Shape sensitivity of eigenvalue functionals for scalar problems: computing the semi-derivative of a minimum

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Abstract

This paper is devoted to the computation of certain *directional semi-derivatives* of eigenvalue functionals of self-adjoint elliptic operators involving a variety of boundary conditions. A uniform treatment of these problems is possible by considering them as a problem of calculating the semi-derivative of a minimum with respect to a parameter. The applicability of this approach, which can be traced back to the works of Danskin [5, 6] and Zolésio [29], to the treatment of eigenvalue problems (where the full shape derivative may not exist, due to multiplicity issues), has been illustrated by Zolésio in [30] (see also [10, Chapter 10] and included references). Despite this, some of the recent literature (see, for example, [1] or [9]) on the shape sensitivity of eigenvalue problems still continue to employ methods such as the *material derivative method* or *Lagrangian methods* which seem less adapted to this class of problems. The Delfour-Zolésio approach does not seem to be fully exploited in the existing literature: we aim to recall the importance and the simplicity of the ideas from [5, 29], by applying it to the analysis of the shape sensitivity for eigenvalue functionals for a class of elliptic operators in the scalar setting (Laplacian or diffusion in heterogeneous media), thus recovering known results in the case of Dirichlet or Neumann boundary conditions and obtaining new results in the case of Steklov or Wentzell boundary conditions.

Keywords: eigenvalues of elliptic operators, shape sensitivity analysis, shape semi-derivatives, generalized boundary condition.

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1 Introduction

Many problems ranging from engineering to physics deal with questions of optimal shapes or designs. An important class of these problems concern optimizing the shape for mechanical vibrations and in these, naturally, the objective functional depends on the eigenvalues of elliptic operators. The sensitivity of the objective functional with respect to shape variations can be understood in terms of certain derivatives or semi-derivatives. For a general discussion of the shape derivative analysis and its typical applications we refer the interested reader to the following texts [10, 21, 24, 26] and the included references.

Depending on the kind of variations that the domain is subjected to, that is, whether we use perturbations of identity or we use general velocity fields, one can give different definitions of semi-derivatives or derivatives. If the definition of the semi-derivative uses perturbations of identity it is more like a directional semi-derivative whereas if it is taken with respect to a velocity field it leads to the Hadamard semi-derivative which has a richer structure. For a discussions of these different notions and their interrelations we refer to Delfour and Zolésio [10]. Although, in this article, we shall restrict ourselves to analyzing the directional semi-derivatives of the eigenvalue functionals in the context of Theorem 2.1, Chapter 10 [10], it is not difficult to see that the same analysis can be successfully completed if, instead, we considered the Hadamard semi-derivatives in these cases.

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We now briefly discuss the notion of the directional semi-derivative of a shape functional. Consider a class of admissible open sets $\mathcal{O}_{\text{ad}}$ (whose closures are contained in an open set $D \subset \mathbb{R}^d$) which is stable with respect to smooth perturbations of the identity, that is, given $\Omega \in \mathcal{O}_{\text{ad}}$ and any $\boldsymbol{V}$ smooth vector field with compact support in a neighborhood of Ω, there exists $\delta > 0$ such that $(\mathbf{I} + t\boldsymbol{V})(\Omega)$ is in $\mathcal{O}_{\text{ad}}$ for all $t \in [0, \delta]$. Consider a shape functional $F : \mathcal{O}_{\text{ad}} \to \mathbb{R}$. The directional semi-derivative of F at $\Omega \in \mathcal{O}_{\text{ad}}$ in the direction of an admissible vector field $\boldsymbol{V}$ is defined as

$$F'(\Omega; \boldsymbol{V}) := \lim_{t \to 0^+} \frac{F(\Omega_t) - F(\Omega)}{t}, \tag{1.1}$$

where

$$\Omega_t := \Psi_t(\Omega), \quad \text{being } \Psi_t(x) := x + t\boldsymbol{V}(x), \tag{1.2}$$

whenever the limit in (1.1) exists.

Frequently, in the applications, the shape functional of interest may be the equilibrium energy for the domain coming from either an unconstrained optimization problem, examples of which are compliance energies, eigenvalue functionals etc or could be of the form

$$F(\Omega) := \int_\Omega f(x, u, \nabla u, \nabla^2 u, \cdots)\, dx + \int_{\partial\Omega} g(x, u, \nabla_\Gamma u, \cdots)\, \mathrm{d}\varsigma(x),$$

where the state $u = u(\Omega)$ is the solution of a boundary value problem in Ω.

There are two-main issues in shape derivative analysis. The first, is the existence of the semi-derivative or the derivative. If the above limit exists, is linear and is continuous with respect to $\boldsymbol{V}$ with respect to a suitable topology and if $\partial\Omega$ is of class $\mathcal{C}^1$ it defines a distribution. Then, by a *structure theorem* (see, for example, [10, Theorem 3.6, Chapter 9]), [21, Proposition 5.9.1]), this distribution is supported in $\partial\Omega$ and depends only on the normal component $\boldsymbol{V} \cdot \mathbf{n}$ of the vector field $\boldsymbol{V}$. So, the second thing of interest is to obtain the boundary expression of $F'(\Omega; \boldsymbol{V})$ which can be of use in studying the evolution of the shapes in the shape optimization problem.

