SHAPE DERIVATIVE FOR SOME EIGENVALUE FUNCTIONALS IN ELASTICITY THEORY*

FABIEN CAUBET†, MARC DAMBRINE‡, AND RAJESH MAHADEVAN§

Abstract. This work is the second part of a previous paper which was devoted to scalar problems. Here we study the shape derivative of eigenvalue problems of elasticity theory for various kinds of boundary conditions, that is Dirichlet, Neumann, Robin, and Wentzell boundary conditions. We also study the case of composite materials, having in mind applications in the sensitivity analysis of mechanical devices manufactured by additive printing.

The main idea, which rests on the computation of the derivative of a minimum with respect to a parameter, was successfully applied in the scalar case in the first part of this paper and is here extended to more interesting situations in the vectorial case (linear elasticity), with applications in additive manufacturing. These computations for eigenvalues in the elasticity problem for generalized boundary conditions and for composite elastic structures constitute the main novelty of this paper. The results obtained here also show the efficiency of this method for such calculations whereas the methods used previously even for classical clamped or transmission boundary conditions are more lengthy or, are based on various simplifying assumptions, such as the simplicity of the eigenvalue or the existence of a shape derivative.

Key words. eigenvalues of elasticity operators, shape derivatives, shape sensitivity analysis, generalized boundary conditions

AMS subject classifications. 49Q10, 35P15, 49R05

1. Introduction.

1.1. Motivations and generalities on shape derivatives.

Many problems ranging from engineering to physics deal with questions of optimal shapes or designs. An important class of these problems involves eigenvalues of elliptic operators since they are important in understanding the vibrating modes of a mechanical structure. A famous example is the so-called *Rayleigh-Faber-Krahn inequality* for the first vibrating mode of a clamped membrane. In recent years, additive manufacturing, or the so-called *3D printing*, has been used in the manufacturing of machine parts with complex geometries or even having a heterogeneous structure. The structural properties of these parts depend on two important features: the distribution of the materials and the effect of thin coatings on the boundary of the device. Of course, engineers would like to optimize the performance of such a printed device by means of an optimal layout of the materials. One of the criteria to consider in the performance of the device are its vibrational properties. In this work, we study the shape sensitivity of eigenvalue problems in linear elasticity for a wide variety of boundary conditions and for both homogeneous and heterogeneous materials with the above applications in mind.

Let $\mathcal{O}_{\rm ad}$ be a family of *admissible open sets* in $\mathbb{R}^d$, $d = 1, 2, 3$, which is stable with respect to a family of diffeomorphisms $(\mathbf{I} + t\boldsymbol{V})$, that is, for a given $\delta > 0$, we

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† Université de Pau et des Pays de l'Adour, E2S UPPA, CNRS, LMAP, UMR 5142, 64000 Pau, France (fabien.caubet@univ-pau.fr, http://fcaubet001.perso.univ-pau.fr).

‡ Université de Pau et des Pays de l'Adour, E2S UPPA, CNRS, LMAP, UMR 5142, 64000 Pau, France (marc.dambrine@univ-pau.fr, http://mdambrin.perso.univ-pau.fr).

§ Departamento de Matemática, Universidad de Concepción, Chile (rmahadevan@udec.cl, https://udec-cl.academia.edu/RajeshMahadevan).

have $(\mathbf{I}+t\boldsymbol{V})(\Omega) \in \mathcal{O}_{\rm ad}$ whenever $\Omega \in \mathcal{O}_{\rm ad}$ for all $t \in [0,\delta]$ and for all $\boldsymbol{V}$ smooth vector fields with compact support in a neighborhood of Ω. Previously and in what follows, $\mathbf{I}$ denotes the identity vector field. The *semi-derivative* a shape functional $F : \mathcal{O}_{\rm ad} \to \mathbb{R}$, in the sense of J. Hadamard [17], at $\Omega \in \mathcal{O}_{\rm ad}$ in the direction of a vector field $\boldsymbol{V}$, is defined as

$$F'(\Omega;\boldsymbol{V}) := \lim_{t\to 0^+} \frac{F(\Omega_t) - F(\Omega)}{t}, \tag{1.1}$$

where

$$\Omega_t := \Psi_t(\Omega), \quad \text{being } \Psi_t(x) := x + t\boldsymbol{V}(x),$$

whenever the limit in (1.1) exists. This is called a *shape derivative* if it exists and is a linear functional with respect to $\boldsymbol{V}$.

1.2. Aim of this work. In the first part of the present work exposed in [8], we have shown how to compute efficiently the semi-derivative of eigenvalue functionals in the scalar case by following a procedure developed initially by M. Delfour and J.P. Zolesio for dealing with the sensitivity with respect to a parameter in minimization problems. In this paper, we focus on vector case and, to be specific, on the study of the semi-derivatives for several families of eigenvalue problems in linear elasticity problems for various kinds of boundary conditions. Whether or not this is linear and continuous with respect to the vector field will not be addressed here. Indeed, the fact that eigenvalues in elasticity problems are not simple makes it very little probable that we could go beyond establishing the existence of a semi-derivative.

Our strategy. For establishing the existence of a semi-derivative in the problems of our interest we shall adopt the following approach. This approach starts with the application of the following version of Theorem 2.1, Chapter 10, M. Delfour and J.P. Zolesio [14] for proving the existence and for obtaining an initial expression for the semi-derivative.

Theorem 1.1. *Let X be a Banach space and let $G : [0,\delta] \times X \to \mathbb{R}$ be a given functional and we set*

$$g(t) = \inf_X G(t,x) \quad \textit{and} \quad X(t) = \{x \in X \,:\, G(t,x) = g(t)\}.$$

If the following hypotheses hold,

- *(H1) $X(t) \neq \emptyset$ for all $t \in [0,\delta]$,*
- *(H2) $\partial_t G(t,x)$ exists in $[0,\delta]$ at all $x \in \cup_{t\in[0,\delta]} X(t)$,*
- *(H3) there exists a topology τ on X such that, for every sequence $\{t_n\} \subset]0,\delta]$ tending to 0 and $x_n \in X(t_n)$, there exists $x_0 \in X(0)$ and a subsequence $\{t_{n_k}\}$ of $\{t_n\}$, for which*
  - *(i) $x_{n_k} \longrightarrow x_0$ with respect to τ*
  - *(ii) $\liminf_{k\longrightarrow\infty} \partial_t G(t_{n_k}, x_{n_k}) \geq \partial_t G(0, x_0)$,*
- *(H4) for all $x \in X(0)$, the function $t \longrightarrow \partial_t G(t,x)$ is upper semi-continuous at $t = 0$,*

then we have that

$$g'(0) = \inf_{x\in X(0)} \partial_t G(0,x).$$

In our setting, the functional $G(t,\cdot)$ will be chosen to be the Rayleigh quotient associated to the original eigenvalue problem on the perturbed domain Ω_t after it is transported back to Ω. We then follow the procedure used in the scalar case [8], for a step-by-step verification of the hypotheses which guarantee the applicability of the theorem. Is follows that we break up the different terms which constitute the Rayleigh quotient and calculate their contributions to the shape derivative through Propositions 3.1 and 3.2 proved below. Then the rather complicated obtained expressions are simplified thanks to a systematic choice of test functions in the variational formulation of the eigenvalue problem. By following this methodical approach, we are able to rigorously establish the existence of the Eulerian semi-derivatives in these problems and obtain the corresponding boundary representations in a simplified manner.

Main novelties of this work. Now a word about some existing results for such derivatives in the elasticity case. Previously, the shape sensitivity of eigenvalue problems of elasticity has been considered for example by J. Sokolowski and J.P. Zolesio in [23] and G. Allaire and F. Jouve in [2]. In these works, the computation for this shape derivative in the presence of Dirichlet and Neumann boundary conditions is given while assuming that the eigenvalue in consideration is simple (and this may be the case for certain domains although not true in general). The arguments therein are based on a suitable adjoint formulation.

In the present work, we avoid the hypothesis of the simplicity of the eigenvalue and, by following our unified and systematic approach, we do not only recover the earlier results but also are able to extend it with a fair amount of ease to other boundary value problems of interest, especially in additive printing, like the so-called *Wentzell boundary conditions*. Let us emphasize that such boundary conditions are not a mathematical curiosity but appear naturally in the context of linear elasticity as soon as the configuration presents discontinuities on the material properties on a submanifold (see, e.g., [9] for a crusted body or [20] for an interface problem). In particular, the Wentzell boundary conditions, coming from asymptotic analysis (see [20, 21, 9] for the mechanical and theoretical justification of such conditions), permit to model *coating or membrane effects*. Notice that this approximation of an original structure with a thin layer by adhering to another domain with new boundary conditions, called *generalized impedance boundary conditions*, is a classical method in order to avoid huge difficulties in the theoretical and numerical analysis of a thin structure (for instance a mesh refinement adapted to the thickness of the layer).

We also underline that we consider two types of eigenvalues problems: the volume and surface types. If the volume type is more classical, at least for classical boundary conditions, the study of shape sensitivity of surface type eigenvalues problem is, up to our knowledge, much less studied although this permits to study transmission problems. Let us emphasize that the surface eigenvalue problems do not model eigenvalues of thin structures like shells. They have been introduced to justify that the asymptotic models derived by M. David, J.J. Marigo and C. Pideri in [20, 21] are well posed in the sense that the problems are of Fredholm type (see [4] for the scalar case and [5] for the elastic case in dimension two). This is why we also deal with these problems in this paper, in order to be as complete as possible.

Motivated by structural optimization of multi-phase material, we consider, in a second step, the eigenvalue problem for a mixture of two isotropic elastic materials. We specify that we use the terminology *composite* to refer to this case. In addition to considering such piecewise constant material properties in the interior of the domain, the effect of a thin coating is also taken into account by allowing a Robin or Wentzell boundary condition.

1.3. Organization of the paper. To conclude this introduction, the paper is organized as follows. The main results of the paper are stated in Section 2. We present first the result in the case of a single isotropic elastic material and, then in the case of a mixture of two phases. The proofs are gathered in Section 3: we first provide the derivatives of the elementary terms arising in Rayleigh quotient in Section 3.2, and then give the proof of the main theorems in Sections 3.3 and 3.4. Finally, we recall (classical) background and technical results in Appendix A.

2. The results.

2.1. Notations. We consider a bounded open subset Ω of $\mathbb{R}^d$ with a $\mathcal{C}^{2,1}$ boundary $\partial\Omega$. Firstly, at each point of $\partial\Omega$, we consider an orthonormal frame $(\boldsymbol{\tau}, \mathbf{n})$ consisting of a family of orthonormal tangential vectors, denoted by $\boldsymbol{\tau}$, and the unit normal vector, denoted by $\mathbf{n}$. Then the tangential projection is given by

$$\Pi_d := \mathrm{I}_d - \mathbf{n} \otimes \mathbf{n}$$

and, in the local frame, has the representation

$$\Pi_d = \begin{pmatrix} \mathrm{I}_{d-1} & 0 \\ 0 & 0 \end{pmatrix},$$

where I_d and I_{d-1} are respectively the identity matrices of size $d{\times}d$ and $(d–1){\times}(d–1)$. More generally, any $d{\times}d$ matrix $\mathcal{M}$ has the following representation in the frame $(\boldsymbol{\tau}, \mathbf{n})$:

$$\begin{pmatrix} \mathcal{M}_{\tau\tau} & \mathcal{M}_{\tau \mathrm{n}} \\ \mathcal{M}_{\mathrm{n}\tau} & \mathcal{M}_{\mathrm{nn}} \end{pmatrix},$$

with the components $\mathcal{M}_{\tau\tau} := \Pi_d \, \mathcal{M} \, \Pi_d$, $\mathcal{M}_{\tau\mathrm{n}} := \Pi_d \, \mathcal{M} \, (\mathrm{I}_d - \Pi_d)$, $\mathcal{M}_{\mathrm{n}\tau} := (\mathrm{I}_d - \Pi_d) \, \mathcal{M} \, \Pi_d$ and $\mathcal{M}_{\mathrm{nn}} := (\mathrm{I}_d - \Pi_d) \, \mathcal{M} \, (\mathrm{I}_d - \Pi_d)$.

Secondly, in the whole paper, we use

$$C^{\mathrm{sym}} := \frac{1}{2} \left(C + {}^t C \right)$$

to denote the symmetric part of a square matrix C. For any vector field $\boldsymbol{u} = (u_i)_{i=1,\dots,d} \in \mathbf{H}^1(\Omega)$, the *strain tensor*

$$e(\boldsymbol{u}) := \frac{1}{2} \left(\nabla \boldsymbol{u} + {}^t \nabla \boldsymbol{u} \right) = (\nabla \boldsymbol{u})^{\mathrm{sym}}$$

is the symmetric part of the Jacobian matrix $\nabla \boldsymbol{u}$ whose rows are ${}^t \nabla u_i$ for $i = 1, \dots, d$. We also introduce, for a scalar function $\phi \in \mathrm{H}^1(\partial\Omega)$, the *tangential gradient*

$$\nabla_\Gamma \phi := \Pi_d \nabla \phi,$$

and, for all vectorial functions $\boldsymbol{\psi} \in \mathbf{H}^1(\partial\Omega)$, the *tangential strain*

$$e_\Gamma(\boldsymbol{\psi}) := \frac{1}{2} \left(\nabla_\Gamma \boldsymbol{\psi} + {}^t \nabla_\Gamma \boldsymbol{\psi} \right) = (\nabla_\Gamma \boldsymbol{\psi})^{\mathrm{sym}},$$

where the rows of $\nabla_\Gamma \boldsymbol{\psi}$ are the tangential gradients of the components ψ_i, $i = 1, \dots, d$. Notice that we have $\nabla_\Gamma \boldsymbol{\psi} = \nabla \boldsymbol{\psi} \, \Pi_d$ since

$$\nabla_\Gamma \boldsymbol{\psi} = \begin{pmatrix} {}^t \nabla_\Gamma \psi_1 \\ \vdots \\ {}^t \nabla_\Gamma \psi_d \end{pmatrix} = \begin{pmatrix} {}^t \nabla \psi_1 \Pi_d \\ \vdots \\ {}^t \nabla \psi_d \Pi_d \end{pmatrix} = \nabla \boldsymbol{\psi} \, \Pi_d,$$

and thus

$$e_\Gamma(\boldsymbol{\psi}) = \Pi_d e(\boldsymbol{\psi}) \Pi_d.$$

Therefore, in the local frame, it is of the form $\begin{pmatrix} e_{\tau\tau} & 0 \\ 0 & 0 \end{pmatrix}$. Then, we denote by

$$\mathrm{div}_\Gamma \boldsymbol{\psi} := \mathrm{Tr}(e_\Gamma(\boldsymbol{\psi}))$$

the *surface or tangential divergence*. The tangential divergence of a matrix field C will be the vector field obtained by taking the tangential divergence of the rows of C, that is, for all $i = 1, \ldots, d$,

$$(\mathrm{div}_\Gamma C)_i := \mathrm{div}_\Gamma C_i.$$

Remark 2.1. We should keep in mind the following basic differences with respect to the scalar case. On the one hand, in the scalar case, we have $\nabla_\Gamma \psi = \Pi_d \nabla \psi$, whereas, on the other hand, in the case of elasticity, we have $\nabla_\Gamma \boldsymbol{\psi} = \nabla \boldsymbol{\psi}\, \Pi_d$. Also, in this case, the tangential strain $e_\Gamma(\boldsymbol{\psi})$ is obtained by reducing the strain $e(\boldsymbol{\psi})$ to the tangent space by multiplying by the projection Π_d on either side. This leads to substantial differences in the formulae for shape derivatives in the scalar case and in the case of elasticity.

Then we introduce the *signed distance* to the boundary $\partial\Omega$ defined by

$$b(x) := \begin{cases} \mathrm{d}(x, \partial\Omega), & \text{if } x \in \Omega, \\ -\mathrm{d}(x, \partial\Omega), & \text{if } x \in \mathbb{R}^d \setminus \overline{\Omega}, \end{cases}$$

and the *mean curvature* at any point on $\partial\Omega$, defined by

$$\mathrm{H} := \mathrm{div}_\Gamma \, \mathbf{n}.$$

Finally, as mentioned in the introduction, given a $\mathcal{C}^{2,1}$ vector field $\boldsymbol{V}$ with compact support in a neighborhood of Ω and a (small) real number $\delta > 0$, we consider the one parameter family of deformations

$$\Psi_t := \mathbf{I} + t\boldsymbol{V}, \tag{2.1}$$

for all $t \in [0, \delta]$, which are in fact diffeomorphisms if δ is sufficiently small. Then we define the *perturbed domain* by

$$\Omega_t := \Psi_t(\Omega). \tag{2.2}$$

We also use the following notation for the normal component of the vector field $\boldsymbol{V}$:

$$V_\mathrm{n} := \boldsymbol{V} \cdot \mathbf{n}.$$

2.2. Shape derivative for eigenvalue problems of linear elasticity-single phase isotropic materials. We assume that Ω is an elastic body and we consider an isotropic elastic medium with *Lamé coefficients* $\mu > 0$ and $\lambda > 0$, and associated *elastic or Hooke tensor* A given by

$$A\xi := 2\,\mu\,\xi + \lambda\,\mathrm{Tr}(\xi)\,\mathrm{I}_d, \qquad \text{for all symmetric matrices } \xi.$$

We also assume that the body Ω is surrounded by a thin layer with an elasticity tensor given by

$$A_\mathrm{c}\xi := 2\,\mu_\mathrm{c}\,\xi + \lambda_\mathrm{c}\,\mathrm{Tr}(\xi)\,\Pi_d, \tag{2.3}$$

where $\mu_c > 0$ and $\lambda_c > 0$ are some (modified) Lamé constants which correspond to a coating (the thin layer). Then, given $\alpha, \beta \geq 0$ two real numbers, we are interested in the following kinds of eigenvalues problems: of *volume type*, where the spectral parameter is in the domain,

$$\begin{cases} -\mathrm{div}\,(Ae(\boldsymbol{u})) & = & \Lambda_\Omega(\Omega)\boldsymbol{u} & \text{in } \Omega, \\ -\beta\mathrm{div}_\Gamma\,(A_c e_\Gamma(\boldsymbol{u})) + \alpha\boldsymbol{u} + Ae(\boldsymbol{u})\mathbf{n} & = & \mathbf{0} & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

and of *surface type*, where the spectral parameter is on the boundary,

$$\begin{cases} -\mathrm{div}\,(Ae(\boldsymbol{u})) & = & \mathbf{0} & \text{in } \Omega, \\ -\beta\mathrm{div}_\Gamma\,(A_c e_\Gamma(\boldsymbol{u})) + \alpha\boldsymbol{u} + Ae(\boldsymbol{u})\mathbf{n} & = & \Lambda_{\partial\Omega}(\Omega)\boldsymbol{u} & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

For different regimes of the parameter, we have different kinds of eigenvalue problems. For the choice $\beta = 0$ and $\alpha = 0$ in (2.4), we obtain *Neumann* (pure traction) eigenvalues. The *Dirichlet* (clamped) eigenvalue problem is obtained from (2.4) in the limiting case $\alpha \to +\infty$. The *Robin* eigenvalue problem is obtained from (2.4) by taking $\beta = 0$. If we take $\beta = 0$ and $\alpha = 0$ in (2.5), we obtain the *Steklov* eigenvalue problem. Finally, for the choice $\beta > 0$, we have the *Wentzell* eigenvalue problem (see [9] for the model and the derivation of the Wentzell boundary conditions in the elasticity case).

