot n) ds,$$
$$\frac{\partial \mathcal{E}}{\partial(u_f, p_f)}(w_f, r_f) = \int_{\Omega_f} 4\nu_f \epsilon(u_f) : \epsilon(w_f) dx,$$
$$\frac{\partial \mathcal{W}}{\partial \theta}(\theta) = \int_{\Omega_{hot}} \operatorname{div}(\theta)\rho c \nabla T \cdot u - \rho c u \cdot (\nabla\theta^t \nabla T) dx - \int_{\Omega_{cold}} \operatorname{div}(\theta) \rho c \nabla T \cdot u - \rho c u \cdot (\nabla \theta^t \nabla T) dx,$$
$$\overline{\frac{\partial \mathcal{W}}{\partial \theta}}(\theta) = 0,$$
$$\frac{\partial \mathcal{W}}{\partial T}(S) = \int_{\Omega_{hot}} \rho c \nabla S \cdot u dx - \int_{\Omega_{cold}} \rho c \nabla S \cdot u dx,$$
$$\frac{\partial \mathcal{W}}{\partial(u,p)}(w, r) = \int_{\Omega_{hot}} \rho c \nabla T \cdot w dx - \int_{\Omega_{cold}} \rho c \nabla T \cdot w dx.$$

Remark 5. These shape derivatives formulas are used in an optimization algorithm: at each iteration, they are used to find a descent direction and then we update the mesh.

§5. Numerical Results

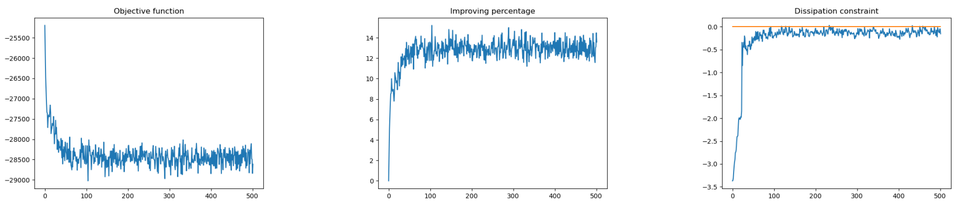
We consider Problem (12) and a similar example to the one in [5]: the square $\Omega = (0, 10) \times (0, 5) \cup (0, 10) \times (5, 10)$, where $\partial\Omega\backslash\Gamma$ are fixed with a counter-current exchange case, this is, the fluids enter from opposite directions. The inlet boundary conditions are $T_{hot} = 100$ and $T_{cold} = 0$, with a parabolic profile v_0 such that $||v_0||_\infty = 1$ at $\Gamma_{in,hot}$ and $\Gamma_{in,cold}$, respectively. Since this work is based on the order zero model, where we have obtained the classical transmissions conditions for the interface, we consider different physical properties: $\rho_{hot} = 1$, $\rho_{cold} = 2$, $k_{hot} = 10$, $k_{cold} = 20$, $\nu_{hot} = 0.16$, $\nu_{cold} = 0.08$, $c_{hot} = 100k_{hot}$, $c_{cold} = 100k_{cold}$.

The initial and last design are depicted in Fig. 3 and the convergence plots in Fig. 4: the dissipation tends to be saturated meanwhile the constraint is satisfied. The objective functions improve between 12% and 14%. We can see that to minimize the exchanged heat with $\rho_{hot}c_{hot} < \rho_{cold}c_{cold}$, the interface moves such that $|\Omega_{cold}| \geq |\Omega_{hot}|$, so in that way the temperature in the cold output will be hotter and the hot temperature will be colder (with respect to the initialization).



(a) Initial temperature field (b) Final temperature field

Figure 3: Plots of the temperature at the first and last iteration



(a) Objective function J (b) Improving percentage of J (c) Dissipation constraint

Figure 4: Convergence plots, where the dissipation constraint is the difference between the dissipation and the maximum value allowed.

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