A general idea to deal with the question of the existence of the semi-derivative might be to look at the problem after first transporting the shape functional to a fixed domain by a domain transformation. This requires us to examine the differentiability, with respect to t, of the composite function $u^t = u_t \circ \Psi_t$ on the fixed domain Ω, where u_t is the state on Ω_t in a constrained problem or the minimizer for the domain Ω_t in an unconstrained minimization problem. The consequent analysis is called the *material derivative method* since the following limit

$$\dot{u} := \lim_{t \to 0} \frac{u^t - u}{t},$$

is named the *material derivative.* For PDE constrained problems sometimes it is possible to prove the existence of the material derivative by a suitable use of the implicit function theorem. However, in the case of eigenvalue problems, when the eigenvalue is not simple it is not possible to apply such a technique. In fact, the eigenvalue functional does not admit a shape derivative at a multiple eigenvalue and it only makes sense to consider the semi-derivatives in such a case.

Nevertheless, for shape functionals such as the eigenvalue functionals which, for elliptic self-adjoint operators, arise through minimizing certain functionals such as the Rayleigh quotients over suitable spaces, the existence of the semi-derivative can be treated in the framework of the sensitivity analysis of a minimum with respect to a parameter. Precisely, we refer to the following result (see [10, Theorem 2.1, Chapter 10]).

Theorem 1.1. *Let X be a Banach space and let $G : [0, \delta] \times X \to \mathbb{R}$ be a given functional and we set*

$$g(t) := \inf_{u \in X} G(t, u) \quad \text{and} \quad X(t) := \{u \in X \,:\, G(t, u) = g(t)\}.$$

If the following hypotheses hold,

(H1) $X(t) \neq \emptyset$ *for all* $t \in [0, \delta]$,

(H2) $\partial_t G(t, u)$ *exists in* $[0, \delta]$ *at all* $u \in \bigcup_{t \in [0,\delta]} X(t)$,

(H3) there exists a topology τ on X such that, for every sequence $\{t_n\} \subset]0, \delta]$ tending to 0 and $u_n \in X(t_n)$, there exists $u_0 \in X(0)$ and a subsequence $\{t_{n_k}\}$ of $\{t_n\}$, for which

(i) $u_{n_k} \longrightarrow u_0$ *with respect to* τ*,*

(ii) $\liminf_{k\longrightarrow\infty} \partial_t G(t_{n_k}, u_{n_k}) \geq \partial_t G(0, u_0)$*,*

(H4) for all $u \in X(0)$*, the function* $t \longrightarrow \partial_t G(t, u)$ *is upper semi-continuous at* $t = 0$*,*

then we have

$$g'(0+) = \lim_{t\to 0^+} \frac{g(t) - g(0)}{t} = \inf_{u\in X(0)} \partial_t G(0, u).$$

Zolésio, in his pioneering works [29, 30], has shown that this principle can be applied successfully to investigate the shape sensitivity in several eigenvalue problems of interest. For earlier applications of the above idea we refer to Danskin [5, 6]. But this does not seem to have claimed sufficient attention, since even in recent times (see, for example, Allaire *et al.* [1] or Dambrine *et al.* [9]) this natural approach is not used. We would like, in this article, to restore the importance of the above idea for treating the shape sensitivity issues in eigenvalue problems.

To illustrate that strategy and in order to convince the reader of its efficiency, we consider a broad class of eigenvalue problems involving the Laplacian and other diffusion operators involving composite materials and for a wide range of boundary conditions. The models considered here, in addition to including well known and classical eigenvalue problems for the Laplacian spectrum with Dirichlet or Neumann boundary conditions, apply to completely new situations such as the Laplacian eigenvalue problem for the Wentzell boundary condition which is motivated by coating problems. We would like to mention that the Wentzell boundary condition is a natural asymptotic boundary condition on the limiting domain when a domain has a thin outer layer (see [17, 18, 27, 28] and also [4]). With modern methods of manufacturing such as 3-d printing it is now quite easy to fabricate material pieces with a thin coating or with a spatially varying filling. This fact motivates both the study of the Wentzell boundary condition and of the mixture case.

Typically, in order to apply Theorem 1.1 for obtaining the sensitivity of the eigenvalue with respect to domain variations, we will choose the functional $G(t, u)$ to be the Rayleigh quotient associated to the eigenvalue problem on the perturbed domain Ω_t transported back to Ω. The verification of the hypotheses of the theorem can be done in a few steps in a systematic way. After showing the applicability of the theorem to the problem in question, it will be enough to use the derivatives of typical elementary terms which constitute the Rayleigh quotient, calculated separately, to obtain an initial expression for the semi-derivative. We will also how to transform the unwieldy initial expression first obtained as a domain integral to a simpler boundary integral thanks to a systematic choice of test functions in the variational formulation of the eigenvalue problem. This procedure is similar to previous treatments (see [10]) of the sensitivity of eigenvalue problems except that it was considered necessary to use the Auchmuty variational principle for reformulating the eigenvalue problem. Our examples show that it is enough to work with the original Rayleigh quotients.