These eigenvalues problems arise as minimization of the associated *Rayleigh quotient* given by

$$\Lambda_\Omega(\Omega) = \inf_{\boldsymbol{u}\in\boldsymbol{\mathcal{H}}(\Omega)} \left\{ \frac{1}{\displaystyle\int_\Omega |\boldsymbol{u}|^2\,\mathrm{d}x} \left(\int_\Omega Ae(\boldsymbol{u}) : e(\boldsymbol{u})\,\mathrm{d}x + \alpha \int_{\partial\Omega} |\boldsymbol{u}|^2\,\mathrm{d}\varsigma(x) + \beta \int_{\partial\Omega} A_c e_\Gamma \boldsymbol{u} : e_\Gamma \boldsymbol{u}\,\mathrm{d}\varsigma(x) \right) \right\}, \tag{2.6}$$

and

$$\Lambda_{\partial\Omega}(\Omega) = \inf_{\boldsymbol{u}\in\boldsymbol{\mathcal{H}}(\Omega)} \left\{ \frac{1}{\displaystyle\int_{\partial\Omega} |\boldsymbol{u}|^2\,\mathrm{d}x} \left(\int_\Omega Ae(\boldsymbol{u}) : e(\boldsymbol{u})\,\mathrm{d}x + \alpha \int_{\partial\Omega} |\boldsymbol{u}|^2\,\mathrm{d}\varsigma(x) + \beta \int_{\partial\Omega} A_c e_\Gamma \boldsymbol{u} : e_\Gamma \boldsymbol{u}\,\mathrm{d}\varsigma(x) \right) \right\}. \tag{2.7}$$

Notice that, in the various eigenvalue problems, an appropriate choice of a subspace of $\mathbf{H}^1(\Omega)$ has to be made for $\boldsymbol{\mathcal{H}}(\Omega)$. For example, in the case of the first Dirichlet eigenvalue, we may choose $\beta = 0$ and $\boldsymbol{\mathcal{H}}(\Omega) = \mathbf{H}^1_0(\Omega)$ in (2.6). In the case of the first non-trivial Neumann or Steklov eigenvalue, we may choose $\alpha = 0$ and $\beta = 0$ and take $\boldsymbol{\mathcal{H}}(\Omega)$ to be the quotient space of $\mathbf{H}^1(\Omega)$ modulo the rigid transformations. In the case of the Wentzell eigenvalue problem, we are in the situation where $\beta > 0$ and we need to choose $\boldsymbol{\mathcal{H}}(\Omega)$ to be $\{\boldsymbol{u} \in \mathbf{H}^1(\Omega); \boldsymbol{u}_{|\partial\Omega} \in \mathbf{H}^1(\partial\Omega)\}$ with the associated norm $\left(\|\boldsymbol{u}\|^2_{\mathbf{H}^1(\Omega)} + \|\boldsymbol{u}\|^2_{\mathbf{H}^1(\partial\Omega)}\right)^{1/2}$ (quotiented over the subspace of rigid transformations if $\alpha = 0$).

We now state the results for these problems.

THEOREM 2.2. *Given a $\mathcal{C}^{2,1}$ domain Ω and $\boldsymbol{V}$ a smooth vector field, the semi-derivative $\Lambda'_\Omega(\Omega;\boldsymbol{V})$ of $\Lambda_\Omega(\Omega)$ in the direction of the vector field $\boldsymbol{V}$ exists and is given by*

$$\begin{aligned}\Lambda'_\Omega(\Omega;\boldsymbol{V}) = \inf\Bigg\{ \int_{\partial\Omega} \Big(& Ae(\boldsymbol{u}):e(\boldsymbol{u}) - 4Ae(\boldsymbol{u})\mathbf{n}\cdot\Pi_d e(\boldsymbol{u})\mathbf{n} \\ & + \alpha\boldsymbol{u}\cdot(\mathrm{H}\,\boldsymbol{u} + 2\partial_{\mathrm{n}}\boldsymbol{u} - 4\Pi_d e(\boldsymbol{u})\mathbf{n}) \\ & + \beta\left(\mathrm{H}A_{\mathrm{c}}e_\Gamma(\boldsymbol{u}):e_\Gamma(\boldsymbol{u}) - A_{\mathrm{c}}e_\Gamma(\boldsymbol{u}):\Pi_d\left(\nabla\boldsymbol{u}\,\mathrm{D}^2 b + \mathrm{D}^2 b\,{}^t\nabla\boldsymbol{u}\right)\Pi_d\right) \\ & + 2\beta\Big(A_{\mathrm{c}}e_\Gamma(\partial_{\mathrm{n}}\boldsymbol{u}):e_\Gamma(\boldsymbol{u}) - 2A_{\mathrm{c}}e_\Gamma(\boldsymbol{u}):\nabla_\Gamma\left(\Pi_d e(\boldsymbol{u})\mathbf{n}\right)\Big) - \Lambda_\Omega(\Omega)\,|\boldsymbol{u}|^2\Big)V_{\mathrm{n}}\;\mathrm{d}\varsigma(x)\Bigg\}.\end{aligned}$$

In the above, the inf *is taken with respect to all functions $\boldsymbol{u}\in\boldsymbol{\mathcal{H}}(\Omega)$ for which the value $\Lambda_\Omega(\Omega)$ is attained in* (2.6).

THEOREM 2.3. *Given a $\mathcal{C}^{2,1}$ domain Ω and $\boldsymbol{V}$ a smooth vector field, the semi-derivative $\Lambda'_{\partial\Omega}(\Omega;\boldsymbol{V})$ of $\Lambda_{\partial\Omega}(\Omega)$ in the direction of the vector field $\boldsymbol{V}$ exists and is given by*

$$\begin{aligned}\Lambda'_{\partial\Omega}(\Omega;\boldsymbol{V}) = \inf\Bigg\{ \int_{\partial\Omega} \Big(& Ae(\boldsymbol{u}):e(\boldsymbol{u}) - 4Ae(\boldsymbol{u})\mathbf{n}\cdot\Pi_d e(\boldsymbol{u})\mathbf{n} \\ & + \alpha\boldsymbol{u}\,(\mathrm{H}\,\boldsymbol{u} + 2\partial_{\mathrm{n}}\boldsymbol{u} - 4\Pi_d e(\boldsymbol{u})\mathbf{n}) \\ & + \beta\left(\mathrm{H}A_{\mathrm{c}}e_\Gamma(\boldsymbol{u}):e_\Gamma(\boldsymbol{u}) - A_{\mathrm{c}}e_\Gamma(\boldsymbol{u}):\Pi_d\left(\nabla\boldsymbol{u}\,\mathrm{D}^2 b + \mathrm{D}^2 b\,{}^t\nabla\boldsymbol{u}\right)\Pi_d\right) \\ & + 2\beta\Big(A_{\mathrm{c}}e_\Gamma(\partial_{\mathrm{n}}\boldsymbol{u}):e_\Gamma(\boldsymbol{u}) - 2A_{\mathrm{c}}e_\Gamma(\boldsymbol{u}):\nabla_\Gamma\left(\Pi_d e(\boldsymbol{u})\mathbf{n}\right)\Big) \\ & - \Lambda_{\partial\Omega}(\Omega)\,\boldsymbol{u}\cdot(\mathrm{H}\,\boldsymbol{u} + 2\partial_{\mathrm{n}}\boldsymbol{u} - 4\Pi_d e(\boldsymbol{u})\mathbf{n})\Big)V_{\mathrm{n}}\;\mathrm{d}\varsigma(x)\Bigg\},\end{aligned}$$

where the inf *is taken with respect to all functions $\boldsymbol{u}\in\boldsymbol{\mathcal{H}}(\Omega)$ for which the value $\Lambda_{\partial\Omega}(\Omega)$ is attained in* (2.7).

Apart from the case of Dirichlet and Neumann boundary conditions for the volume case stated in Theorem 2.2, the remaining others results are, completely new to our best knowledge, new even in the case of a simple eigenvalue. We finally underline that, even if it is not the same expression, the given formula given above coincides with the known expression in the corresponding to Dirichlet and Neumann case: this can be done with checked by direct a computation.

2.3. Shape derivative for eigenvalue problems of linear elasticity - composite materials. Consider now a subset Ω_1 of Ω with a $\mathcal{C}^{2,1}$ boundary and set $\Omega_2 := \Omega\setminus\overline{\Omega_1}$. We assume that there exists $\rho > 0$ such that $\|x-y\|\geq\rho$ for all $x\in\Omega_1$ and $y\in\partial\Omega$. We consider two isotropic elastic materials, with elasticity tensors $A_1\neq A_2$ given by (for $i=1,2$)

$$A_i\xi := 2\,\mu_i\,\xi + \lambda_i\,\mathrm{Tr}(\xi)\,\mathrm{I}_d,$$

with Lamé coefficients $\mu_i > 0$ and $\lambda_i > 0$, which occupy respectively the domains Ω_1 and Ω_2 with respective densities $\rho_1 > 0$ and $\rho_2 > 0$ (with $\rho_1\neq\rho_2$). We set

$$\rho := \rho_1\chi_{\Omega_1} + \rho_2\chi_{\Omega_2} \qquad \text{and} \qquad A := A_1\chi_{\Omega_1} + A_2\chi_{\Omega_2}.$$

As previously, $\mathbf{n}$ denotes the exterior unit normal to $\partial\Omega$. Moreover Γ stands for the interface between Ω_1 and Ω_2, that is

$$\Gamma := \partial\Omega_1 \cap \partial\Omega_2 = \partial\Omega_1,$$

and, on Γ, the notation $\mathbf{n}$ will represent the unit normal pointing outward from Ω_1, that is

$$\mathbf{n} = \mathbf{n}_1 = -\mathbf{n}_2$$

(where $\mathbf{n}_i$, $i = 1, 2$, represent the exterior unit normal to $\partial\Omega_i$). We summarize the notations in Figure 1. We also use the notation $[\cdot]$ in order to represent the jump on the interface Γ, that is, for a function u and a point $x \in \Gamma$,

$$[u](x) := \lim_{\varepsilon \to 0^+} \left(u(x - \varepsilon\mathbf{n}(x)) - u(x + \varepsilon\mathbf{n}(x))\right) = u_1 - u_2.$$

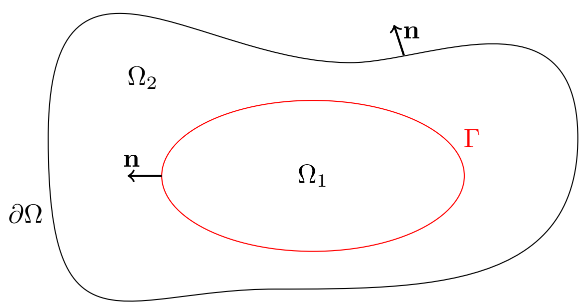

FIG. 1. *Notations*

We consider the eigenvalue problem of *volume type*

$$(2.8) \qquad \begin{cases} -\mathrm{div}\,(A(x)e(\boldsymbol{u})) = \mathfrak{M}_\Omega(\Omega)\rho(x)\boldsymbol{u} & \text{in } \Omega, \\ -\beta\mathrm{div}_\Gamma\,(A_c e_\Gamma(\boldsymbol{u})) + \alpha\boldsymbol{u} + Ae(\boldsymbol{u})\mathbf{n} = \mathbf{0} & \text{on } \partial\Omega, \end{cases}$$

and of *surface type*

$$(2.9) \qquad \begin{cases} -\mathrm{div}\,(A(x)e(\boldsymbol{u})) = \mathbf{0} & \text{in } \Omega, \\ -\beta\mathrm{div}_\Gamma\,(A_c e_\Gamma(\boldsymbol{u})) + \alpha\boldsymbol{u} + Ae(\boldsymbol{u})\mathbf{n} = \mathfrak{M}_{\partial\Omega}(\Omega)\boldsymbol{u} & \text{on } \partial\Omega. \end{cases}$$

As previsouly, for different regimes of the parameters α and β, we obtain different kinds of boundary conditions and the eigenvalues are associated to minimization of the Rayleigh quotients

$$(2.10) \quad \mathfrak{M}_\Omega(\Omega) = \inf_{\boldsymbol{u} \in \boldsymbol{\mathcal{H}}(\Omega)} \left\{ \frac{1}{\int_\Omega \rho\,|\boldsymbol{u}|^2} \left(\int_\Omega A(x)e(\boldsymbol{u}) : e(\boldsymbol{u})\,\mathrm{d}x \right.\right.$$
$$\left.\left. +\alpha \int_{\partial\Omega} |\boldsymbol{u}|^2\,\mathrm{d}\varsigma(x) + \beta \int_{\partial\Omega} A_c e_\Gamma(\boldsymbol{u}) : e_\Gamma(\boldsymbol{u})\,\mathrm{d}\varsigma(x) \right) \right\},$$

and

$$(2.11)\quad \mathfrak{M}_{\partial\Omega}(\Omega) = \inf_{\boldsymbol{u}\in\boldsymbol{\mathcal{H}}(\Omega)} \left\{ \frac{1}{\displaystyle\int_{\partial\Omega} |\boldsymbol{u}|^2 \,\mathrm{d}\varsigma(x)} \left(\int_\Omega A(x) e(\boldsymbol{u}) : e(\boldsymbol{u}) \,\mathrm{d}x \right.\right.$$
$$\left.\left. +\alpha \int_{\partial\Omega} |\boldsymbol{u}|^2 \,\mathrm{d}\varsigma(x) + \beta \int_{\partial\Omega} A_{\mathrm{c}} e_\Gamma(\boldsymbol{u}) : e_\Gamma(\boldsymbol{u}) \,\mathrm{d}\varsigma(x) \right) \right\},$$

where $\boldsymbol{\mathcal{H}}(\Omega)$ is an appropriate subspace of $\mathbf{H}^1(\Omega)$ as discussed above.

We now state the results for these problems.