The shape sensitivity results are stated in Section 2: we present first the result in the case of the Laplace operator and then in the case of a mixture of two phases. The proofs are gathered in Section 3: we first provide the derivatives of the elementary terms arising in Rayleigh quotient in Section 3.2 (see Propositions 3.1 and 3.2) and then the proofs of the theorems are given. We recall (classical) background results in Section A.1. Finally, we give in Section A.2 an alternative proof of Theorem 2.5 in a particular case following the material derivative method in order to highlight the advantage of the approach employed here over the material derivative method in this class of problems.

2 The results

We state the shape sensitivity results, at first, in the case of the eigenvalues of the Laplace operator. Then we focus on eigenvalue problems related to the structural optimization of multi-phase materials which is our original motivation, more specifically, eigenvalues of elliptic operators of the type $-\text{div}\,(\sigma\nabla\cdot)$ in the specific case where σ only takes two values $0 < \sigma_1 \leq \sigma_2$. Although the Laplacian operator constitutes a special case of the latter situation we prefer to treat this separately since the development of the ideas are easier to follow in this simpler situation.

We start with a brief note on some of the expressions or notations which appear in the presentation of the problem and the statement of the results. Ω will be a bounded open set Ω of $\mathbb{R}^d$, $d = 1, 2, 3$, with a $\mathcal{C}^{2,1}$ boundary $\partial\Omega$. The unit exterior normal of $\partial\Omega$ is denoted by $\mathbf{n}$. V_{n} is then the normal component $\boldsymbol{V} \cdot \mathbf{n}$ of the vector field $\boldsymbol{V}$. We use ∇_Γ to denote the tangential gradient and use Δ_Γ to denote the *Laplace-Beltrami operator* on $\partial\Omega$. If we denote by b the *signed distance* to the boundary $\partial\Omega$, then $\mathrm{H} = \mathrm{Tr}(\mathrm{D}^2 b)$ is the *mean curvature function* on $\partial\Omega$. In the appendix we, briefly, recall some elements of the intrinsic tangential calculus used here. In the following, the function space $\mathcal{H}(\Omega)$ will be an appropriate subspace of $\mathrm{H}^1(\Omega)$ or $\mathrm{H}^2(\Omega)$ depending on whether the parameter β is not active (that is. $\beta = 0$) or active (that is, $\beta > 0$) and having the boundary condition in mind (Dirichlet, Neumann, Robin, Steklov, Wentzell etc.). We use I_d to denote the identity matrix of size $d \times d$. $\alpha, \beta \geq 0$ are two fixed real numbers.

2.1 Shape sensitivity for Laplacian eigenvalues

We are interested in two families of eigenvalues problems for the Laplacian which cover an ample range of boundary conditions. First we consider the least eigenvalue in eigenvalue problems of *volume type*, that is, the spectral parameter is in the domain:

$$\begin{cases} -\Delta u &= \Lambda_\Omega(\Omega) u \quad \text{in } \Omega, \\ -\beta \Delta_\Gamma u + \alpha u + \partial_{\mathrm{n}} u &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Then we look at the least eigenvalue in eigenvalue problems of *surface type*, that is, the spectral parameter is on the boundary:

$$\begin{cases} -\Delta u &= 0 \quad \text{in } \Omega, \\ -\beta \Delta_\Gamma u + \alpha u + \partial_{\mathrm{n}} u &= \Lambda_{\partial\Omega}(\Omega) u \quad \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

In the first of the situations, the eigenvalue problem results from a minimization of the Rayleigh quotient given by

$$\Lambda_\Omega(\Omega) = \inf_{u \in \mathcal{H}(\Omega)} \left\{ \frac{\displaystyle\int_\Omega |\nabla u|^2 \, \mathrm{d}x + \alpha \int_{\partial\Omega} u^2 \, \mathrm{d}\varsigma(x) + \beta \int_{\partial\Omega} |\nabla_\Gamma u|^2 \, \mathrm{d}\varsigma(x)}{\displaystyle\int_\Omega u^2 \, \mathrm{d}x} \right\}. \tag{2.3}$$

In the surface type eigenvalue problem, it comes from the minimization of the Rayleigh quotient given by

$$\Lambda_{\partial\Omega}(\Omega) = \inf_{u \in \mathcal{H}(\Omega)} \left\{ \frac{\displaystyle\int_\Omega |\nabla u|^2 \, \mathrm{d}x + \alpha \int_{\partial\Omega} u^2 \, \mathrm{d}\varsigma(x) + \beta \int_{\partial\Omega} |\nabla_\Gamma u|^2 \, \mathrm{d}\varsigma(x)}{\displaystyle\int_{\partial\Omega} u^2 \, \mathrm{d}\varsigma(x)} \right\}. \tag{2.4}$$