THEOREM 2.4. *Let Ω be a $\mathcal{C}^{2,1}$ domain and $\boldsymbol{V}$ a smooth vector field. Let $\boldsymbol{u}$ be a normalized eigenfunction corresponding to $\mathfrak{M}_\Omega(\Omega)$. Then the semi-derivative $\mathfrak{M}'_\Omega(\Omega;\boldsymbol{V})$ of $\mathfrak{M}_\Omega(\Omega)$ in the direction of the vector field $\boldsymbol{V}$ exists and is given by*

$$\mathfrak{M}'_\Omega(\Omega;\boldsymbol{V}) = \inf \left\{ \int_{\partial\Omega_1} \Big([A e(\boldsymbol{u}) : e(\boldsymbol{u})] - 2 A e(\boldsymbol{u})\mathbf{n} \cdot [\partial_{\mathrm{n}} \boldsymbol{u}] - \mathfrak{M}_\Omega(\Omega)[\rho] |\boldsymbol{u}|^2 \Big) V_{\mathrm{n}} \,\mathrm{d}\varsigma(x) \right.$$
$$+ \int_{\partial\Omega} \Big(A e(\boldsymbol{u}) : e(\boldsymbol{u}) - 4 A e(\boldsymbol{u})\mathbf{n} \cdot \Pi_d e(\boldsymbol{u})\mathbf{n} + \alpha \boldsymbol{u} \cdot (\mathrm{H}\,\boldsymbol{u} + 2\partial_{\mathrm{n}}\boldsymbol{u} - 4\Pi_d e(\boldsymbol{u})\mathbf{n})$$
$$+ \beta \left(\mathrm{H} A_{\mathrm{c}} e_\Gamma(\boldsymbol{u}) : e_\Gamma(\boldsymbol{u}) - A_{\mathrm{c}} e_\Gamma(\boldsymbol{u}) : \Pi_d \left(\nabla\boldsymbol{u}\, \mathrm{D}^2 b + \mathrm{D}^2 b\, {}^t\nabla\boldsymbol{u} \right) \Pi_d \right)$$
$$\left. +2\beta \Big(A_{\mathrm{c}} e_\Gamma(\partial_{\mathrm{n}}\boldsymbol{u}) : e_\Gamma(\boldsymbol{u}) - 2 A_{\mathrm{c}} e_\Gamma(\boldsymbol{u}) : \nabla_\Gamma \left(\Pi_d e(\boldsymbol{u})\mathbf{n} \right) \Big) - \mathfrak{M}_\Omega(\Omega) \rho_2 |\boldsymbol{u}|^2 \Big) V_{\mathrm{n}} \,\mathrm{d}\varsigma(x) \Big) \right\},$$

where the inf *is taken over all functions $\boldsymbol{u} \in \boldsymbol{\mathcal{H}}(\Omega)$ for which the value $\Lambda_\Omega(\Omega)$ is attained in (2.10).*

THEOREM 2.5. *Let Ω be a $\mathcal{C}^{2,1}$ domain and $\boldsymbol{V}$ a smooth vector field. Let $\boldsymbol{u}$ be a normalized eigenfunction corresponding to $\mathfrak{M}_{\partial\Omega}$. Then the semi-derivative $\mathfrak{M}'_{\partial\Omega}(\Omega;\boldsymbol{V})$ of $\mathfrak{M}_{\partial\Omega}(\Omega)$ in the direction of the vector field $\boldsymbol{V}$ exists and is given by*

$$\mathfrak{M}'_{\partial\Omega}(\Omega;\boldsymbol{V}) = \inf \left\{ \int_{\partial\Omega_1} \Big([A e(\boldsymbol{u}) : e(\boldsymbol{u})] - 2 A e(\boldsymbol{u})\mathbf{n} \cdot [\partial_{\mathrm{n}} \boldsymbol{u}] \Big) V_{\mathrm{n}} \,\mathrm{d}\varsigma(x) \right.$$
$$+ \int_{\partial\Omega} \Big(A e(\boldsymbol{u}) : e(\boldsymbol{u}) - 4 A e(\boldsymbol{u})\mathbf{n} \cdot \Pi_d e(\boldsymbol{u})\mathbf{n} + \alpha \boldsymbol{u} \cdot (\mathrm{H}\,\boldsymbol{u} + 2\partial_{\mathrm{n}}\boldsymbol{u} - 4\Pi_d e(\boldsymbol{u})\mathbf{n})$$
$$+ \beta \left(\mathrm{H} A_{\mathrm{c}} e_\Gamma(\boldsymbol{u}) : e_\Gamma(\boldsymbol{u}) - A_{\mathrm{c}} e_\Gamma(\boldsymbol{u}) : \Pi_d \left(\nabla\boldsymbol{u}\, \mathrm{D}^2 b + \mathrm{D}^2 b\, {}^t\nabla\boldsymbol{u} \right) \Pi_d \right)$$
$$+ 2\beta \Big(A_{\mathrm{c}} e_\Gamma(\partial_{\mathrm{n}}\boldsymbol{u}) : e_\Gamma(\boldsymbol{u}) - 2 A_{\mathrm{c}} e_\Gamma(\boldsymbol{u}) : \nabla_\Gamma \left(\Pi_d e(\boldsymbol{u})\mathbf{n} \right) \Big)$$
$$\left. - \mathfrak{M}_{\partial\Omega}(\Omega) \rho_2\, \boldsymbol{u} \cdot (\mathrm{H}\,\boldsymbol{u} + 2\partial_{\mathrm{n}}\boldsymbol{u} - 4\Pi_d e(\boldsymbol{u})\mathbf{n}) \Big) V_{\mathrm{n}} \,\mathrm{d}\varsigma(x) \Big) \right\},$$

where the inf *is taken over all $\boldsymbol{u} \in \boldsymbol{\mathcal{H}}(\Omega)$ for which the value $\mathfrak{M}_{\partial\Omega}(\Omega)$ is attained in (2.11).*

Obviously, Theorem 2.2 (respectively Theorem 2.3) can be obtained as a particular case of Theorem 2.4 (respectively Theorem 2.5) by letting $A_1 = A_2 = A$

and $\rho_1 = \rho_2 = 1$. Even so, we present the proofs of Theorem 2.2 (respectively Theorem 2.3) since the main ideas can be illustrated more clearly in these particular cases.

Remark 2.6. We can notice that the formulae of the above Theorems 2.2, 2.3, 2.4 and 2.5 are a little bit different and more complicated that the scalar case exposed in [8]. Indeed, in the scalar case, some simplifications occur, which is not the case in the elasticity case: this is linked with the differences underlined in Remark 2.1 (see also Remark 3.4 below).

3. Proofs. The shape derivative results stated in the previous section will be established in this section in the framework of Theorem 1.1 by following a general strategy which we employed in the scalar problems (see [8]) and is recalled below for the benefit of the reader.

3.1. General strategy. The first step is to reformulate the eigenvalue problem for the perturbed domain Ω_t, which is obtained by the minimization of a Rayleigh quotient, as a minimization problem for a functional $G(t,\cdot)$ in a space $\boldsymbol{\mathcal{H}}(\Omega)$ which is independent of the parameter t.

The next step consists in verifying that the assumptions of Theorem 1.1 are satisfied. For verifying the hypothesis $(H3)$, in the class of eigenvalue problems, we usually need to show the Γ-*convergence* (see Appendix A for some reminders on this notion) of $G(t,\cdot)$ to $G(0,\cdot)$ as $t \to 0^+$ in the weak topology of $\boldsymbol{\mathcal{H}}(\Omega)$ and later the strong convergence of a sequence of minimizers.

Then, Theorem 1.1 allows us to immediately calculate the shape derivative by evaluating $\inf_{\boldsymbol{u}\in \boldsymbol{X}(0)} \partial_t G(0,\boldsymbol{u})$ where $\boldsymbol{X}(0)$ is, generally, an eigenspace for the problem on Ω. In the case of a simple eigenfunction, it is enough to evaluate at a normalized eigenfunction. An initial expression for $\partial_t G(0,\boldsymbol{u})$ is obtained by using the propositions given in the following subsection and this gives an integral over the domain Ω.

As a last step, we transform and simplify the initial calculation of $\partial_t G(0,\boldsymbol{u})$, to get a boundary expression for $\partial_t G(0,\boldsymbol{u})$. This can be usually achieved by choosing $-\nabla \boldsymbol{u}\boldsymbol{V}$ as a test function in the governing equation, provided that it has enough regularity.

3.2. Preliminary computations. Before computing the shape derivatives, we first prove some preliminary results. We compute the separate contributions of the different terms of the Rayleigh quotient to the derivatives $\partial_t G(0,\boldsymbol{u})$ in the various problems. For this, we rely on the classical formulae in the calculation of shape derivatives which are recalled in Lemma A.1 and Lemma A.2 in the appendix.

PROPOSITION 3.1. *For* $\boldsymbol{u} \in \mathbf{H}^1(\Omega)$, *we have*

$$\partial_t \left(\int_{\Omega_t} Ae(\boldsymbol{u}\circ\Psi_t^{-1}) : e(\boldsymbol{u}\circ\Psi_t^{-1}) \ \mathrm{d}x \right) \Big|_{t=0} = \int_{\partial\Omega} Ae(\boldsymbol{u}):e(\boldsymbol{u})\, V_{\mathrm{n}} \ \mathrm{d}\varsigma(x) + 2\int_{\Omega} Ae(\boldsymbol{u}):e(-\nabla\boldsymbol{u}\boldsymbol{V}) \ \mathrm{d}x, \tag{3.1}$$

$$\partial_t \left(\int_{\Omega_t} |(\boldsymbol{u}\circ\Psi_t^{-1})|^2 \mathrm{d}x \right) \Big|_{t=0} = \int_{\partial\Omega} |\boldsymbol{u}|^2\, V_{\mathrm{n}} \ \mathrm{d}\varsigma(x) + 2\int_{\Omega} \boldsymbol{u}\cdot(-\nabla\boldsymbol{u}\boldsymbol{V}) \ \mathrm{d}x \tag{3.2}$$

and

$$\partial_t \left(\int_{\partial\Omega_t} |(\boldsymbol{u}\circ\Psi_t^{-1})|^2 \mathrm{d}x \right) \Big|_{t=0} = \tag{3.3}$$

$$\int_{\partial\Omega} \left(\mathrm{H}\, |\boldsymbol{u}|^2 + \partial_{\mathrm{n}} |\boldsymbol{u}|^2\right) V_{\mathrm{n}}\ \mathrm{d}\varsigma(x) + 2\int_{\partial\Omega} \boldsymbol{u}\cdot(-\nabla\boldsymbol{u}\boldsymbol{V})\ \mathrm{d}\varsigma(x).$$

Proof. The above formulae are obtained by a straighforward application of the formulae for derivatives of domain and boundary integrals given in Lemma A.1 and Lemma A.2 in the appendix and the fact that $\partial_t(\boldsymbol{u}\circ\Psi_t^{-1})|_{t=0} = -\nabla\boldsymbol{u}\boldsymbol{V}$ since $\partial_t(\Psi_t^{-1})|_{t=0} = -\boldsymbol{V}$ (see [19, equation (5.7)]). ∎

Proposition 3.2. *For $\boldsymbol{u}\in\mathbf{H}^2(\Omega)$, we have*

$$\begin{aligned}(3.4)\quad \partial_t\Big(\int_{\partial\Omega_t} A_{\mathrm{c}} e_{\Gamma_t}(\boldsymbol{u}\circ\Psi_t^{-1}) : e_{\Gamma_t}(\boldsymbol{u}\circ\Psi_t^{-1})\ \mathrm{d}x\Big)\Big|_{t=0} \\ = \int_{\partial\Omega}\Big(\mathrm{H}\, A_{\mathrm{c}} e_\Gamma(\boldsymbol{u}) : e_\Gamma(\boldsymbol{u}) + 2A_{\mathrm{c}} e_\Gamma(\partial_{\mathrm{n}}\boldsymbol{u}) : e_\Gamma(\boldsymbol{u}) \\ - A_{\mathrm{c}} e_\Gamma(\boldsymbol{u}) : \Pi_d\left(\nabla\boldsymbol{u}\,\mathrm{D}^2 b + \mathrm{D}^2 b\,{}^t\nabla\boldsymbol{u}\right)\Pi_d\Big) V_{\mathrm{n}}\ \mathrm{d}\varsigma(x) \\ + 2\int_{\partial\Omega} A_{\mathrm{c}} e_\Gamma(\boldsymbol{u}) : \Big(e_\Gamma(-\nabla\boldsymbol{u}\boldsymbol{V}) + \Pi_d e(\boldsymbol{u})(\mathbf{n}\otimes\nabla_\Gamma V_{\mathrm{n}} + \nabla_\Gamma V_{\mathrm{n}}\otimes\mathbf{n}) \\ + (\mathbf{n}\otimes\nabla_\Gamma V_{\mathrm{n}} + \nabla_\Gamma V_{\mathrm{n}}\otimes\mathbf{n}) e(\boldsymbol{u})\Pi_d\Big)\ \mathrm{d}\varsigma(x).\end{aligned}$$

Proof. By applying the classical derivation formula recalled in Lemma A.2, we get

$$\begin{aligned}\partial_t\Big(\int_{\partial\Omega_t} A_{\mathrm{c}} e_{\Gamma_t}(\boldsymbol{u}\circ\Psi_t^{-1}) : e_{\Gamma_t}(\boldsymbol{u}\circ\Psi_t^{-1})\ \mathrm{d}x\Big)\Big|_{t=0} \\ = \int_{\partial\Omega}\left(\mathrm{H}\, A_{\mathrm{c}} e_\Gamma(\boldsymbol{u}) : e_\Gamma(\boldsymbol{u}) + \partial_{\mathrm{n}}(A_{\mathrm{c}} e_\Gamma(\boldsymbol{u}) : e_\Gamma(\boldsymbol{u}))\right) V_{\mathrm{n}}\ \mathrm{d}\varsigma(x) \\ + 2\int_{\partial\Omega} A_{\mathrm{c}} e_\Gamma(\boldsymbol{u}) : \partial_t\big(e_{\Gamma_t}(\boldsymbol{u}\circ\Phi_t^{-1})\big)\,|_{t=0}\mathrm{d}\varsigma(x).\end{aligned}$$

We conclude using the fact (see respectively Lemma A.3 and Lemma A.4)

$$\partial_{\mathrm{n}}\left(A_{\mathrm{c}} e_\Gamma(\boldsymbol{u}) : e_\Gamma(\boldsymbol{u})\right) = 2A_{\mathrm{c}} e_\Gamma(\partial_{\mathrm{n}}\boldsymbol{u}) : e_\Gamma(\boldsymbol{u}) - A_{\mathrm{c}} e_\Gamma(\boldsymbol{u}) : \Pi_d\left(\nabla\boldsymbol{u}\,\mathrm{D}^2 b + \mathrm{D}^2 b\,{}^t\nabla\boldsymbol{u}\right)\Pi_d,$$

and

$$\begin{aligned}\partial_t\big(e_{\Gamma_t}(\boldsymbol{u}\circ\Psi_t^{-1})\big)\,|_{t=0} = e_\Gamma(-\nabla\boldsymbol{u}\boldsymbol{V}) \\ + \Pi_d e(\boldsymbol{u})(\mathbf{n}\otimes\nabla_\Gamma V_{\mathrm{n}} + \nabla_\Gamma V_{\mathrm{n}}\otimes\mathbf{n}) + (\mathbf{n}\otimes\nabla_\Gamma V_{\mathrm{n}} + \nabla_\Gamma V_{\mathrm{n}}\otimes\mathbf{n}) e(\boldsymbol{u})\Pi_d.\end{aligned}$$ ∎

We also need the following proposition concerning the case of eigenvalue problems for composites. Let Ω_1 and Ω_2 be a subdivision of Ω as presented in Section 2.3 with the corresponding notations for normal vectors and for the jumps of functions. Then we define following perturbed elasticity tensor and density

$$A_t := A_1\chi_{\Omega_{1,t}} + A_2\chi_{\Omega_{2,t}} \qquad \text{and} \qquad \rho_t := \rho_1\chi_{\Omega_{1,t}} + \rho_2\chi_{\Omega_{2,t}}$$

with

$$\Omega_{1,t} := \Psi_t(\Omega_1) \qquad \text{and} \qquad \Omega_{2,t} := \Psi_t(\Omega_2).$$

PROPOSITION 3.3. *For $\boldsymbol{u} \in \mathbf{H}^1(\Omega)$, we have*

$$\partial_t \left(\int_{\Omega_t} A_t e(\boldsymbol{u} \circ \Psi_t^{-1}) : e(\boldsymbol{u} \circ \Psi_t^{-1}) \, \mathrm{d}x \right) \Big|_{t=0} = \int_\Gamma [Ae(\boldsymbol{u}) : e(\boldsymbol{u})] V_\mathrm{n} \, \mathrm{d}\varsigma(x) + 2 \int_{\Omega_1} A_1 e(\boldsymbol{u}) : e(-\nabla \boldsymbol{u} \boldsymbol{V}) \, \mathrm{d}x + 2 \int_{\Omega_2} A_2 e(\boldsymbol{u}) : e(-\nabla \boldsymbol{u} \boldsymbol{V}) \, \mathrm{d}x + \int_{\partial\Omega} A_2 e(\boldsymbol{u}) : e(\boldsymbol{u}) V_\mathrm{n} \, \mathrm{d}\varsigma(x) \tag{3.5}$$

and

$$\partial_t \left(\int_{\Omega_t} \rho_t |(\boldsymbol{u} \circ \Psi_t^{-1})|^2 \mathrm{d}x \right) \Big|_{t=0} = \int_\Gamma [\rho |\boldsymbol{u}|^2] V_\mathrm{n} \, \mathrm{d}\varsigma(x) + 2 \int_{\Omega_1} \rho_1 \boldsymbol{u} \cdot (-\nabla \boldsymbol{u} \boldsymbol{V}) \, \mathrm{d}x + 2 \int_{\Omega_2} \rho_2 \boldsymbol{u} \cdot (-\nabla \boldsymbol{u} \boldsymbol{V}) \, \mathrm{d}x + \int_{\partial\Omega} \rho_2 |\boldsymbol{u}|^2 V_\mathrm{n} \, \mathrm{d}\varsigma(x). \tag{3.6}$$

Proof. The above formulae are obtained by an application of Lemma A.1 to each of the terms on the right hand side after writing

$$\int_{\Omega_t} A_t e(\boldsymbol{u} \circ \Psi_t^{-1}) : e(\boldsymbol{u} \circ \Psi_t^{-1}) \, \mathrm{d}x = \int_{\Omega_{1,t}} A_1 e(\boldsymbol{u} \circ \Psi_t^{-1}) : e(\boldsymbol{u} \circ \Psi_t^{-1}) \, \mathrm{d}x + \int_{\Omega_{2,t}} A_2 e(\boldsymbol{u} \circ \Psi_t^{-1}) : e(\boldsymbol{u} \circ \Psi_t^{-1}) \, \mathrm{d}x \quad \square$$

and

$$\int_{\Omega_t} \rho_t |(\boldsymbol{u} \circ \Psi_t^{-1})|^2 \mathrm{d}x = \int_{\Omega_{1,t}} \rho_1 |(\boldsymbol{u} \circ \Psi_t^{-1})|^2 \mathrm{d}x + \int_{\Omega_{2,t}} \rho_2 |(\boldsymbol{u} \circ \Psi_t^{-1})|^2 \mathrm{d}x.$$

3.3. Semi-derivatives for single phase isotropic materials.

3.3.1. Proof of Theorem 2.2. The considered eigenvalue functional on the perturbed domain is

$$\Lambda_\Omega(\Omega_t) = \inf_{\boldsymbol{v} \in \boldsymbol{\mathcal{H}}(\Omega_t)} \left\{ \frac{1}{\displaystyle\int_{\Omega_t} |\boldsymbol{v}|^2 \, \mathrm{d}x} \left(\int_{\Omega_t} Ae(\boldsymbol{v}) : e(\boldsymbol{v}) \, \mathrm{d}x + \alpha \int_{\partial\Omega_t} |\boldsymbol{v}|^2 \, \mathrm{d}\varsigma(x) + \beta \int_{\partial\Omega_t} A_\mathrm{c} e_\Gamma(\boldsymbol{v}) : e_\Gamma(\boldsymbol{v}) \, \mathrm{d}\varsigma(x) \right) \right\}, \tag{3.7}$$

where $\boldsymbol{\mathcal{H}}(\Omega_t)$ is a suitable subspace of $\mathbf{H}^1(\Omega_t)$ as discussed in Section 2.2. Since the function space $\boldsymbol{\mathcal{H}}(\Omega_t)$ gets mapped to a function space $\boldsymbol{\mathcal{H}}(\Omega)$ which is independent of t under the isomorphism $\boldsymbol{v} \mapsto \boldsymbol{v} \circ \Psi_t$, the above functional can be obtained as a minimization problem over $\boldsymbol{\mathcal{H}}(\Omega)$ as follows

$$\Lambda_\Omega(\Omega_t) = \inf_{\boldsymbol{u} \in \boldsymbol{\mathcal{H}}(\Omega)} G_\Omega(t, \boldsymbol{u})$$

where the functional G_Ω is defined by

$$G_\Omega(t,\boldsymbol{u}) := \frac{1}{\displaystyle\int_{\Omega_t} \left|\boldsymbol{u}\circ\Psi_t^{-1}\right|^2 \mathrm{d}x}\Bigg(\int_{\Omega_t} Ae(\boldsymbol{u}\circ\Psi_t^{-1}):e(\boldsymbol{u}\circ\Psi_t^{-1})\,\mathrm{d}x \tag{3.8}$$
$$+\alpha\int_{\partial\Omega_t}\left|\boldsymbol{u}\circ\Psi_t^{-1}\right|^2\mathrm{d}\varsigma(x)+\beta\int_{\partial\Omega_t} A_{\mathrm{c}}e_\Gamma(\boldsymbol{u}\circ\Psi_t^{-1}):e_\Gamma(\boldsymbol{u}\circ\Psi_t^{-1})\,\mathrm{d}\varsigma(x)\Bigg).$$

Existence of the semi-derivative. First, we check that the assumptions of Theorem 1.1 are satisfied for the above functional G_Ω.

Let us start by Assumption $(H1)$. The arguments to show that the set of minimizers of (3.7) is non-empty for each t is classical and is based on the direct method of calculus of variations. In fact, the functional is lower semi-continuous for the weak topology on $\boldsymbol{\mathcal{H}}(\Omega_t)$, since the numerator is convex and continuous for the strong topology on $\boldsymbol{\mathcal{H}}(\Omega_t)$ (and therefore weakly lower semi-continuous), and since the denominator is continuous due to the compact inclusion of $\mathbf{H}^1(\Omega_t)$ into $\mathbf{L}^2(\Omega_t)$. As concerns the coercivity of the functional for given t, it is enough to show that the numerator dominates square of the norm or a quotient norm on $\boldsymbol{\mathcal{H}}(\Omega_t)$. In the case of Dirichlet eigenvalue problem, this can be obtained from the coercivity of the tensor A and by the use of *Korn's inequality* (see, e.g., [1, Lemma 2.25] or [16, Theorem 3.1]). In the case of the first non-trivial Neumann eigenvalue problem, one uses the coercivity of the tensor A and the generalized Korn's inequality, that is, Korn's inequality modulo rigid transformations. When $\alpha > 0$ it is enough, once again, to use Korn's inequality without quotienting. The set $\boldsymbol{X}(t)$, defined in Theorem 1.1, of minimizers for $G_\Omega(t,\cdot)$ is obtained by transporting the minimizers in (3.7) to Ω by composition with Ψ_t. Therefore Assumption $(H1)$ is satisfied.

Let us now check Assumption $(H2)$. Since $\nabla(\boldsymbol{u}\circ\Psi_t^{-1}) = \left(\nabla\boldsymbol{u}\,\mathrm{D}\Psi_t^{-1}\right)\circ\Psi_t^{-1}$, we have

$$e(\boldsymbol{u}\circ\Psi_t^{-1}) = \left(\nabla\boldsymbol{u}\,\mathrm{D}\Psi_t^{-1}\right)^{\mathrm{sym}}\circ\Psi_t^{-1}.$$

and

$$e_\Gamma(\boldsymbol{u}\circ\Psi_t^{-1}) = (\mathrm{I}_d-\mathbf{n}_t\otimes\mathbf{n}_t)\left(\nabla\boldsymbol{u}\,\mathrm{D}\Psi_t^{-1}\right)^{\mathrm{sym}}\circ\Psi_t^{-1}\,(\mathrm{I}_d-\mathbf{n}_t\otimes\mathbf{n}_t),$$

where $\mathbf{n}_t$ the normal vector field on $\partial\Omega_t$. Therefore,

$$G_\Omega(t,\boldsymbol{u})$$
$$= \frac{1}{\displaystyle\int_{\Omega_t}\left|\boldsymbol{u}\circ\Psi_t^{-1}\right|^2\mathrm{d}x}\Bigg(\int_{\Omega_t} A\left(\nabla\boldsymbol{u}\,\mathrm{D}\Psi_t^{-1}\right)^{\mathrm{sym}}\circ\Psi_t^{-1} : \left(\nabla\boldsymbol{u}\,\mathrm{D}\Psi_t^{-1}\right)^{\mathrm{sym}}\circ\Psi_t^{-1}\ \mathrm{d}x$$
$$+\alpha\int_{\partial\Omega_t}\left|\boldsymbol{u}\circ\Psi_t^{-1}\right|^2\ \mathrm{d}\varsigma(x)+\beta\int_{\partial\Omega_t} A_{\mathrm{c}}(\mathrm{I}_d-\mathbf{n}_t\otimes\mathbf{n}_t)\left(\nabla\boldsymbol{u}\,\mathrm{D}\Psi_t^{-1}\right)^{\mathrm{sym}}\circ\Psi_t^{-1}\,(\mathrm{I}_d-\mathbf{n}_t\otimes\mathbf{n}_t)$$
$$:(\mathrm{I}_d-\mathbf{n}_t\otimes\mathbf{n}_t)\left(\nabla\boldsymbol{u}\,\mathrm{D}\Psi_t^{-1}\right)^{\mathrm{sym}}\circ\Psi_t^{-1}\,(\mathrm{I}_d-\mathbf{n}_t\otimes\mathbf{n}_t)\,\mathrm{d}\varsigma(x)\Bigg).$$

Then, by a change of variables, this can be written as

$$G_\Omega(t,\boldsymbol{u}) = \frac{1}{\displaystyle\int_\Omega|\boldsymbol{u}|^2 j(t)\mathrm{d}x}\Bigg(\int_\Omega A\left(\nabla\boldsymbol{u}\,\mathrm{D}\Psi_t^{-1}\right)^{\mathrm{sym}} : \left(\nabla\boldsymbol{u}\,\mathrm{D}\Psi_t^{-1}\right)^{\mathrm{sym}} j(t)\mathrm{d}x \tag{3.9}$$

$$+\alpha\int_{\partial\Omega}|\boldsymbol{u}|^2\omega(t)\mathrm{d}\varsigma(x)+\beta\int_{\partial\Omega}A_{\mathrm{c}}(\mathrm{I}_d-\boldsymbol{\nu}_t\otimes\boldsymbol{\nu}_t)\left(\nabla\boldsymbol{u}\ \mathrm{D}\Psi_t^{-1}\right)^{\mathrm{sym}}(\mathrm{I}_d-\boldsymbol{\nu}_t\otimes\boldsymbol{\nu}_t)$$

$$:(\mathrm{I}_d-\boldsymbol{\nu}_t\otimes\boldsymbol{\nu}_t)\left(\nabla\boldsymbol{u}\ \mathrm{D}\Psi_t^{-1}\right)^{\mathrm{sym}}(\mathrm{I}_d-\boldsymbol{\nu}_t\otimes\boldsymbol{\nu}_t)\omega(t)\mathrm{d}\varsigma(x)\Bigg),$$

where

$$j(t):=\det(\mathrm{D}\Psi_t(x))\qquad\text{and}\qquad\omega(t):=\det(\mathrm{D}\Psi_t(x))\|{}^t(\mathrm{D}\Psi_t^{-1})(x)\mathbf{n}(x)\|$$

are respectively the Jacobian and the surface Jacobian and where

$$\boldsymbol{\nu}_t:=\mathbf{n}_t\circ\Psi_t.$$

Clearly, by the definition (2.1), Ψ_t depends smoothly on t and, for t small enough, Ψ_t is a diffeomorphism by which we have that $j(t)$ and $\omega(t)$ are smooth functions of t. Also, since $\partial\Omega$ is smooth, it follows that Ω_t has the same smoothness of Ω and therefore, $\mathbf{n}_t$ is differentiable with respect to t for t small enough (see, e.g., [19, Proposition 5.4.14]). Therefore, we are able to conclude from the previous expression (3.9) that $G_\Omega(\cdot,\boldsymbol{u})$ is derivable for all t small enough, for all $\boldsymbol{u}\in\boldsymbol{\mathcal{H}}(\Omega)$, and this gives the hypothesis $(H2)$ of Theorem 1.1.

Before proving Assumption $(H3)$, let us focus briefly on Assumption $(H4)$. The derivative $\partial_t G_\Omega(\cdot,\boldsymbol{u})$ may be obtained by deriving under the integral sign in the previous equation and since all the integrands are $\mathcal{C}^1$ functions of t, it follows that $\partial_t G_\Omega(\cdot,\boldsymbol{u})$ is also continuous with respect to t, for t small enough. This gives Assumption $(H4)$.

We now proceed to show that the hypothesis $(H3)$ holds for the strong topology on $\boldsymbol{\mathcal{H}}(\Omega)$. This will be achieved through the following steps. First, we show that $G_\Omega(t,\cdot)$ converges, in the sense of Γ-limit, to $G_\Omega(0,\cdot)$ as $t\to 0^+$, in the weak topology on $\boldsymbol{\mathcal{H}}(\Omega)$ (see Definition A.6 and Proposition A.7 in the Appendix for some reminders on this notion; also refer to [11]).

(i) Consider a sequence $\{\boldsymbol{u}^t\}$ which converges weakly to a $\boldsymbol{u}$ in $\boldsymbol{\mathcal{H}}(\Omega)$. We obtain the estimate

$$G_\Omega(t,\boldsymbol{u}^t)=G_\Omega(0,\boldsymbol{u}^t)+(G_\Omega(t,\boldsymbol{u}^t)-G_\Omega(0,\boldsymbol{u}^t))\geq G_\Omega(0,\boldsymbol{u}^t)+O(t).$$

Indeed, since any weakly convergent sequence $\{\boldsymbol{u}^t\}$ is bounded in $\boldsymbol{\mathcal{H}}(\Omega)$ and the coefficients in both the numerator and denominator of G_Ω given by (3.9) are continuous in t, we obtain that $G_\Omega(t,\boldsymbol{u}^t)-G_\Omega(0,\boldsymbol{u}^t)$ is $O(t)$ (that is, goes to 0 as $t\to 0^+$). Then, to conclude the $\Gamma-\liminf$ inequality of Definition A.6, it is enough to use the already observed fact that $G_\Omega(0,\cdot)$ is lower semi-continuous for the weak topology on $\boldsymbol{\mathcal{H}}(\Omega)$.

(ii) The $\Gamma-\limsup$ inequality of Definition A.6 is obtained by taking the constant sequence $\boldsymbol{u}$, for any given $\boldsymbol{u}\in\boldsymbol{\mathcal{H}}(\Omega)$, and observing as previously that $G_\Omega(t,\boldsymbol{u})\to G_\Omega(0,\boldsymbol{u})$ as $t\to 0^+$.

Having obtained the Γ-convergence of $G_\Omega(t,\cdot)$, Proposition A.7 ensures that $\Lambda_\Omega(\Omega_t)\to\Lambda_\Omega(\Omega)$ as $t\to 0^+$ since the minimum of $G_\Omega(t,\cdot)$ converges to the minimum of $G_\Omega(0,\cdot)$. Moreover, the 0-homogeneity of the Rayleigh quotients $G_\Omega(t,\cdot)$ means that, for each t, it is enough to consider a minimizer $\boldsymbol{u}^t$ for which the denominator is 1. Under this normalization, we have the equi-coercivity of $G_\Omega(t,\cdot)$ using the coercivity of the tensor A and Korn's inequality by the same arguments used during the verification of the hypothesis $(H1)$: there exists constant a positive constant C such that

$$C\|\boldsymbol{u}^t\|^2_{\boldsymbol{\mathcal{H}}(\Omega)}\leq G_\Omega(t,\boldsymbol{u}^t),\qquad\text{for all } t.$$

This implies, by Proposition A.7, that $\{\boldsymbol{u}^t\}$ converges weakly in $\boldsymbol{\mathcal{H}}(\Omega)$ to a minimizer $\boldsymbol{u}$ of $G_\Omega(0,\cdot)$. To conclude this part, we will prove the strong convergence of $\{\boldsymbol{u}^t\}$ to $\boldsymbol{u}$ in $\boldsymbol{\mathcal{H}}(\Omega)$. The equi-coercivity can be used once again to give us the following inequality:

$$C\|\boldsymbol{u}^t-\boldsymbol{u}\|^2_{\boldsymbol{\mathcal{H}}(\Omega)} \leq G_\Omega(t,\boldsymbol{u}^t-\boldsymbol{u}).$$

It remains to prove that $G_\Omega(t,\boldsymbol{u}^t-\boldsymbol{u})\to 0$ when $t\to 0$. Expanding the quadratic function $G_\Omega(t,\cdot)$ on the right hand side we get

$$\begin{aligned}
G_\Omega(t,\boldsymbol{u}^t-\boldsymbol{u}) &= G_\Omega(t,\boldsymbol{u}^t)+G_\Omega(t,\boldsymbol{u})\\
&-2\Bigg(\int_\Omega A\left(\nabla\boldsymbol{u}^t\ \mathrm{D}\Psi_t^{-1}\right)^{\mathrm{sym}} : \left(\nabla\boldsymbol{u}\ \mathrm{D}\Psi_t^{-1}\right)^{\mathrm{sym}} j(t)\mathrm{d}x+\alpha\int_{\partial\Omega}\boldsymbol{u}^t:\boldsymbol{u}\ \omega(t)\mathrm{d}\varsigma(x)\\
&\qquad+\beta\int_{\partial\Omega}A_{\mathrm{c}}(\mathrm{I}_d-\boldsymbol{\nu}_t\otimes\boldsymbol{\nu}_t)\left(\nabla\boldsymbol{u}^t\ \mathrm{D}\Psi_t^{-1}\right)^{\mathrm{sym}}(\mathrm{I}_d-\boldsymbol{\nu}_t\otimes\boldsymbol{\nu}_t)\\
&\qquad\qquad:(\mathrm{I}_d-\boldsymbol{\nu}_t\otimes\boldsymbol{\nu}_t)\left(\nabla\boldsymbol{u}\ \mathrm{D}\Psi_t^{-1}\right)^{\mathrm{sym}}(\mathrm{I}_d-\boldsymbol{\nu}_t\otimes\boldsymbol{\nu}_t)\omega(t)\mathrm{d}\varsigma(x)\Bigg).
\end{aligned}$$

Then we use the uniform convergence of the coefficients, the weak convergence of $\{\boldsymbol{u}^t\}$ to $\boldsymbol{u}$ and the convergence of $\Lambda_\Omega(\Omega_t)$ to $\Lambda_\Omega(\Omega)$ to obtain that

$$\begin{aligned}
G_\Omega(t,\boldsymbol{u}^t-\boldsymbol{u}) &\longrightarrow \Lambda_\Omega(\Omega)+\Lambda_\Omega(\Omega)\\
&\quad-2\left\{\int_\Omega Ae(\boldsymbol{u}):e(\boldsymbol{u})\mathrm{d}x+\alpha\int_{\partial\Omega}2|\mathbf{u}|^2\mathrm{d}\varsigma(x)+\beta\int_{\partial\Omega}A_{\mathrm{c}}e_\Gamma(\boldsymbol{u}):e_\Gamma(\boldsymbol{u})\mathrm{d}\varsigma(x)\right\}\\
&= \Lambda_\Omega(\Omega)+\Lambda_\Omega(\Omega)-2\Lambda_\Omega(\Omega)=0.
\end{aligned}$$

Hence $\{\boldsymbol{u}^t\}$ converges strongly to $\boldsymbol{u}$ in $\boldsymbol{\mathcal{H}}(\Omega)$ and, since we have seen that $\partial_t G_\Omega(\cdot,\boldsymbol{u})$ is continuous with respect to t, this proves hypothesis $(H3)$.

The existence of the semi-derivative $\Lambda'_\Omega(\Omega;\boldsymbol{V})$ follows from Theorem 1.1 since we have proved above that the four assumptions of the theorem are satisfied for G_Ω.