As mentioned earlier, the above formulations include a variety of eigenvalue problems. For example, choosing $\beta = 0$ and $\alpha = 0$ in (2.1) and $\mathcal{H}(\Omega)$ to be the subspace of functions in $\mathrm{H}^1(\Omega)$ whose mean value is 0, we obtain the first non-trivial *Neumann* eigenvalue. The *Dirichlet* eigenvalue problem is obtained from (2.1) in the limiting case $\alpha \to +\infty$ or alternately by taking $\beta = 0$ and choosing $\mathcal{H}(\Omega) = \mathrm{H}^1_0(\Omega)$ in (2.3). The *Robin* eigenvalue problem is obtained from (2.1) by taking $\beta = 0$ and $\mathcal{H}(\Omega) = \mathrm{H}^1(\Omega)$. Moreover, if we take $\beta = 0$ and $\alpha = 0$ in (2.2) while working on $\mathcal{H}(\Omega) = \mathrm{H}^1(\Omega)$, we obtain the *Steklov* eigenvalue problem. Finally, the choice $\beta > 0$ and the space $\mathcal{H}(\Omega) = \{u \in \mathrm{H}^1(\Omega);\ u_{|\partial\Omega} \in \mathrm{H}^1(\partial\Omega)\}$ with the associated norm $\left(\|u\|^2_{\mathrm{H}^1(\Omega)} + \|u\|^2_{\mathrm{H}^1(\partial\Omega)}\right)^{1/2}$ give rise to the *Wentzell* problem for the Laplacian.

We obtain the following results (see Section 3.3 for the proofs).

Theorem 2.1. *Let Ω be a $\mathcal{C}^{2,1}$ domain, and $\boldsymbol{V}$ be a smooth vector field. Then the semi-derivative $\Lambda'_\Omega(\Omega; \boldsymbol{V})$ of $\Lambda_\Omega(\Omega)$ in the direction of the vector field $\boldsymbol{V}$ exists and is given by*

$$\Lambda'_\Omega(\Omega; \boldsymbol{V}) = \inf \left\{ \int_{\partial\Omega} \left(|\nabla_\Gamma u|^2 - (\partial_{\mathrm{n}} u)^2 + \alpha \mathrm{H} \, |u|^2 + \beta (\mathrm{H} \mathrm{I}_d - 2\mathrm{D}^2 b) \nabla_\Gamma u \cdot \nabla_\Gamma u - \Lambda_\Omega(\Omega) |u|^2 \right) V_{\mathrm{n}} \, \mathrm{d}\varsigma(x) \right\},$$

where the inf *is taken with respect to all normalized eigenfunctions $u \in \mathcal{H}(\Omega)$ for which $\Lambda_\Omega(\Omega)$ is attained in* (2.3).

Remark 2.2. *Such a result in the particular cases of Dirichlet, Neumann and Robin eigenvalues has been known since a long time (see, for example, [10, 20]). The result for the Wentzell eigenvalue ($\beta > 0$) for $\Lambda_\Omega(\Omega)$ is new to the best of our knowledge.*

Theorem 2.3. *Let Ω be a $\mathcal{C}^{2,1}$ domain Ω, and $\boldsymbol{V}$ be a smooth vector field. Then the semi-derivative $\Lambda'_{\partial\Omega}(\Omega;\boldsymbol{V})$ of $\Lambda_{\partial\Omega}(\Omega)$ in the direction of the vector field $\boldsymbol{V}$ exists and is given by*

$$\Lambda'_{\partial\Omega}(\Omega;\boldsymbol{V}) = \inf\left\{\int_{\partial\Omega}\left(|\nabla_\Gamma u|^2 - (\partial_{\mathrm{n}} u)^2 + \alpha \mathrm{H}\,|u|^2 + \beta(\mathrm{HI}_d - 2\mathrm{D}^2 b)\,\nabla_\Gamma u\cdot\nabla_\Gamma u - \Lambda_{\partial\Omega}(\Omega)\mathrm{H}|u|^2\right)V_{\mathrm{n}}\ \mathrm{d}\varsigma(x)\right\},$$

where the inf *is taken with respect to all normalized eigenfunctions $u\in\mathcal{H}(\Omega)$ for which $\Lambda_{\partial\Omega}(\Omega)$ is attained in* (2.4)*.*

Remark 2.4. *The expression for the shape derivative of the Wentzell eigenvalue $\Lambda_{\partial\Omega}(\Omega)$ has been obtained in [9] using the material derivative approach and then used to study the problem of maximizing the first eigenvalue. It will be seen in the paper that the approach developed by Zolésio in [29, 30] (see also the work of Danskin [5, 6]) allows us to recover the results given in [9] in a more efficient way. Notice also the connection of the above results with the works of Desaint and Zolésio [14, 15, 16] on the Laplace-Beltrami operator.*

2.2 Shape sensitivity for eigenvalue problems for composites

Consider an open subset Ω_1 of Ω with a $\mathcal{C}^{2,1}$ boundary and set $\Omega_2 = \Omega\setminus\overline{\Omega_1}$. We assume that there exists $r>0$ such that $\|x-y\|\geq r$ for all $x\in\Omega_1$ and $y\in\partial\Omega$. We consider two conducting materials with coefficients $0<\sigma_1\leq\sigma_2$ which occupy respectively the regions Ω_1 and Ω_2 according to respective densities $0<\rho_1\leq\rho_2$. We set

$$\rho = \rho_1\chi_{\Omega_1} + \rho_2\chi_{\Omega_2} \qquad \text{and} \qquad \sigma = \sigma_1\chi_{\Omega_1} + \sigma_2\chi_{\Omega_2},$$

with χ_{Ω_1} and χ_{Ω_2} denoting the characteristic functions of the sets Ω_1 and Ω_2 respectively.