Computation of the directional shape derivatives. We want to obtain a suitable expression for $\partial_t G_\Omega(0,\boldsymbol{u})$ whenever $\boldsymbol{u}$ is a normalized eigenfunction for $\Lambda_\Omega(\Omega)$ since, by the theorem,

$$\Lambda'_\Omega(\Omega;\boldsymbol{V})=\inf\{\partial_t G_\Omega(0,\boldsymbol{u});\ \Lambda_\Omega(\Omega)\text{ is attained at }\boldsymbol{u}\}.$$

First, using the expressions (3.1)-(3.4) evaluated at $t=0$, we get

$$\begin{aligned}
\partial_t G(0,\boldsymbol{u}) &= \int_{\partial\Omega}Ae(\boldsymbol{u}):e(\boldsymbol{u})V_{\mathrm{n}}\ \mathrm{d}\varsigma(x)+2\int_\Omega Ae(\boldsymbol{u}):e(-\nabla\boldsymbol{u}\boldsymbol{V})\ \mathrm{d}x\\
&\quad+\alpha\left(\int_{\partial\Omega}\left(\mathrm{H}\,|\boldsymbol{u}|^2+\partial_{\mathrm{n}}\,|\boldsymbol{u}|^2\right)V_{\mathrm{n}}\ \mathrm{d}\varsigma(x)+2\int_{\partial\Omega}\boldsymbol{u}\cdot(-\nabla\boldsymbol{u}\boldsymbol{V})\ \mathrm{d}\varsigma(x)\right)\\
&\qquad+\beta\int_{\partial\Omega}\Big(\mathrm{H}\,A_{\mathrm{c}}e_\Gamma(\boldsymbol{u}):e_\Gamma(\boldsymbol{u})+2A_{\mathrm{c}}e_\Gamma(\partial_{\mathrm{n}}\boldsymbol{u}):e_\Gamma(\boldsymbol{u})\\
&\qquad\quad-A_{\mathrm{c}}e_\Gamma(\boldsymbol{u}):\Pi_d\left(\nabla\boldsymbol{u}\,\mathrm{D}^2b+\mathrm{D}^2b\,{}^t\nabla\boldsymbol{u}\right)\Pi_d\Big)V_{\mathrm{n}}\ \mathrm{d}\varsigma(x)\\
&\quad+2\beta\int_{\partial\Omega}A_{\mathrm{c}}e_\Gamma(\boldsymbol{u}):\Big(e_\Gamma(-\nabla\boldsymbol{u}\boldsymbol{V})+\Pi_d e(\boldsymbol{u})(\mathbf{n}\otimes\nabla_\Gamma V_{\mathrm{n}}+\nabla_\Gamma V_{\mathrm{n}}\otimes\mathbf{n})
\end{aligned}$$

$$+ (\mathbf{n} \otimes \nabla_\Gamma V_\mathrm{n} + \nabla_\Gamma V_\mathrm{n} \otimes \mathbf{n}) e(\boldsymbol{u}) \Pi_d \Big) \,\mathrm{d}\varsigma(x)$$
$$- \Lambda_\Omega(\Omega) \left(\int_{\partial\Omega} |\boldsymbol{u}|^2 \, V_\mathrm{n} \,\mathrm{d}\varsigma(x) + 2 \int_\Omega \boldsymbol{u} \cdot (-\nabla \boldsymbol{u} \boldsymbol{V}) \,\mathrm{d}x \right).$$

Using $-\nabla \boldsymbol{u}\boldsymbol{V}$ as a test function in (2.4), we observe that

$$\int_\Omega A e(\boldsymbol{u}) : e(-\nabla \boldsymbol{u}\boldsymbol{V}) + \alpha \int_{\partial\Omega} \boldsymbol{u} \cdot (-\nabla \boldsymbol{u}\boldsymbol{V}) + \beta \int_{\partial\Omega} A_\mathrm{c} e_\Gamma(\boldsymbol{u}) : e_\Gamma(-\nabla \boldsymbol{u}\boldsymbol{V})$$
$$= \Lambda_\Omega(\Omega) \int_\Omega \boldsymbol{u} \cdot (-\nabla \boldsymbol{u}\boldsymbol{V}).$$

Notice that the function $-\nabla \boldsymbol{u}\boldsymbol{V}$ belongs to $\mathbf{H}^1(\Omega)$. Indeed $\boldsymbol{V}$ is assumed to be smooth and the boundary $\partial\Omega$ has a $\mathcal{C}^{2,1}$ regularity and then $\boldsymbol{u} \in \mathbf{H}^2(\Omega)$ by usual a priori estimates (see [5, Theorem 1.1 and its proof]. Also, observe that for symmetric matrix B and any square matrix C, we have $B : C = B : {}^tC$ and choose $B = A_\mathrm{c} e_\Gamma(\boldsymbol{u})$ along with $C = (\mathbf{n} \otimes \nabla_\Gamma V_\mathrm{n} + \nabla_\Gamma V_\mathrm{n} \otimes \mathbf{n})\, e(\boldsymbol{u}) \Pi_d$. We use these to get that

$$\partial_t G(0, \boldsymbol{u}) = \int_{\partial\Omega} A e(\boldsymbol{u}) : e(\boldsymbol{u})\, V_\mathrm{n} \,\mathrm{d}\varsigma(x) + \alpha \int_{\partial\Omega} \left(\mathrm{H}\, |\boldsymbol{u}|^2 + 2\boldsymbol{u} \cdot \partial_\mathrm{n} \boldsymbol{u} \right) V_\mathrm{n} \,\mathrm{d}\varsigma(x) \tag{3.10}$$
$$+ \beta \int_{\partial\Omega} \Big(\mathrm{H}\, A_\mathrm{c} e_\Gamma(\boldsymbol{u}) : e_\Gamma(\boldsymbol{u}) + 2 A_\mathrm{c} e_\Gamma(\partial_\mathrm{n} \boldsymbol{u}) : e_\Gamma(\boldsymbol{u})$$
$$- A_\mathrm{c} e_\Gamma(\boldsymbol{u}) : \Pi_d \left(\nabla \boldsymbol{u}\, \mathrm{D}^2 b + \mathrm{D}^2 b\, {}^t\nabla \boldsymbol{u} \right) \Pi_d \Big) V_\mathrm{n} \,\mathrm{d}\varsigma(x)$$
$$+ 4\beta \int_{\partial\Omega} A_\mathrm{c} e_\Gamma(\boldsymbol{u}) : \Big((\mathbf{n} \otimes \nabla_\Gamma V_\mathrm{n} + \nabla_\Gamma V_\mathrm{n} \otimes \mathbf{n}) e(\boldsymbol{u}) \Pi_d \Big) \,\mathrm{d}\varsigma(x) - \Lambda_\Omega(\Omega) \int_{\partial\Omega} |\boldsymbol{u}|^2 \, V_\mathrm{n} \,\mathrm{d}\varsigma(x).$$

Remark 3.4. Notice that we have a factor 4 as compared to 2 in the corresponding term in the scalar case owing to the fact that the tangential strain is obtained by multiplying the strain by Π_d on either side and while deriving with respect to t, we obtain an additional term as observed in Remark A.5 (see also Remark 2.1).

We also observe that

$$A_\mathrm{c} e_\Gamma(\boldsymbol{u}) : (\mathbf{n} \otimes \nabla_\Gamma V_\mathrm{n})\; e(\boldsymbol{u})\, \Pi_d = 0. \tag{3.11}$$

Indeed, after setting, $\mathcal{A} := A_\mathrm{c} e_\Gamma(\boldsymbol{u})$, $\mathcal{B} := \mathbf{n} \otimes \nabla_\Gamma V_\mathrm{n}$, $\mathcal{C} := e(\boldsymbol{u})$ and $\mathcal{D} := \Pi_d$, and writing these in the local frame, we obtain

$$A_\mathrm{c} e_\Gamma(\boldsymbol{u}) : \mathbf{n} \otimes \nabla_\Gamma V_\mathrm{n} e(\boldsymbol{u}) \Pi_d = \mathcal{A} : \mathcal{B}\mathcal{C}\mathcal{D}$$
$$= \begin{pmatrix} \mathcal{A}_{\tau\tau} & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} 0 & 0 \\ \mathcal{B}_{\mathrm{n}\tau} & 0 \end{pmatrix} \begin{pmatrix} \mathcal{C}_{\tau\tau} & \mathcal{C}_{\tau\mathrm{n}} \\ \mathcal{C}_{\mathrm{n}\tau} & \mathcal{C}_{\mathrm{nn}} \end{pmatrix} \begin{pmatrix} \mathrm{I}_{d-1} & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

Next we remark that $C : ((\boldsymbol{v} \otimes \boldsymbol{w})\, B) = \boldsymbol{v} \cdot (C\, {}^tB\, \boldsymbol{w})$ (for any matrices C and B and any vectors $\boldsymbol{v}$ and $\boldsymbol{w}$) and apply this to $A_\mathrm{c} e_\Gamma(\boldsymbol{u}) : \nabla_\Gamma V_\mathrm{n} \otimes \mathbf{n}\, e(\boldsymbol{u})\, \Pi_d$. We also remark that $A_\mathrm{c} e_\Gamma(\boldsymbol{u})\, \Pi_d\, e(\boldsymbol{u})\mathbf{n}$ is a tangential vector since Π_d commutes with $A_\mathrm{c} e_\Gamma(\boldsymbol{u})$ and so we can apply the tangential Stokes formula without any curvature term (see, e.g., [14, Equation (5.27)]) and obtain that

$$\int_{\partial\Omega} A_\mathrm{c} e_\Gamma(\boldsymbol{u}) : (\nabla_\Gamma V_\mathrm{n} \otimes \mathbf{n})\; e(\boldsymbol{u})\, \Pi_d \,\mathrm{d}\varsigma(x) = \int_{\partial\Omega} \nabla_\Gamma V_\mathrm{n} \cdot (A_\mathrm{c} e_\Gamma(\boldsymbol{u})\, \Pi_d\, e(\boldsymbol{u})\mathbf{n}) \,\mathrm{d}\varsigma(x) \tag{3.12}$$

$$= -\int_{\partial\Omega} \operatorname{div}_\Gamma \left(A_c e_\Gamma(\boldsymbol{u})\, \Pi_d\, e(\boldsymbol{u})\mathbf{n}\right) V_n \, d\varsigma(x)$$

$$= -\int_{\partial\Omega} \left(\operatorname{div}_\Gamma \left(A_c e_\Gamma(\boldsymbol{u})\right) \cdot \Pi_d e(\boldsymbol{u})\mathbf{n} + A_c e_\Gamma(\boldsymbol{u}) : \nabla_\Gamma \left(\Pi_d e(\boldsymbol{u})\mathbf{n}\right)\right) V_n \, d\varsigma(x).$$

Therefore, inserting (3.11) and (3.12) in (3.10), we get

$$\begin{aligned}
\partial_t G(0,\boldsymbol{u}) = &\int_{\partial\Omega} Ae(\boldsymbol{u}) : e(\boldsymbol{u})\, V_n \, d\varsigma(x) + \alpha \int_{\partial\Omega} \left(\mathrm{H}\, |\boldsymbol{u}|^2 + 2\boldsymbol{u} \cdot \partial_n \boldsymbol{u}\right) V_n \, d\varsigma(x) \\
&+ \beta \int_{\partial\Omega} \Big(\mathrm{H}\, A_c e_\Gamma(\boldsymbol{u}) : e_\Gamma(\boldsymbol{u}) + 2A_c e_\Gamma(\partial_n \boldsymbol{u}) : e_\Gamma(\boldsymbol{u}) \\
&- A_c e_\Gamma(\boldsymbol{u}) : \Pi_d \left(\nabla \boldsymbol{u}\, \mathrm{D}^2 b + \mathrm{D}^2 b\, {}^t\nabla \boldsymbol{u}\right) \Pi_d \Big) V_n \, d\varsigma(x) \\
&- 4\beta \int_{\partial\Omega} \left(\operatorname{div}_\Gamma \left(A_c e_\Gamma(\boldsymbol{u})\right) \cdot \Pi_d e(\boldsymbol{u})\mathbf{n} + A_c e_\Gamma(\boldsymbol{u}) : \nabla_\Gamma \left(\Pi_d e(\boldsymbol{u})\mathbf{n}\right)\right) V_n \, d\varsigma(x) \\
&- \Lambda_\Omega(\Omega) \int_{\partial\Omega} |\boldsymbol{u}|^2 \, V_n \, d\varsigma(x).
\end{aligned}$$

Then using the boundary condition in (2.4), to replace the term $\operatorname{div}_\Gamma \left(A_c e_\Gamma(\boldsymbol{u})\right)$, we obtain

$$\begin{aligned}
\partial_t G(0,\boldsymbol{u}) = &\int_{\partial\Omega} Ae(\boldsymbol{u}) : e(\boldsymbol{u})\, V_n \, d\varsigma(x) + \alpha \int_{\partial\Omega} \left(\mathrm{H}\, |\boldsymbol{u}|^2 + 2\boldsymbol{u} \cdot \partial_n \boldsymbol{u}\right) V_n \, d\varsigma(x) \\
&+ \beta \int_{\partial\Omega} \left(\mathrm{H}\, A_c e_\Gamma(\boldsymbol{u}) : e_\Gamma(\boldsymbol{u}) - A_c e_\Gamma(\boldsymbol{u}) : \Pi_d \left(\nabla \boldsymbol{u}\, \mathrm{D}^2 b + \mathrm{D}^2 b\, {}^t\nabla \boldsymbol{u}\right) \Pi_d\right) V_n \, d\varsigma(x) \\
&+ 2\beta \int_{\partial\Omega} \left(A_c e_\Gamma(\partial_n \boldsymbol{u}) : e_\Gamma(\boldsymbol{u}) - 2A_c e_\Gamma(\boldsymbol{u}) : \nabla_\Gamma \left(\Pi_d e(\boldsymbol{u})\mathbf{n}\right)\right) V_n \, d\varsigma(x) \\
&+ 4 \int_{\partial\Omega} \left(-\alpha \boldsymbol{u} - Ae(\boldsymbol{u})\mathbf{n}\right) \cdot \Pi_d e(\boldsymbol{u})\mathbf{n}\, V_n \, d\varsigma(x) - \Lambda_\Omega(\Omega) \int_{\partial\Omega} |\boldsymbol{u}|^2 \, V_n \, d\varsigma(x).
\end{aligned}$$

This may be further rearranged to obtain the expression announced in Theorem 2.2.

3.3.2. Proof of Theorem 2.3. The eigenvalue functional over the perturbed domain reads

$$\Lambda_{\partial\Omega}(\Omega_t) = \inf_{\boldsymbol{v} \in \boldsymbol{\mathcal{H}}(\Omega_t)} \left\{ \frac{1}{\displaystyle\int_{\partial\Omega_t} |\boldsymbol{v}|^2 \, dx} \left(\int_{\Omega_t} Ae(\boldsymbol{v}) : e(\boldsymbol{v}) \, dx + \alpha \int_{\partial\Omega_t} |\boldsymbol{v}|^2 \, d\varsigma(x) + \beta \int_{\partial\Omega_t} A_c e_\Gamma(\boldsymbol{v}) : e_\Gamma(\boldsymbol{v}) \, d\varsigma(x) \right) \right\}, \tag{3.13}$$

and this may be reformulated over a function space $\boldsymbol{\mathcal{H}}(\Omega)$ which is independent of t using the isomorphism $\boldsymbol{v} \mapsto \boldsymbol{v} \circ \Psi_t$ and setting $\boldsymbol{u} = \boldsymbol{v} \circ \Psi_t$ as

$$\Lambda_{\partial\Omega}(\Omega_t) = \inf_{\boldsymbol{u} \in \boldsymbol{\mathcal{H}}(\Omega)} G_{\partial\Omega}(t, \boldsymbol{u})$$

where

$$G_{\partial\Omega}(t,\boldsymbol{u}) := \frac{1}{\displaystyle\int_{\partial\Omega_t} \left|\boldsymbol{u}\circ\Psi_t^{-1}\right|^2 \mathrm{d}x}\left(\int_{\Omega_t} Ae(\boldsymbol{u}\circ\Psi_t^{-1}) : e(\boldsymbol{u}\circ\Psi_t^{-1})\ \mathrm{d}x\right.$$
$$\left. + \alpha\int_{\partial\Omega_t}\left|\boldsymbol{u}\circ\Psi_t^{-1}\right|^2 \mathrm{d}\varsigma(x) + \beta\int_{\partial\Omega_t} A_{\mathrm{c}}e_\Gamma(\boldsymbol{u}\circ\Psi_t^{-1}) : e_\Gamma(\boldsymbol{u}\circ\Psi_t^{-1})\ \mathrm{d}\varsigma(x)\right). \tag{3.14}$$

Existence of directional shape derivatives. We go over the main arguments needed for applying Theorem 1.1 to problem (2.7) to obtain the existence of the directional shape derivative.

Let us start by Assumption $(H1)$. The arguments are exactly the same as in the previous case concerning G_Ω, except that which is needed for the continuity of the denominator. In this case, it is enough to use the compact inclusion of $\mathbf{H}^1(\Omega_t)$ into $\mathbf{L}^2(\partial\Omega_t)$ (for which we refer, e.g., to [3]). Then, as in the previous case, the set $X(t)$ of minimizers for $G_{\partial\Omega}(t,\cdot)$ is obtained by transporting the minimizers in (3.13) to the domain Ω by composition with Ψ_t. Therefore Assumption $(H1)$ is satisfied.

Concerning Assumption $(H2)$, we first get the following expression for $G_{\partial\Omega}$

(3.15)

$$G_{\partial\Omega}(t,\boldsymbol{u}) = \frac{1}{\displaystyle\int_{\partial\Omega}|\boldsymbol{u}|^2\omega(t)\mathrm{d}\varsigma(x)}\left(\int_\Omega A\left(\nabla\boldsymbol{u}\ \mathrm{D}\Psi_t^{-1}\right)^{\mathrm{sym}} : \left(\nabla\boldsymbol{u}\ \mathrm{D}\Psi_t^{-1}\right)^{\mathrm{sym}} j(t)\mathrm{d}x\right.$$
$$+ \alpha\int_{\partial\Omega}|\boldsymbol{u}|^2\omega(t)\mathrm{d}\varsigma(x) + \beta\int_{\partial\Omega} A_{\mathrm{c}}(\mathrm{I}_d-\boldsymbol{\nu}_t\otimes\boldsymbol{\nu}_t)\left(\nabla\boldsymbol{u}\ \mathrm{D}\Psi_t^{-1}\right)^{\mathrm{sym}}(\mathrm{I}_d-\boldsymbol{\nu}_t\otimes\boldsymbol{\nu}_t)$$
$$\left. : (\mathrm{I}_d-\boldsymbol{\nu}_t\otimes\boldsymbol{\nu}_t)\left(\nabla\boldsymbol{u}\ \mathrm{D}\Psi_t^{-1}\right)^{\mathrm{sym}}(\mathrm{I}_d-\boldsymbol{\nu}_t\otimes\boldsymbol{\nu}_t)\omega(t)\mathrm{d}\varsigma(x)\right).$$

Due to the smooth dependence in t of the coefficients appearing in (3.15), we conclude that $G_{\partial\Omega}(\cdot,\boldsymbol{u})$ is derivable with respect to t for all $\boldsymbol{u}\in\boldsymbol{\mathcal{H}}(\Omega)$, which gives Assumption $(H2)$.