The interface between Ω_1 and Ω_2 will be denoted by Γ, that is $\Gamma = \partial\Omega_1\cap\partial\Omega_2 = \partial\Omega_1$. The exterior normal on $\partial\Omega$ as well as the unit normal pointing outward from Ω_1 will be denoted $\mathbf{n}$. We also use the notation $[\cdot]$ in order to represent the jump on the interface Γ, that is, for a function u and a point $x\in\Gamma$:

$$[u]\,(x) = \lim_{\varepsilon\to 0^+}\left(u(x-\varepsilon\mathbf{n}(x)) - u(x+\varepsilon\mathbf{n}(x))\right).$$

We summarize the notations in Figure 1. We consider the eigenvalue problem of *volume type*

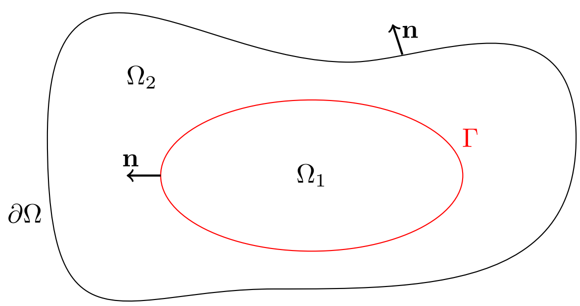

Figure 1: Notations

$$\left\{\begin{array}{rcll} -\mathrm{div}\,(\sigma(x)\nabla u) &=& \mathfrak{M}_\Omega(\Omega)\rho(x)u & \text{in } \Omega,\\ -\beta\Delta_\Gamma u + \alpha u + \sigma_2\partial_{\mathrm{n}} u &=& 0 & \text{on } \partial\Omega,\end{array}\right. \tag{2.5}$$

whose first eigenvalue is given by

$$\mathfrak{M}_\Omega(\Omega) = \inf_{u\in\mathcal{H}(\Omega)}\left\{\frac{\displaystyle\int_\Omega \sigma(x)|\nabla u|^2\ \mathrm{d}x + \alpha\int_{\partial\Omega} u^2\ \mathrm{d}\varsigma(x) + \beta\int_{\partial\Omega}|\nabla_\Gamma u|^2\ \mathrm{d}\varsigma(x)}{\displaystyle\int_\Omega \rho\,|u|^2}\right\}. \tag{2.6}$$

We also consider the eigenvalue problem of *surface type*

$$\left\{ \begin{array}{rcll} -\text{div }(\sigma(x)\nabla u) & = & 0 & \text{in } \Omega, \\ -\beta\Delta_\Gamma u + \alpha u + \sigma_2 \partial_{\rm n} u & = & \mathfrak{M}_{\partial\Omega}(\Omega) u & \text{on } \partial\Omega, \end{array} \right. \tag{2.7}$$

whose first eigenvalue is given by

$$\mathfrak{M}_{\partial\Omega}(\Omega) = \inf_{u\in\mathcal{H}(\Omega)} \left\{ \frac{\displaystyle\int_\Omega \sigma(x)|\nabla u|^2 \, {\rm d}x + \alpha \int_{\partial\Omega} u^2 \, {\rm d}\varsigma(x) + \beta \int_{\partial\Omega} |\nabla_\Gamma u|^2 \, {\rm d}\varsigma(x)}{\displaystyle\int_{\partial\Omega} |u|^2 \, {\rm d}\varsigma(x)} \right\}. \tag{2.8}$$

We obtain the following results (see Section 3.4 for the proofs). To the best of our knowledge, the following shape sensitivity results which allows for variations in the diffusivity σ and *also* in the density ρ are new. A particular case of the above problem was studied in [23, 8]. We also allow for boundary variations of Ω while considering general boundary conditions (which includes Dirichlet boundary conditions as a particular case).

Theorem 2.5. *Let Ω be a $\mathcal{C}^{2,1}$ domain, and $\boldsymbol{V}$ be a smooth vector field. Then the semi-derivative $\mathfrak{M}'_\Omega(\Omega;\boldsymbol{V})$ of $\mathfrak{M}_\Omega(\Omega)$ in the direction of the vector field $\boldsymbol{V}$ exists and is given by*

$$\mathfrak{M}'_\Omega(\Omega;\boldsymbol{V}) = \inf \left\{ \int_\Gamma \Big([\sigma]|\nabla_\Gamma u|^2 - [\sigma(\partial_{\rm n} u)^2] - \mathfrak{M}_\Omega(\Omega)[\rho]|u|^2 \Big) V_{\rm n} \, {\rm d}\varsigma(x) \right.$$
$$\left. + \int_{\partial\Omega} \Big(\sigma_2(|\nabla_\Gamma u|^2 - (\partial_{\rm n} u)^2) + \alpha {\rm H}|u|^2 + \beta \left({\rm H}{\rm I}_d - 2{\rm D}^2 b \right) \nabla_\Gamma u \cdot \nabla_\Gamma u - \mathfrak{M}_\Omega(\Omega)\rho_2|u|^2 \Big) V_{\rm n} \, {\rm d}\varsigma(x) \right\},$$

where the inf *is taken with respect to all normalized eigenfunctions $u \in \mathcal{H}(\Omega)$ for which $\Lambda_\Omega(\Omega)$ is attained in* (2.6).