As in the previous case of G_Ω, the derivative of the individual terms may be obtained by deriving under the integrals which lead to the fact that $\partial_t G_{\partial\Omega}(\cdot,\boldsymbol{u})$ is also continuous with respect to t, for all $\boldsymbol{u}\in\boldsymbol{\mathcal{H}}(\Omega)$, due to the $\mathcal{C}^1$ nature of the coefficients. This gives Assumption $(H4)$.

Finally we prove that assumption $(H3)$ is also satisfied by showing, as in the case of G_Ω, that $G_{\partial\Omega}(t,\cdot)$ converges to $G_{\partial\Omega}(0,\cdot)$ as $t\to 0$ in the sense of Γ-limit in the weak topology on $\boldsymbol{\mathcal{H}}(\Omega)$ and that the minimizers converge in the strong topology.

Thus the existence of the semi-derivative $\Lambda'_{\partial\Omega}(\Omega;\boldsymbol{V})$ follows from Theorem 1.1.

Computation of directional shape derivatives. We only need to get a suitable expression for $\partial_t G_{\partial\Omega}(0,\boldsymbol{u})$ whenever $\boldsymbol{u}$ is a normalized eigenfunction for $\Lambda_{\partial\Omega}(\Omega)$ since, by the theorem,

$$\Lambda'_{\partial\Omega}(\Omega;\boldsymbol{V}) = \inf\{\partial_t G_{\partial\Omega}(0,\boldsymbol{u});\ \Lambda_{\partial\Omega}(\Omega) \text{ is attained at } \boldsymbol{u}\}.$$

Using the expressions (3.1)-(3.4) evaluated at $t=0$, we get

$$\partial_t G_{\partial\Omega}(0,\boldsymbol{u}) = \int_{\partial\Omega} Ae(\boldsymbol{u}):e(\boldsymbol{u})V_{\rm n}\ {\rm d}\varsigma(x) + 2\int_\Omega Ae(\boldsymbol{u}):e(-\nabla\boldsymbol{u}\boldsymbol{V})\ {\rm d}x$$
$$+ \alpha\left(\int_{\partial\Omega}\left({\rm H}\,|\boldsymbol{u}|^2 + \partial_{\rm n}\,|\boldsymbol{u}|^2\right)V_{\rm n}\ {\rm d}\varsigma(x) + 2\int_{\partial\Omega}\boldsymbol{u}\cdot(-\nabla\boldsymbol{u}\boldsymbol{V})\ {\rm d}\varsigma(x)\right)$$
$$+ \beta\int_{\partial\Omega}\Big({\rm H}\,A_{\rm c}e_\Gamma(\boldsymbol{u}):e_\Gamma(\boldsymbol{u}) + 2A_{\rm c}e_\Gamma(\partial_{\rm n}\boldsymbol{u}):e_\Gamma(\boldsymbol{u})$$
$$- A_{\rm c}e_\Gamma(\boldsymbol{u}):\Pi_d\left(\nabla\boldsymbol{u}\,{\rm D}^2b + {\rm D}^2b\,{}^t\nabla\boldsymbol{u}\right)\Pi_d\Big)V_{\rm n}\ {\rm d}\varsigma(x)$$
$$+ 2\beta\int_{\partial\Omega}A_{\rm c}e_\Gamma(\boldsymbol{u}):\Big(e_\Gamma(-\nabla\boldsymbol{u}\boldsymbol{V}) + \Pi_d e(\boldsymbol{u})(\mathbf{n}\otimes\nabla_\Gamma V_{\rm n} + \nabla_\Gamma V_{\rm n}\otimes\mathbf{n})$$
$$+ (\mathbf{n}\otimes\nabla_\Gamma V_{\rm n} + \nabla_\Gamma V_{\rm n}\otimes\mathbf{n})e(\boldsymbol{u})\Pi_d\Big)\ {\rm d}\varsigma(x)$$
$$- \Lambda_{\partial\Omega}(\Omega)\left(\int_{\partial\Omega}\left({\rm H}\,|\boldsymbol{u}|^2 + \partial_{\rm n}\,|\boldsymbol{u}|^2\right)V_{\rm n}\ {\rm d}\varsigma(x) + 2\int_{\partial\Omega}\boldsymbol{u}\cdot(-\nabla\boldsymbol{u}\boldsymbol{V})\ {\rm d}\varsigma(x)\right).$$

Now, given u an eigenfunction in (2.5) whose L^2 norm is 1, we use $-\nabla\boldsymbol{u}\boldsymbol{V}$ as a test function in (2.5) since $\boldsymbol{u}\in\mathbf{H}^2(\Omega)$ by usual a priori estimates (see [5, Theorem 1.1 and its proof] and we observe that

$$\int_\Omega Ae(\boldsymbol{u}):e(-\nabla\boldsymbol{u}\boldsymbol{V})\ {\rm d}x + \alpha\int_{\partial\Omega}\boldsymbol{u}\cdot(-\nabla\boldsymbol{u}\boldsymbol{V})\ {\rm d}\varsigma(x)$$
$$+ \beta\int_{\partial\Omega}A_{\rm c}e_\Gamma(\boldsymbol{u}):e_\Gamma(-\nabla\boldsymbol{u}\boldsymbol{V})\ {\rm d}\varsigma(x) = \Lambda_{\partial\Omega}(\Omega)\int_{\partial\Omega}\boldsymbol{u}\cdot(-\nabla\boldsymbol{u}\boldsymbol{V})\ {\rm d}x,$$

and then arguing as in the previous subsection while using the boundary condition in (2.5), we get

$$\partial_t G(0,\boldsymbol{u}) = \int_{\partial\Omega}Ae(\boldsymbol{u}):e(\boldsymbol{u})\,V_{\rm n}\ {\rm d}\varsigma(x) + \alpha\int_{\partial\Omega}\left({\rm H}\,|\boldsymbol{u}|^2 + 2\boldsymbol{u}\cdot\partial_{\rm n}\boldsymbol{u}\right)V_{\rm n}\ {\rm d}\varsigma(x)$$
$$+ \beta\int_{\partial\Omega}\left({\rm H}\,A_{\rm c}e_\Gamma(\boldsymbol{u}):e_\Gamma(\boldsymbol{u}) - A_{\rm c}e_\Gamma(\boldsymbol{u}):\Pi_d\left(\nabla\boldsymbol{u}\,{\rm D}^2b + {\rm D}^2b\,{}^t\nabla\boldsymbol{u}\right)\Pi_d\right)V_{\rm n}\ {\rm d}\varsigma(x)$$
$$+ 2\beta\int_{\partial\Omega}\left(A_{\rm c}e_\Gamma(\partial_{\rm n}\boldsymbol{u}):e_\Gamma(\boldsymbol{u}) - 2A_{\rm c}e_\Gamma(\boldsymbol{u}):\nabla_\Gamma\left(\Pi_d e(\boldsymbol{u})\mathbf{n}\right)\right)V_{\rm n}\ {\rm d}\varsigma(x)$$
$$+ 4\int_{\partial\Omega}(-\alpha\boldsymbol{u} - Ae(\boldsymbol{u})\mathbf{n})\cdot\Pi_d e(\boldsymbol{u})\mathbf{n}\,V_{\rm n}\ {\rm d}\varsigma(x)$$
$$- \Lambda_{\partial\Omega}(\Omega)\int_{\partial\Omega}\left({\rm H}|\boldsymbol{u}|^2 + 2\boldsymbol{u}\cdot\partial_{\rm n}\boldsymbol{u} - 4\boldsymbol{u}\cdot\Pi_d e(\boldsymbol{u})\mathbf{n}\right)V_{\rm n}\ {\rm d}\varsigma(x),$$

and by rearranging the terms we get the desired expression.

3.4. Shape derivative for eigenvalue problems for composite materials. We recall that we use the following notation: $A_t = A_1\chi_{\Omega_{1,t}} + A_2\chi_{\Omega_{2,t}}$ and $\rho_t = \rho_1\chi_{\Omega_{1,t}} + \rho_2\chi_{\Omega_{2,t}}$, with $\Omega_t = \Psi_t(\Omega)$, $\Omega_{1,t} = \Psi_t(\Omega_1)$ and $\Omega_{2,t} = \Psi_t(\Omega_2)$.

3.4.1. Proof of Theorem 2.4. The considered perturbed problem on Ω_t reads

$$\mathfrak{M}_\Omega(\Omega_t) = \inf_{\boldsymbol{v}\in\boldsymbol{\mathcal{H}}(\Omega_t)} \left\{ \frac{1}{\displaystyle\int_{\Omega_t} \rho_t \left|\boldsymbol{v}\right|^2 \mathrm{d}x} \left(\int_{\Omega_t} A_t(x) e(\boldsymbol{v}) : e(\boldsymbol{v}) \mathrm{d}x + \alpha \int_{\partial\Omega_t} |\boldsymbol{v}|^2 \,\mathrm{d}\varsigma(x) \right.\right.$$
$$\left.\left. + \beta \int_{\partial\Omega_t} A_{\mathrm{c}} e_\Gamma(\boldsymbol{v}) : e_\Gamma(\boldsymbol{v}) \,\mathrm{d}\varsigma(x) \right) \right\}.$$

The above can be formulated as

$$\mathfrak{M}_\Omega(\Omega_t) = \inf_{\boldsymbol{u}\in\boldsymbol{\mathcal{H}}(\Omega)} G_\Omega(t,\boldsymbol{u}), \tag{3.16}$$

with

$$G_\Omega(t,\boldsymbol{u}) := \frac{1}{\displaystyle\int_{\Omega_t} \rho_t \left|(\boldsymbol{u}\circ\Psi_t^{-1})\right|^2 \mathrm{d}x} \left(\int_{\Omega_t} A_t(x) e(\boldsymbol{u}\circ\Psi_t^{-1}) : e(\boldsymbol{u}\circ\Psi_t^{-1}) \mathrm{d}x \right.$$
$$\left. + \alpha \int_{\partial\Omega_t} \left|(\boldsymbol{u}\circ\Psi_t^{-1})\right|^2 \mathrm{d}\varsigma(x) + \beta \int_{\partial\Omega_t} A_{\mathrm{c}} e_\Gamma(\boldsymbol{u}\circ\Psi_t^{-1}) : e_\Gamma(\boldsymbol{u}\circ\Psi_t^{-1}) \,\mathrm{d}\varsigma(x) \right).$$

We proceed as in the proofs of the previous theorems.

Existence of directional shape derivatives. The existence of the semi-derivative $\mathfrak{M}'_\Omega(\Omega;\boldsymbol{V})$ follows from Theorem 1.1 once the hypothesis of the theorem are verified.

The verification of the hypothesis $(H1)$ is like in the previous subsections due to the coercivity of the tensor A_t.

The differentiability of $G_\Omega(\cdot,\boldsymbol{u})$ with respect to t for any $\boldsymbol{u}\in\boldsymbol{\mathcal{H}}(\Omega)$ is seen once we use a change of variables to rewrite $G_\Omega(t,\boldsymbol{u})$ as

$$G_\Omega(t,\boldsymbol{u}) = \frac{1}{\displaystyle\int_\Omega |\boldsymbol{u}|^2 j(t)\mathrm{d}y} \left(\int_\Omega C_t(y) \left(\nabla\boldsymbol{u}\ \mathrm{D}\Psi_t^{-1}\right)^{\mathrm{sym}} : \left(\nabla\boldsymbol{u}\ \mathrm{D}\Psi_t^{-1}\right)^{\mathrm{sym}} j(t)\mathrm{d}y \right. \tag{3.17}$$
$$+ \alpha \int_{\partial\Omega} |\boldsymbol{u}|^2 \omega(t)\mathrm{d}\varsigma(y) + \beta \int_{\partial\Omega} A_{\mathrm{c}} (\mathrm{I}_d - \boldsymbol{\nu}_t\otimes\boldsymbol{\nu}_t) \left(\nabla\boldsymbol{u}\ \mathrm{D}\Psi_t^{-1}\right)^{\mathrm{sym}} (\mathrm{I}_d - \boldsymbol{\nu}_t\otimes\boldsymbol{\nu}_t)$$
$$\left. : (\mathrm{I}_d - \boldsymbol{\nu}_t\otimes\boldsymbol{\nu}_t) \left(\nabla\boldsymbol{u}\ \mathrm{D}\Psi_t^{-1}\right)^{\mathrm{sym}} (\mathrm{I}_d - \boldsymbol{\nu}_t\otimes\boldsymbol{\nu}_t)\omega(t)\mathrm{d}\varsigma(y) \right),$$

while observing that

$$C_t(y) := A_t(\Psi_t(y)) = A_1\chi_{\Omega_{1,t}}(\Psi_t(y)) + A_2\chi_{\Omega_{2,t}}(\Psi_t(y)) = A_1\chi_{\Omega_1}(y) + A_2\chi_{\Omega_2}(y)$$

is independent of t. The differentiability with respect to t, that is hypothesis (H2), then follows due to the smooth dependence of the coefficients with respect to t, and also the hypothesis $(H4)$ follows.

The hypothesis $(H3)$ is proved by showing, similarly as in the subsection 2.2, that $G_\Omega(t,\cdot)$ converges to $G_\Omega(0,\cdot)$ as $t\to 0$ in the sense of Γ-limit in the weak topology on $\boldsymbol{\mathcal{H}}(\Omega)$ and that the minimizers converge in the strong topology.

Computation of directional shape derivatives. Thus, we now only need to get a suitable expression for $\partial_t G_\Omega(0,\boldsymbol{u})$ given any normalized eigenfunction $\boldsymbol{u}$ for $\mathfrak{M}_\Omega(\Omega)$ since, by the theorem,

$$\mathfrak{M}'_\Omega(\Omega;\boldsymbol{V}) = \inf\{\partial_t G_\Omega(0,\boldsymbol{u});\ \mathfrak{M}_\Omega(\Omega) \text{ is attained at } \boldsymbol{u}\}.$$

Using the calculated expressions in (3.5), (3.6), (3.3) and (3.4), we get

$$\begin{aligned}
\partial_t G_\Omega(0,\boldsymbol{u}) = &\int_\Gamma [Ae(\boldsymbol{u}):e(\boldsymbol{u})]V_{\mathrm{n}}\ \mathrm{d}\varsigma(x) + 2\int_{\Omega_1} A_1 e(\boldsymbol{u}):e(-\nabla\boldsymbol{u}\boldsymbol{V})\ \mathrm{d}x\\
&+2\int_{\Omega_2} A_2 e(\boldsymbol{u}):e(-\nabla\boldsymbol{u}\boldsymbol{V})\ \mathrm{d}x + \int_{\partial\Omega} A_2 e(\boldsymbol{u}):e(\boldsymbol{u})V_{\mathrm{n}}\ \mathrm{d}\varsigma(x)\\
&+\alpha\left(\int_{\partial\Omega}\left(\mathrm{H}\,|\boldsymbol{u}|^2 + \partial_{\mathrm{n}}\,|\boldsymbol{u}|^2\right)V_{\mathrm{n}}\ \mathrm{d}\varsigma(x) + 2\int_{\partial\Omega}\boldsymbol{u}\cdot(-\nabla\boldsymbol{u}\boldsymbol{V})\ \mathrm{d}\varsigma(x)\right)\\
&+\beta\int_{\partial\Omega}\Big(\mathrm{H}\,A_{\mathrm{c}}e_\Gamma(\boldsymbol{u}):e_\Gamma(\boldsymbol{u}) + 2A_{\mathrm{c}}e_\Gamma(\partial_{\mathrm{n}}\boldsymbol{u}):e_\Gamma(\boldsymbol{u})\\
&\qquad - A_{\mathrm{c}}e_\Gamma(\boldsymbol{u}):\Pi_d\left(\nabla\boldsymbol{u}\,\mathrm{D}^2 b + \mathrm{D}^2 b\,{}^t\nabla\boldsymbol{u}\right)\Pi_d\Big)\,V_{\mathrm{n}}\ \mathrm{d}\varsigma(x)\\
&+2\beta\int_{\partial\Omega} A_{\mathrm{c}}e_\Gamma(\boldsymbol{u}):\Big(e_\Gamma(-\nabla\boldsymbol{u}\boldsymbol{V}) + \Pi_d e(\boldsymbol{u})(\mathbf{n}\otimes\nabla_\Gamma(V_{\mathrm{n}}) + \nabla_\Gamma(V_{\mathrm{n}})\otimes\mathbf{n})\\
&\qquad + (\mathbf{n}\otimes\nabla_\Gamma(V_{\mathrm{n}}) + \nabla_\Gamma(V_{\mathrm{n}})\otimes\mathbf{n})e(\boldsymbol{u})\Pi_d\Big)\ \mathrm{d}\varsigma(x)\\
&-\mathfrak{M}_\Omega(\Omega)\Bigg(\int_\Gamma[\rho|\boldsymbol{u}|^2]V_{\mathrm{n}}\ \mathrm{d}\varsigma(x) + 2\int_{\Omega_1}\rho_1\boldsymbol{u}\cdot(-\nabla\boldsymbol{u}\boldsymbol{V})\ \mathrm{d}x\\
&\qquad +2\int_{\Omega_2}\rho_2\boldsymbol{u}\cdot(-\nabla\boldsymbol{u}\boldsymbol{V})\ \mathrm{d}x + \int_{\partial\Omega}\rho_2|\boldsymbol{u}|^2 V_{\mathrm{n}}\ \mathrm{d}\varsigma(x)\Bigg).
\end{aligned}$$

Notice that the eigenmode $\boldsymbol{u}$ does not belong to $\mathbf{H}^2(\Omega)$ due to the jumps on the interface. Therefore the function $-\nabla\boldsymbol{u}\boldsymbol{V}$ does not belong anymore to $\mathbf{H}^1(\Omega)$ and hence cannot be used as test function directly. However the restriction of $\boldsymbol{u}$ to each Ω_i, for $i=1,2$, belongs to $\mathbf{H}^2(\Omega_i)$ thanks to regularity assumptions on both the outer boundary and the interface. Then, multiplying (2.8) by $-\nabla\boldsymbol{u}\boldsymbol{V}\in\mathbf{H}^1(\Omega_i)$ in each Ω_i and integrating by part on Ω_i, for $i=1,2$, we obtain that

$$\begin{aligned}
&\int_{\Omega_1} A_1 e(\boldsymbol{u}):e(-\nabla\boldsymbol{u}\boldsymbol{V})\ \mathrm{d}x + \int_{\Omega_2} A_2 e(\boldsymbol{u}):e(-\nabla\boldsymbol{u}\boldsymbol{V})\ \mathrm{d}x\\
&+\int_{\partial\Omega_1}[Ae(\boldsymbol{u})\mathbf{n}\cdot(-\nabla\boldsymbol{u}\boldsymbol{V})]\ \mathrm{d}\varsigma(x) + \alpha\int_{\partial\Omega}\boldsymbol{u}\cdot(-\nabla\boldsymbol{u}\boldsymbol{V}) + \beta\int_{\partial\Omega}A_{\mathrm{c}}e_\Gamma(\boldsymbol{u}):e_\Gamma(-\nabla\boldsymbol{u}\boldsymbol{V})\\
&\qquad = \mathfrak{M}_\Omega(\Omega)\left(\int_{\Omega_1}\rho_1\boldsymbol{u}\cdot(-\nabla\boldsymbol{u}\boldsymbol{V})\ \mathrm{d}x + \int_{\Omega_2}\rho_2\boldsymbol{u}\cdot(-\nabla\boldsymbol{u}\boldsymbol{V})\ \mathrm{d}x\right).
\end{aligned}$$

Then, noticing that $\nabla_\Gamma u$ has a continuous trace on $\partial\Omega_1$ as aslo $Ae(\boldsymbol{u})\mathbf{n}$ we obtain

$$\begin{aligned}
-\int_{\partial\Omega_1}[Ae(\boldsymbol{u})\mathbf{n}\cdot(-\nabla\boldsymbol{u}\boldsymbol{V})]\ \mathrm{d}\varsigma(x) &= \int_{\partial\Omega_1}Ae(\boldsymbol{u})\mathbf{n}\cdot[(\nabla\boldsymbol{u}\boldsymbol{V})]\ \mathrm{d}\varsigma(x)\\
&= \int_{\partial\Omega_1}Ae(\boldsymbol{u})\mathbf{n}\cdot[\partial_{\mathrm{n}}\boldsymbol{u}]\,V_{\mathrm{n}}\ \mathrm{d}\varsigma(x),
\end{aligned}$$

Using the above, we get

$$\begin{aligned}
\partial_t G_\Omega(0,\boldsymbol{u}) = &\int_\Gamma [Ae(\boldsymbol{u}):e(\boldsymbol{u})]V_\mathrm{n}\ \mathrm{d}\varsigma(x) - 2\int_{\partial\Omega_1} Ae(\boldsymbol{u})\mathbf{n}\cdot[\partial_\mathrm{n}\boldsymbol{u}]\ \mathrm{d}\varsigma(x)\\
&+\int_{\partial\Omega} A_2 e(\boldsymbol{u}):e(\boldsymbol{u})V_\mathrm{n}\ \mathrm{d}\varsigma(x) + \alpha\int_{\partial\Omega}\left(\mathrm{H}\,|\boldsymbol{u}|^2 + 2\boldsymbol{u}\cdot\partial_\mathrm{n}\boldsymbol{u}\right)V_\mathrm{n}\ \mathrm{d}\varsigma(x)\\
&+\beta\int_{\partial\Omega}\Big(\mathrm{H}\,A_\mathrm{c}e_\Gamma(\boldsymbol{u}):e_\Gamma(\boldsymbol{u}) + 2A_\mathrm{c}e_\Gamma(\partial_\mathrm{n}\boldsymbol{u}):e_\Gamma(\boldsymbol{u})\\
&\quad - A_\mathrm{c}e_\Gamma(\boldsymbol{u}):\Pi_d\left(\nabla\boldsymbol{u}\,\mathrm{D}^2 b + \mathrm{D}^2 b\,{}^t\nabla\boldsymbol{u}\right)\Pi_d\Big)V_\mathrm{n}\ \mathrm{d}\varsigma(x)\\
&+4\beta\int_{\partial\Omega} A_\mathrm{c}e_\Gamma(\boldsymbol{u}):\Big((\mathbf{n}\otimes\nabla_\Gamma V_\mathrm{n} + \nabla_\Gamma V_\mathrm{n}\otimes\mathbf{n})e(\boldsymbol{u})\Pi_d\Big)\ \mathrm{d}\varsigma(x)\\
&-\mathfrak{M}_\Omega(\Omega)\left(\int_\Gamma[\rho|\boldsymbol{u}|^2]V_\mathrm{n}\ \mathrm{d}\varsigma(x) + \int_{\partial\Omega}\rho_2|\boldsymbol{u}|^2 V_\mathrm{n}\ \mathrm{d}\varsigma(x)\right).
\end{aligned}$$

An argument which shows that $A_\mathrm{c}e_\Gamma(\boldsymbol{u}):\mathbf{n}\otimes\nabla_\Gamma V_\mathrm{n} e(\boldsymbol{u})\Pi_d = 0$ (see (3.11)), then leads to

$$\begin{aligned}
\partial_t G_\Omega(0,\boldsymbol{u}) = &\int_{\partial\Omega_1} [Ae(\boldsymbol{u}):e(\boldsymbol{u})V_\mathrm{n}] - 2\int_{\partial\Omega_1} Ae(\boldsymbol{u})\mathbf{n}\cdot[\partial_\mathrm{n}\boldsymbol{u}]\ \mathrm{d}\varsigma(x)\\
&+\int_{\partial\Omega} A_2 e(\boldsymbol{u}):e(\boldsymbol{u})V_\mathrm{n}\ \mathrm{d}\varsigma(x) + \alpha\int_{\partial\Omega}\left(\mathrm{H}\,|\boldsymbol{u}|^2 + 2\boldsymbol{u}\cdot\partial_\mathrm{n}\boldsymbol{u}\right)V_\mathrm{n}\ \mathrm{d}\varsigma(x)\\
&+\beta\int_{\partial\Omega}\Big(\mathrm{H}\,A_\mathrm{c}e_\Gamma(\boldsymbol{u}):e_\Gamma(\boldsymbol{u}) + 2A_\mathrm{c}e_\Gamma(\partial_\mathrm{n}\boldsymbol{u}):e_\Gamma(\boldsymbol{u})\\
&\quad - A_\mathrm{c}e_\Gamma(\boldsymbol{u}):\Pi_d\left(\nabla\boldsymbol{u}\,\mathrm{D}^2 b + \mathrm{D}^2 b\,{}^t\nabla\boldsymbol{u}\right)\Pi_d\Big)V_\mathrm{n}\ \mathrm{d}\varsigma(x)\\
&+4\beta\int_{\partial\Omega} A_\mathrm{c}e_\Gamma(\boldsymbol{u}):\Big((\nabla_\Gamma V_\mathrm{n}\otimes\mathbf{n})e(\boldsymbol{u})\Pi_d\Big)\ \mathrm{d}\varsigma(x)\\
&-\mathfrak{M}_\Omega(\Omega)\left(\int_{\partial\Omega_1}[\rho]\,|\boldsymbol{u}|^2 V_\mathrm{n}\ \mathrm{d}\varsigma(x) + \int_{\partial\Omega}\rho_2|\boldsymbol{u}|^2 V_\mathrm{n}\ \mathrm{d}\varsigma(x)\right).
\end{aligned}$$

Then, applying the tangential Stokes formula, similarly as in (3.12), we get

$$\begin{aligned}
\partial_t G_\Omega(0,\boldsymbol{u}) = &\int_{\partial\Omega_1} [Ae(\boldsymbol{u}):e(\boldsymbol{u})V_\mathrm{n}] - 2\int_{\partial\Omega_1} Ae(\boldsymbol{u})\mathbf{n}\cdot[\partial_\mathrm{n}\boldsymbol{u}]\ \mathrm{d}\varsigma(x)\\
&+\int_{\partial\Omega} A_2 e(\boldsymbol{u}):e(\boldsymbol{u})V_\mathrm{n}\ \mathrm{d}\varsigma(x) + \alpha\int_{\partial\Omega}\left(\mathrm{H}\,|\boldsymbol{u}|^2 + 2\boldsymbol{u}\cdot\partial_\mathrm{n}\boldsymbol{u}\right)V_\mathrm{n}\ \mathrm{d}\varsigma(x)\\
&+\beta\int_{\partial\Omega}\left(\mathrm{H}\,A_\mathrm{c}e_\Gamma(\boldsymbol{u}):e_\Gamma(\boldsymbol{u}) - A_\mathrm{c}e_\Gamma(\boldsymbol{u}):\Pi_d\left(\nabla\boldsymbol{u}\,\mathrm{D}^2 b + \mathrm{D}^2 b\,{}^t\nabla\boldsymbol{u}\right)\Pi_d\right)V_\mathrm{n}\ \mathrm{d}\varsigma(x)\\
&+2\beta\int_{\partial\Omega}\Big(A_\mathrm{c}e_\Gamma(\partial_\mathrm{n}\boldsymbol{u}):e_\Gamma(\boldsymbol{u}) - 2A_\mathrm{c}e_\Gamma(\boldsymbol{u}):\nabla_\Gamma(\Pi_d e(\boldsymbol{u})\mathbf{n})\Big)V_\mathrm{n}\Big)\ \mathrm{d}\varsigma(x)\\
&+4\int_{\partial\Omega}(-\alpha\boldsymbol{u} - Ae(\boldsymbol{u})\mathbf{n})\cdot\Pi_d e(\boldsymbol{u})\mathbf{n}V_\mathrm{n}\mathrm{d}\varsigma(x)\\
&-\mathfrak{M}_\Omega(\Omega)\left(\int_{\partial\Omega_1}[\rho]\,|\boldsymbol{u}|^2 V_\mathrm{n}\ \mathrm{d}\varsigma(x) + \int_{\partial\Omega}\rho_2|\boldsymbol{u}|^2 V_\mathrm{n}\ \mathrm{d}\varsigma(x)\right)
\end{aligned}$$

and after rearranging the terms we get the announced expression.

3.4.2. Proof of Theorem 2.5. We will now calculate the sensitivity of $\mathfrak{M}_{\partial\Omega}(\Omega)$ with respect to variations of the domain Ω and of the interface Γ. The perturbed problem then reads

$$\mathfrak{M}_{\partial\Omega}(\Omega_t) = \inf_{\boldsymbol{v}\in\boldsymbol{\mathcal{H}}(\Omega_t)} \left\{ \frac{1}{\displaystyle\int_{\partial\Omega_t} |\boldsymbol{v}|^2 \,\mathrm{d}\varsigma(x)} \left(\int_{\Omega_t} A_t(x) e(\boldsymbol{v}) : e(\boldsymbol{v}) \mathrm{d}x + \alpha \int_{\partial\Omega_t} |\boldsymbol{v}|^2 \,\mathrm{d}\varsigma(x) \right.\right.$$
$$\left.\left. + \beta \int_{\partial\Omega_t} A_{\mathrm{c}} e_\Gamma(\boldsymbol{v}) : e_\Gamma(\boldsymbol{v}) \,\mathrm{d}\varsigma(x) \right) \right\}.$$

The above can be formulated as

$$\mathfrak{M}_{\partial\Omega}(\Omega_t) = \inf_{\boldsymbol{u}\in\boldsymbol{\mathcal{H}}(\Omega)} G_{\partial\Omega}(t, \boldsymbol{u}), \tag{3.18}$$

with

$$G_{\partial\Omega}(t,\boldsymbol{u}) := \frac{1}{\displaystyle\int_{\partial\Omega_t} \left|(\boldsymbol{u}\circ\Psi_t^{-1})\right|^2 \mathrm{d}\varsigma(x)} \left(\int_{\Omega_t} A_t(x) e(\boldsymbol{u}\circ\Psi_t^{-1}) : e(\boldsymbol{u}\circ\Psi_t^{-1}) \mathrm{d}x \right.$$
$$\left. + \alpha \int_{\partial\Omega_t} \left|(\boldsymbol{u}\circ\Psi_t^{-1})\right|^2 \mathrm{d}\varsigma(x) + \beta \int_{\partial\Omega_t} A_{\mathrm{c}} e_\Gamma(\boldsymbol{u}\circ\Psi_t^{-1}) : e_\Gamma(\boldsymbol{u}\circ\Psi_t^{-1}) \,\mathrm{d}\varsigma(x) \right).$$

The verification of the hypotheses of Theorem 1.1 which guarantee the existence of the semi-derivative $\mathfrak{M}'_{\partial\Omega}(\Omega;\boldsymbol{V})$ can be shown using arguments from the previous subsections. We now get a suitable expression for $\partial_t G_{\partial\Omega}(0,\boldsymbol{u})$ given a normalized eigenfunction $\boldsymbol{u}$ for $\mathfrak{M}_{\partial\Omega}(\Omega)$. To begin with we have

$$\begin{aligned}
\partial_t G_{\partial\Omega}(0,u) = {}& \int_\Gamma [Ae(\boldsymbol{u}) : e(\boldsymbol{u})] V_{\mathrm{n}} \,\mathrm{d}\varsigma(x) + 2\int_{\Omega_1} A_1 e(\boldsymbol{u}) : e(-\nabla\boldsymbol{u}\boldsymbol{V}) \,\mathrm{d}x \\
& + 2\int_{\Omega_2} A_2 e(\boldsymbol{u}) : e(-\nabla\boldsymbol{u}\boldsymbol{V}) \,\mathrm{d}x + \int_{\partial\Omega} A_2 e(\boldsymbol{u}) : e(\boldsymbol{u}) V_{\mathrm{n}} \,\mathrm{d}\varsigma(x) \\
& + \alpha \left(\int_{\partial\Omega} \left(\mathrm{H}\,|\boldsymbol{u}|^2 + \partial_{\mathrm{n}} |\boldsymbol{u}|^2 \right) V_{\mathrm{n}} \,\mathrm{d}\varsigma(x) + 2\int_{\partial\Omega} \boldsymbol{u}\cdot(-\nabla\boldsymbol{u}\boldsymbol{V}) \,\mathrm{d}\varsigma(x) \right) \\
& + \beta \int_{\partial\Omega} \Big(\mathrm{H}\, A_{\mathrm{c}} e_\Gamma(\boldsymbol{u}) : e_\Gamma(\boldsymbol{u}) + 2A_{\mathrm{c}} e_\Gamma(\partial_{\mathrm{n}}\boldsymbol{u}) : e_\Gamma(\boldsymbol{u}) \\
& - A_{\mathrm{c}} e_\Gamma(\boldsymbol{u}) : \Pi_d \left(\nabla\boldsymbol{u}\, \mathrm{D}^2 b + \mathrm{D}^2 b\, {}^t\nabla\boldsymbol{u} \right) \Pi_d \Big) V_{\mathrm{n}} \,\mathrm{d}\varsigma(x) \\
& + 2\beta \int_{\partial\Omega} A_{\mathrm{c}} e_\Gamma(\boldsymbol{u}) : \Big(e_\Gamma(-\nabla\boldsymbol{u}\boldsymbol{V}) + \Pi_d e(\boldsymbol{u}) (\mathbf{n}\otimes\nabla_\Gamma(V_{\mathrm{n}}) + \nabla_\Gamma(V_{\mathrm{n}})\otimes\mathbf{n}) \\
& + (\mathbf{n}\otimes\nabla_\Gamma(V_{\mathrm{n}}) + \nabla_\Gamma(V_{\mathrm{n}})\otimes\mathbf{n}) e(\boldsymbol{u}) \Pi_d \Big) \,\mathrm{d}\varsigma(x) \\
& - \mathfrak{M}_{\partial\Omega}(\Omega) \left(\int_{\partial\Omega} \left(\mathrm{H}\,|\boldsymbol{u}|^2 + \partial_{\mathrm{n}} |\boldsymbol{u}|^2 \right) V_{\mathrm{n}} \,\mathrm{d}\varsigma(x) + 2\int_{\partial\Omega} \boldsymbol{u}\cdot(-\nabla\boldsymbol{u}\boldsymbol{V}) \,\mathrm{d}\varsigma(x) \right).
\end{aligned}$$

Then, multiplying (2.9) by $-\nabla\boldsymbol{u}\boldsymbol{V}$ in each Ω_i, for $i = 1, 2$, we observe that

$$\int_{\Omega_1} A_1 e(\boldsymbol{u}) : e(-\nabla \boldsymbol{u} \boldsymbol{V}) \, \mathrm{d}x + \int_{\Omega_2} A_2 e(\boldsymbol{u}) : e(-\nabla \boldsymbol{u} \boldsymbol{V}) \, \mathrm{d}x - \int_{\partial\Omega_1} Ae(\boldsymbol{u})\mathbf{n} \cdot [\partial_{\mathrm{n}} \boldsymbol{u}] \, \mathrm{d}\varsigma(x)$$

$$+\alpha \int_{\partial\Omega} \boldsymbol{u} \cdot (-\nabla \boldsymbol{u} \boldsymbol{V}) + \beta \int_{\partial\Omega} A_{\mathrm{c}} e_\Gamma(\boldsymbol{u}) : e_\Gamma(-\nabla \boldsymbol{u} \boldsymbol{V}) = \mathfrak{M}_{\partial\Omega}(\Omega) \int_{\partial\Omega} \boldsymbol{u} \cdot (-\nabla \boldsymbol{u} \boldsymbol{V}) \, \mathrm{d}\varsigma(x).$$