Theorem 2.6. *Let Ω be a $\mathcal{C}^{2,1}$ domain and $\boldsymbol{V}$ be a smooth vector field. Then the semi-derivative $\mathfrak{M}'_{\partial\Omega}(\Omega;\boldsymbol{V})$ of $\mathfrak{M}_{\partial\Omega}(\Omega)$ in the direction of the vector field $\boldsymbol{V}$ exists and is given by*

$$\mathfrak{M}'_{\partial\Omega}(\Omega;\boldsymbol{V}) = \inf \left\{ \int_\Gamma \Big([\sigma] \, |\nabla_\Gamma u|^2 - \left[\sigma(\partial_{\rm n} u)^2 \right] \Big) V_{\rm n} \, {\rm d}\varsigma(x) \right.$$
$$\left. + \int_{\partial\Omega} \Big(\sigma_2(|\nabla_\Gamma u|^2 - (\partial_{\rm n} u)^2) + \alpha {\rm H} \, |u|^2 + \beta \left({\rm H}{\rm I}_d - 2{\rm D}^2 b \right) \nabla_\Gamma u \cdot \nabla_\Gamma u - \mathfrak{M}_{\partial\Omega}(\Omega){\rm H} \, |u|^2 \Big) V_{\rm n} \, {\rm d}\varsigma(x) \right\},$$

where the inf *is taken with respect to all normalized eigenfunctions $u \in \mathcal{H}(\Omega)$ for which $\mathfrak{M}_{\partial\Omega}(\Omega)$ is attained in* (2.8).

3 Proofs

The shape sensitivity results stated in the previous section will be established in the framework of Theorem 1.1. We will give a full treatment of one of the results, Theorem 2.1, following the general strategy outlined below. In the other cases we restrict ourselves to, more or less, obtaining the computation of the expressions for the semi-derivatives. This approach proposed in [29, 30, 5, 6] can be seen to be effective (see [10, Chapter 10, Section 5] for studying, systematically, the existence of directional semi-derivatives of eigenvalue functionals, without having to worry about multiplicity issues. Whereas, it was considered advantageous to work with the Auchmuty's dual principle in these problems (see [10]) we show that the same can be achieved working only with the Rayleigh quotient in such problems. We also show how to compute, in a systematic and efficient, useful expressions of these semi-derivatives.

In the sequel we consider a smooth vector field $\boldsymbol{V}$ and we recall that the perturbed Ω_t and the diffeomorphism Ψ_t are defined by (1.2).

3.1 General strategy

The least eigenvalue problem on the perturbed domain Ω_t obtained by the minimization of a Rayleigh quotient will need to be formulated in a space independent of the parameter t giving rise to family of functions $G(t,\cdot)$. For this, we will use the fact that $u \mapsto u \circ \Psi_t^{-1}$ is, usually, an isomorphism between $\mathcal{H}(\Omega)$ and $\mathcal{H}(\Omega_t)$. Recall that $\mathcal{H}(\Omega_t)$ is an appropriate subspace of the Sobolev spaces $\mathrm{H}^1(\Omega_t)$ or $\mathrm{H}^2(\Omega_t)$ for the eigenvalue problem concerned.

The next step will consist in verifying that the assumptions of Theorem 1.1 are satisfied. For verifying the hypothesis $(H3)$, in the class of eigenvalue problems, we will usually need to show the Γ-*convergence* (see Appendix A.1 for some reminders on this notion) of $G(t,\cdot)$ to $G(0,\cdot)$ as $t \to 0^+$ in the weak topology of $\mathcal{H}(\Omega)$ and later the strong convergence of a sequence of minimizers.

Then Theorem 1.1 will allow us to calculate the shape derivative by evaluating $\inf_{u\in X(0)} \partial_t G(0,u)$ where $X(0)$ will be the eigenspace for the problem over the domain Ω. In the case of a simple eigenfunction, it is enough to evaluate at a normalized eigenfunction. Finally, it is shown how to obtain a fairly simple expression for $\partial_t G(0,u)$ by making the canonical choice of $-\nabla u \cdot \boldsymbol{V}$ as a test function in the governing equation and using the intrinsic tangential calculus developed Delfour and Zolésio [12, 13] for handling complicated surface terms such as in Wentzell eigenvalue problems.

3.2 Preliminary computations

In this subsection we gather together preliminary calculations of derivatives with respect to t of some typical integrals which will constitute the functionals $G(t,\cdot)$. For this we will rely on the classical derivative with respect to the shape formulae recalled in Lemma A.1 and Lemma A.2 in the appendix. The regularity necessary on u for applying these lemma will be guaranteed by the classical regularity of the eigenfunctions in the problems considered.