Using the above we get

$$\begin{aligned}
\partial_t G_\Omega(0, \boldsymbol{u}) = &\int_\Gamma [Ae(\boldsymbol{u}) : e(\boldsymbol{u})] V_{\mathrm{n}} \, \mathrm{d}\varsigma(x) - 2 \int_{\partial\Omega_1} Ae(\boldsymbol{u})\mathbf{n} \cdot [\partial_{\mathrm{n}} \boldsymbol{u}] \, \mathrm{d}\varsigma(x) \\
&+ \int_{\partial\Omega} A_2 e(\boldsymbol{u}) : e(\boldsymbol{u}) V_{\mathrm{n}} \, \mathrm{d}\varsigma(x) + \alpha \int_{\partial\Omega} \left(\mathrm{H} \, |\boldsymbol{u}|^2 + 2\boldsymbol{u} \cdot \partial_{\mathrm{n}} \boldsymbol{u}\right) V_{\mathrm{n}} \, \mathrm{d}\varsigma(x) \\
&+ \beta \int_{\partial\Omega} \Big(\mathrm{H} \, A_{\mathrm{c}} e_\Gamma(\boldsymbol{u}) : e_\Gamma(\boldsymbol{u}) + 2A_{\mathrm{c}} e_\Gamma(\partial_{\mathrm{n}} \boldsymbol{u}) : e_\Gamma(\boldsymbol{u}) \\
&\qquad - A_{\mathrm{c}} e_\Gamma(\boldsymbol{u}) : \Pi_d \left(\nabla \boldsymbol{u} \, \mathrm{D}^2 b + \mathrm{D}^2 b \, {}^t \nabla \boldsymbol{u}\right) \Pi_d \Big) V_{\mathrm{n}} \, \mathrm{d}\varsigma(x) \\
&+ 4\beta \int_{\partial\Omega} A_{\mathrm{c}} e_\Gamma(\boldsymbol{u}) : \Big((\mathbf{n} \otimes \nabla_\Gamma(V_{\mathrm{n}}) + \nabla_\Gamma(V_{\mathrm{n}}) \otimes \mathbf{n}) e(\boldsymbol{u}) \Pi_d \Big) \, \mathrm{d}\varsigma(x) \\
&- \mathfrak{M}_\Omega(\Omega) \int_{\partial\Omega} \left(\mathrm{H} \, |\boldsymbol{u}|^2 + 2\boldsymbol{u} \cdot \partial_{\mathrm{n}} \boldsymbol{u}\right) V_{\mathrm{n}} \, \mathrm{d}\varsigma(x).
\end{aligned}$$

Then, continuing as in the proof of Theorem 2.4, we obtain

$$\begin{aligned}
\partial_t G_\Omega(0, \boldsymbol{u}) = &\int_{\partial\Omega_1} [Ae(\boldsymbol{u}) : e(\boldsymbol{u}) V_{\mathrm{n}}] - 2 \int_{\partial\Omega_1} Ae(\boldsymbol{u})\mathbf{n} \cdot [\partial_{\mathrm{n}} \boldsymbol{u}] \, \mathrm{d}\varsigma(x) \\
&+ \int_{\partial\Omega} A_2 e(\boldsymbol{u}) : e(\boldsymbol{u}) V_{\mathrm{n}} \, \mathrm{d}\varsigma(x) + \alpha \int_{\partial\Omega} \left(\mathrm{H} \, |\boldsymbol{u}|^2 + 2\boldsymbol{u} \cdot \partial_{\mathrm{n}} \boldsymbol{u}\right) V_{\mathrm{n}} \, \mathrm{d}\varsigma(x) \\
&+ \beta \int_{\partial\Omega} \left(\mathrm{H} \, A_{\mathrm{c}} e_\Gamma(\boldsymbol{u}) : e_\Gamma(\boldsymbol{u}) - A_{\mathrm{c}} e_\Gamma(\boldsymbol{u}) : \Pi_d \left(\nabla \boldsymbol{u} \, \mathrm{D}^2 b + \mathrm{D}^2 b \, {}^t \nabla \boldsymbol{u}\right) \Pi_d \right) V_{\mathrm{n}} \, \mathrm{d}\varsigma(x) \\
&+ 2\beta \int_{\partial\Omega} \Big(A_{\mathrm{c}} e_\Gamma(\partial_{\mathrm{n}} \boldsymbol{u}) : e_\Gamma(\boldsymbol{u}) - 2A_{\mathrm{c}} e_\Gamma(\boldsymbol{u}) : \nabla_\Gamma(\Pi_d e(\boldsymbol{u})\mathbf{n}) \Big) V_{\mathrm{n}} \Big) \, \mathrm{d}\varsigma(x) \\
&+ 4 \int_{\partial\Omega} (-\alpha \boldsymbol{u} - Ae(\boldsymbol{u})\mathbf{n}) \cdot \Pi_d e(\boldsymbol{u}) \mathbf{n} V_{\mathrm{n}} \mathrm{d}\varsigma(x) \\
&- \mathfrak{M}_\Omega(\Omega) \int_{\partial\Omega} \left(\mathrm{H} \, |u|^2 + 2\boldsymbol{u} \cdot (\partial_{\mathrm{n}} \boldsymbol{u} - 2\Pi_d e(\boldsymbol{u})\mathbf{n})\right) V_{\mathrm{n}} \, \mathrm{d}\varsigma(x).
\end{aligned}$$

Appendix A. Auxiliary results on shape derivatives.

The purpose of this subsection is to recall some auxiliary results or notions used in the calculations of the shape sensitivities.

A.1. Classical derivative formulæ with respect to the domain.

LEMMA A.1 (See, e.g., [19]). *Let $\delta > 0$. Let a vector field $\boldsymbol{V} \in \mathbf{W}^{1,\infty}(\mathbb{R}^d)$ and let*

$$\Psi \; : \; t \in [0, \delta) \mapsto \Psi_t = \mathbf{I} + t\boldsymbol{V} \in \mathbf{W}^{1,\infty}(\mathbb{R}^d).$$

Let a bounded Lipschitz open set Ω in $\mathbb{R}^d$ and let $\Omega_t := \Psi_t(\Omega)$ for all $t \in [0, \delta)$. We consider a function f such that $t \in [0, \delta) \mapsto f(t) \in \mathrm{L}^1(\mathbb{R}^d)$ is differentiable at 0 with $f(0) \in \mathrm{W}^{1,1}(\mathbb{R}^d)$. Then the function

$$t \in [0, \delta) \mapsto F(t) = \int_{\Omega_t} f(t, x) \, \mathrm{d}x$$

is differentiable at 0 *(we say that* F *admits a semi-derivative) and we have*

$$F'(0) = \int_{\partial\Omega} f(0,x) V_{\rm n} \, {\rm d}\varsigma(x) + \int_{\Omega} f'(0,x) \, {\rm d}x,$$

where $V_{\rm n} = \boldsymbol{V} \cdot \mathbf{n}$.

LEMMA A.2 (See, e.g., [19]). *Let* $\delta > 0$. *Let a vector field* $\boldsymbol{V} \in \boldsymbol{C}^{1,\infty}(\mathbb{R}^d)$ *and let*

$$\Psi \ : \ t \in [0,\delta) \mapsto \Psi_t = \mathbf{I} + t\boldsymbol{V} \in \boldsymbol{C}^{1,\infty}(\mathbb{R}^d).$$

Let a bounded open set Ω *in* $\mathbb{R}^d$ *of classe* $\mathcal{C}^2$ *and let* $\Omega_t := \Psi_t(\Omega)$ *for all* $t \in [0,\delta)$. *We consider a function* g *such that* $t \in [0,\delta) \mapsto g(t) \circ \Psi_t \in {\rm W}^{1,1}(\Omega)$ *is differentiable at* 0 *with* $g(0) \in {\rm W}^{2,1}(\Omega)$. *Then the function*

$$t \in [0,\delta) \mapsto G(t) = \int_{\partial\Omega_t} g(t,x) \, {\rm d}x$$

is differentiable at 0 *(we say that* G *admits a semi-derivative), the function* $t \in [0,\delta) \mapsto g(t)|_{\omega} \in {\rm W}^{1,1}(\omega)$ *is differentiable at* 0 *for all open set* $\omega \subset \overline{\omega} \subset \Omega$ *and the derivative* $g'(0)$ *belongs to* $W^{1,1}(\Omega)$ *and we have*

$$G'(0) = \int_{\partial\Omega} \left(g'(0,x) + ({\rm H}\, g(0,x) + \partial_{\rm n} g)\, V_{\rm n}\right) {\rm d}\varsigma(x),$$

where $V_{\rm n} = \boldsymbol{V} \cdot \mathbf{n}$ *and where* H *is the mean curvature function on* $\partial\Omega$.

A.2. Decomposition formulæ.

LEMMA A.3. *Given a bounded open set* Ω *in* $\mathbb{R}^d$ *of class* $\mathcal{C}^2$ *and* $\boldsymbol{u} \in \mathbf{H}^2(\mathbb{R}^d)$ *we have*

$$\partial_{\rm n} \left(A_{\rm c} e_\Gamma(\boldsymbol{u}) : e_\Gamma(\boldsymbol{u})\right) = 2A_{\rm c} e_\Gamma(\partial_{\rm n}\boldsymbol{u}) : e_\Gamma(\boldsymbol{u}) - A_{\rm c} e_\Gamma(\boldsymbol{u}) : \Pi_d \left(\nabla\boldsymbol{u}\, {\rm D}^2 b + {\rm D}^2 b\, {}^t\nabla\boldsymbol{u}\right) \Pi_d,$$

where b *is the signed distance to the boundary* $\partial\Omega$.

Proof. Let us first notice that $\partial_{\rm n}\Pi_d = 0$ and that $\Pi_d {\rm D}^2 b = {\rm D}^2 b \Pi_d = {\rm D}^2 b$, and we underline the fact that $\nabla_\Gamma \boldsymbol{u} = \nabla\boldsymbol{u}\, \Pi_d$ since

$$\nabla_\Gamma \boldsymbol{u} = \begin{pmatrix} {}^t\nabla_\Gamma u_1 \\ \vdots \\ {}^t\nabla_\Gamma u_d \end{pmatrix} = \begin{pmatrix} {}^t\nabla u_1 \Pi_d \\ \vdots \\ {}^t\nabla u_d \Pi_d \end{pmatrix} = \nabla\boldsymbol{u}\, \Pi_d.$$

Then we have

$$\partial_{\rm n}(\nabla_\Gamma\boldsymbol{u}) = \partial_{\rm n}(\nabla\boldsymbol{u}\, \Pi_d) = \partial_{\rm n}(\nabla\boldsymbol{u})\, \Pi_d \qquad \text{and} \qquad \partial_{\rm n}(\nabla\boldsymbol{u}) = {\rm D}^2\boldsymbol{u}\, \mathbf{n}.$$

Thus $\nabla(\partial_{\rm n}\boldsymbol{u}) = \nabla(\nabla\boldsymbol{u}\, \mathbf{n}) = {\rm D}^2\boldsymbol{u}\, \mathbf{n} + \nabla\boldsymbol{u}\, \nabla\mathbf{n} = \partial_{\rm n}(\nabla\boldsymbol{u}) + \nabla\boldsymbol{u}\, {\rm D}^2 b$. Hence we obtain

$$\nabla_\Gamma(\partial_{\rm n}\boldsymbol{u}) = \nabla(\partial_{\rm n}\boldsymbol{u})\Pi_d = \partial_{\rm n}(\nabla\boldsymbol{u})\Pi_d + \nabla\boldsymbol{u}\, {\rm D}^2 b\, \Pi_d = \partial_{\rm n}(\nabla_\Gamma\boldsymbol{u}) + \nabla\boldsymbol{u}\, {\rm D}^2 b.$$

We also obtain, noticing that ${\rm D}^2 b$ is symmetric,

$${}^t\nabla_\Gamma(\partial_{\rm n}\boldsymbol{u}) = \partial_{\rm n}({}^t\nabla_\Gamma\boldsymbol{u}) + {\rm D}^2 b\, {}^t\nabla\boldsymbol{u}.$$

We deduce from the previous computation that

$$\begin{aligned}
\partial_{\rm n}\left(e_\Gamma(\boldsymbol{u})\right) &= \frac{1}{2}\partial_{\rm n}\left(\Pi_d\left(\nabla_\Gamma u + {}^t\nabla_\Gamma \boldsymbol{u}\right)\Pi_d\right) = \frac{1}{2}\partial_{\rm n}\left(\Pi_d\nabla_\Gamma u + {}^t\nabla_\Gamma \boldsymbol{u}\Pi_d\right)\\
&= \frac{1}{2}\left(\Pi_d\partial_{\rm n}\left(\nabla_\Gamma u\right) + \partial_{\rm n}\left({}^t\nabla_\Gamma \boldsymbol{u}\right)\Pi_d\right)\\
&= \frac{1}{2}\left(\Pi_d\left(\nabla_\Gamma(\partial_{\rm n}\boldsymbol{u}) - \nabla\boldsymbol{u}\,\mathrm{D}^2 b\right) + \left({}^t\nabla_\Gamma(\partial_{\rm n}\boldsymbol{u}) - \mathrm{D}^2 b\,{}^t\nabla\boldsymbol{u}\right)\Pi_d\right)\\
&= e_\Gamma(\partial_{\rm n}\boldsymbol{u}) - \frac{1}{2}\left(\Pi_d\nabla\boldsymbol{u}\,\mathrm{D}^2 b + \mathrm{D}^2 b\,{}^t\nabla\boldsymbol{u}\Pi_d\right)\\
&= e_\Gamma(\partial_{\rm n}\boldsymbol{u}) - \frac{1}{2}\Pi_d\left(\nabla\boldsymbol{u}\,\mathrm{D}^2 b + \mathrm{D}^2 b\,{}^t\nabla\boldsymbol{u}\right)\Pi_d.
\end{aligned}$$

Therefore, we obtain that

$$\begin{aligned}
\partial_{\rm n}\left(A_{\rm c} e_\Gamma(\boldsymbol{u}) : e_\Gamma(\boldsymbol{u})\right) &= 2A_{\rm c}\partial_{\rm n}\left(e_\Gamma(\boldsymbol{u})\right) : e_\Gamma(\boldsymbol{u})\\
&= 2A_{\rm c}\left(e_\Gamma(\partial_{\rm n}\boldsymbol{u}) - \frac{1}{2}\Pi_d\left(\nabla\boldsymbol{u}\,\mathrm{D}^2 b + \mathrm{D}^2 b\,{}^t\nabla\boldsymbol{u}\right)\Pi_d\right) : e_\Gamma(\boldsymbol{u}),
\end{aligned}$$

which concludes the proof. ∎

LEMMA A.4. *Given a bounded open set Ω in $\mathbb{R}^d$ of class $\mathcal{C}^2$, $\boldsymbol{V} \in \mathcal{C}^1(\mathbb{R}^d;\mathbb{R}^d)$ and $\boldsymbol{u} \in \mathbf{H}^2(\mathbb{R}^d)$, we have*

$$\begin{aligned}
\partial_t\Big(e_{\Gamma_t}(\boldsymbol{u}\circ\Psi_t^{-1})\Big)\,|_{t=0} &= e_\Gamma(-\nabla\boldsymbol{u}\boldsymbol{V})\\
&\quad + \Pi_d e(\boldsymbol{u})(\mathbf{n}\otimes\nabla_\Gamma V_{\rm n} + \nabla_\Gamma V_{\rm n}\otimes\mathbf{n}) + (\mathbf{n}\otimes\nabla_\Gamma V_{\rm n} + \nabla_\Gamma V_{\rm n}\otimes\mathbf{n})e(\boldsymbol{u})\Pi_d,
\end{aligned}$$

where $V_{\rm n} = \boldsymbol{V}\cdot\mathbf{n}$.

Proof. We first recall that, since $\partial_t \mathbf{n}_{t\,|t=0} = -\nabla_\Gamma V_{\rm n}$, we have $\partial_t\Pi_d|_{t=0} = \mathbf{n}\otimes\nabla_\Gamma V_{\rm n} + \nabla_\Gamma V_{\rm n}\otimes\mathbf{n}$. Hence we obtain the result noticing that $e_\Gamma(\boldsymbol{u}) = \Pi_d e(\boldsymbol{u})\Pi_d$. ∎

Remark A.5. As compared to the scalar case dealt with in our previous paper, since $e_\Gamma(\boldsymbol{u})$ is obtained by mutliplying $e(\boldsymbol{u})$ on both sides by Π_d, when we derive $e_{\Gamma_t}(\boldsymbol{u}\circ\Psi_t^{-1})$ with respect to t we obtain an extra term.

A.3. Γ- convergence. For the convenience of the reader, we recall the definition and the main property of the Γ- convergence. For further details we refer to Dal Maso [11].

DEFINITION A.6. *(Sequential Γ-convergence) A family of functionals $\{F_t\}_{t>0}$ defined on a topological space X is said to be sequentially Γ-convergent to a functional F as $t\to 0^+$ if the two following statements hold.*

(i) $\Gamma-\liminf$ inequality*. For every sequence $\{x_t\}$ converging to $x\in X$, we have:*

$$\liminf_{t\to 0^+} F_t(x_t) \geq F(x). \tag{A.1}$$

(ii) $\Gamma-\limsup$ inequality*. For every $x\in X$, there exists a sequence $\{x_t\}$ converging to x such that*

$$\limsup_{t\to 0^+} F_t(x_t) \leq F(x). \tag{A.2}$$

When properties (i) and (ii) are satisfied, we write $F = \Gamma - \lim\limits_{t\to 0^+} F_t$.

PROPOSITION A.7. *Let $F_t : X \to \mathbb{R}$ be a sequence of functionals on a topological space such that:*
(i) $F = \Gamma - \lim\limits_{t\to 0^+} F_t$,
(ii) $\sup_t F_t(x_t) < +\infty \;\Rightarrow\; \{x_t\}$ *is sequentially relatively compact in* X.
Then we have the convergence: $\inf F_t \to \inf F$ *as* $t \to 0^+$ *and, every cluster point of a minimizing sequence* $\{x_t\}$ *(i.e. such that* $F_t(x_t) = \inf\limits_{x\in X} F_t(x)$*) achieves the minimum of* F.

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