Proposition 3.1. *For sufficiently smooth u and Ω, we have*

$$\partial_t\Big(\int_{\Omega_t} |\nabla(u\circ\Psi_t^{-1})|^2 \,\mathrm{d}x\Big)\Big|_{t=0} = \int_{\partial\Omega} |\nabla u|^2 V_{\mathrm{n}}\,\mathrm{d}\varsigma(x) + 2\int_{\Omega} \nabla u\cdot\nabla(-\nabla u\cdot\boldsymbol{V})\,\mathrm{d}x, \tag{3.1}$$

$$\partial_t\Big(\int_{\Omega_t} |(u\circ\Psi_t^{-1})|^2 \,\mathrm{d}x\Big)\Big|_{t=0} = \int_{\partial\Omega} |u|^2 V_{\mathrm{n}}\,\mathrm{d}\varsigma(x) + 2\int_{\Omega} u\,(-\nabla u\cdot\boldsymbol{V})\,\mathrm{d}x, \tag{3.2}$$

$$\partial_t\Big(\int_{\partial\Omega_t} |(u\circ\Psi_t^{-1})|^2 \,\mathrm{d}x\Big)\Big|_{t=0} = \int_{\partial\Omega} \big(\mathrm{H}\,|u|^2 + \partial_{\mathrm{n}} u^2\big)\,V_{\mathrm{n}} + 2u\,(-\nabla u\cdot\boldsymbol{V})\,\mathrm{d}\varsigma(x). \tag{3.3}$$

Proof. The above formulae are obtained by a straightforward application of the formulae for derivatives of domain and boundary integrals given in Lemma A.1 and Lemma A.2 in the appendix and the fact that $\partial_t(u\circ\Psi_t^{-1})|_{t=0} = -\nabla u\cdot\boldsymbol{V}$ since $\partial_t(\Psi_t^{-1})|_{t=0} = -\boldsymbol{V}$. □

Proposition 3.2. *For sufficiently smooth u and Ω, we have*

$$\begin{aligned}\partial_t\Big(\int_{\partial\Omega_t} |\nabla_{\Gamma_t}(u\circ\Psi_t^{-1})|^2 \,\mathrm{d}x\Big)\Big|_{t=0} &= \int_{\partial\Omega} \big(\mathrm{H}\,|\nabla_\Gamma u|^2 + 2\nabla_\Gamma u\cdot\nabla_\Gamma(\partial_{\mathrm{n}} u) - 2\mathrm{D}^2 b\,\nabla_\Gamma u\cdot\nabla_\Gamma u\big)\,V_{\mathrm{n}}\,\mathrm{d}\varsigma(x)\\ &\quad + 2\int_{\partial\Omega} \nabla_\Gamma u\cdot\big(\nabla_\Gamma(-\nabla u\cdot\boldsymbol{V}) + \nabla u\cdot\nabla_\Gamma V_{\mathrm{n}}\mathbf{n} + \partial_{\mathrm{n}} u\nabla_\Gamma V_{\mathrm{n}}\big)\,\mathrm{d}\varsigma(x).\end{aligned} \tag{3.4}$$

Proof. By applying the classical derivation formula recalled in Lemma A.2, we get

$$\begin{aligned}&\partial_t\Big(\int_{\partial\Omega_t} |\nabla_{\Gamma_t}(u\circ\Psi_t^{-1})|^2 \,\mathrm{d}x\Big)\Big|_{t=0}\\ &\qquad = \int_{\partial\Omega} \big(\mathrm{H}\,|\nabla_\Gamma u|^2 + \partial_{\mathrm{n}}(|\nabla_\Gamma u|^2)\big)\,V_{\mathrm{n}}\,\mathrm{d}\varsigma(x) + 2\int_{\partial\Omega} \nabla_\Gamma u\cdot\partial_t\big(\nabla_{\Gamma_t}(u\circ\Phi_t^{-1})\big)\,|_{t=0}\mathrm{d}\varsigma(x).\end{aligned}$$

We obtain the conclusion using the facts that (see respectively Lemma A.3 and Lemma A.4)

$$\partial_{\mathrm{n}}|\nabla_\Gamma u|^2 = 2\nabla_\Gamma u\cdot\big(\nabla_\Gamma(\partial_{\mathrm{n}} u) - \mathrm{D}^2 b\,\nabla_\Gamma u\big)$$

and

$$\partial_t\big(\nabla_{\Gamma_t}(u\circ\Psi_t^{-1})\big)\,|_{t=0} = \nabla_\Gamma(-\nabla u\cdot\boldsymbol{V}) + \nabla u\cdot\nabla_\Gamma V_{\rm n}\,\mathbf{n} + \partial_{\rm n}u\,\nabla_\Gamma V_{\rm n}.$$

□

Proposition 3.3. *For sufficiently smooth u and Ω, we have*

$$\partial_t\Big(\int_{\Omega_t}\sigma_t|\nabla(u\circ\Psi_t^{-1})|^2\,{\rm d}x\Big)\Big|_{t=0} = \int_\Gamma[\sigma|\nabla u|^2]V_{\rm n}\,{\rm d}\varsigma(x) + 2\int_{\Omega_1}\sigma_1\nabla u\cdot\nabla(-\nabla u\cdot\boldsymbol{V})\,{\rm d}x + \int_{\partial\Omega}\sigma_2|\nabla u|^2V_{\rm n}\,{\rm d}\varsigma(x) + 2\int_{\Omega_2}\sigma_2\nabla u\cdot\nabla(-\nabla u\cdot\boldsymbol{V})\,{\rm d}x \quad (3.5)$$

and

$$\partial_t\Big(\int_{\Omega_t}\rho_t|(u\circ\Psi_t^{-1})|^2\,{\rm d}x\Big)\Big|_{t=0} = \int_\Gamma[\rho]|u|^2V_{\rm n}\,{\rm d}\varsigma(x) + 2\int_{\Omega_1}\rho_1u\,(-\nabla u\cdot\boldsymbol{V})\,{\rm d}x + \int_{\partial\Omega}\rho_2|u|^2V_{\rm n}\,{\rm d}\varsigma(x) + 2\int_{\Omega_2}\rho_2u\,(-\nabla u\cdot\boldsymbol{V})\,{\rm d}x. \quad (3.6)$$

Proof. The above formulae are obtained by an application of the classical derivative formula (see Lemma A.1) after writing

$$\int_{\Omega_t}\sigma_t|\nabla(u\circ\Psi_t^{-1})|^2\,{\rm d}x = \int_{\Omega_{1,t}}\sigma_1|\nabla(u\circ\Psi_t^{-1})|^2\,{\rm d}x + \int_{\Omega_{2,t}}\sigma_2|\nabla(u\circ\Psi_t^{-1})|^2\,{\rm d}x$$

and

$$\int_{\Omega_t}\rho_t|(u\circ\Psi_t^{-1})|^2\,{\rm d}x = \int_{\Omega_{1,t}}\rho_1|(u\circ\Psi_t^{-1})|^2\,{\rm d}x + \int_{\Omega_{2,t}}\rho_2|(u\circ\Psi_t^{-1})|^2\,{\rm d}x,$$

where $\Omega_{1,t} := \Psi_t(\Omega_1)$ and $\Omega_{2,t} := \Psi_t(\Omega_2)$. To begin with one has

$$\partial_t\Big(\int_{\Omega_t}\sigma_t|\nabla(u\circ\Psi_t^{-1})|^2\,{\rm d}x\Big)\Big|_{t=0} = \int_{\partial\Omega_1}\sigma_1|\nabla u|^2V_{\rm n}\,{\rm d}\varsigma(x) + 2\int_{\Omega_1}\sigma_1\nabla u\cdot\nabla(-\nabla u\cdot\boldsymbol{V})\,{\rm d}x - \int_{\partial\Omega_1}\sigma_2|\nabla u|^2V_{\rm n}\,{\rm d}\varsigma(x) + 2\int_{\Omega_2}\sigma_2\nabla u\cdot\nabla(-\nabla u\cdot\boldsymbol{V})\,{\rm d}x + \int_{\partial\Omega}\sigma_2|\nabla u|^2V_{\rm n}\,{\rm d}\varsigma(x)$$

and

$$\partial_t\Big(\int_{\Omega_t}\rho_t|(u\circ\Psi_t^{-1})|^2\,{\rm d}x\Big)\Big|_{t=0} = \int_{\partial\Omega_1}\rho_1|u|^2V_{\rm n}\,{\rm d}\varsigma(x) + 2\int_{\Omega_1}\rho_1u\,(-\nabla u\cdot\boldsymbol{V})\,{\rm d}x - \int_{\partial\Omega_2\cap\partial\Omega_1}\rho_2|u|^2V_{\rm n}\,{\rm d}\varsigma(x) + 2\int_{\Omega_2}\rho_2u\,(-\nabla u\cdot\boldsymbol{V})\,{\rm d}x + \int_{\partial\Omega}\rho_2|u|^2V_{\rm n}\,{\rm d}\varsigma(x).$$

In the above, notice that the domain Ω_2 has two boundaries, $\partial\Omega_2\cap\partial\Omega_1$ and $\partial\Omega$. The boundary $\partial\Omega_2\cap\partial\Omega_1$ is identified with $\partial\Omega_1$ but the outward pointing normal on $\partial\Omega_2\cap\partial\Omega_1$ with respect to Ω_2 is just $-\mathbf{n}$ with $\mathbf{n}$ being the outward pointing normal to $\partial\Omega_1$ with respect to Ω_1. Although we denote by the same letter u the eigenfunction on either side of the common boundary $\partial\Omega_1$, it must be kept in mind that ∇u in $\int_{\partial\Omega_1}\sigma_1|\nabla u|^2V_{\rm n}\,{\rm d}\varsigma(x)$ is the trace of ∇u coming from Ω_1 and ∇u in $-\int_{\partial\Omega_1}\sigma_2|\nabla u|^2V_{\rm n}\,{\rm d}\varsigma(x)$ is the trace of ∇u coming from Ω_2 and are not, necessarily, equal. However, the traces of u on $\partial\Omega_1$ coming from Ω_2 or Ω_1 are equal. □

3.3 Shape derivatives for the Laplacian eigenvalues

3.3.1 Proof of Theorem 2.1

The considered eigenvalue functional on the perturbed domain is

$$\Lambda_\Omega(\Omega_t) = \inf_{v\in\mathcal{H}(\Omega_t)}\left\{\frac{\displaystyle\int_{\Omega_t}|\nabla v|^2\,{\rm d}x + \alpha\int_{\partial\Omega_t}v^2\,{\rm d}\varsigma(x) + \beta\int_{\partial\Omega_t}|\nabla_\Gamma v|^2\,{\rm d}\varsigma(x)}{\displaystyle\int_{\Omega_t}v^2\,{\rm d}x}\right\}, \quad (3.7)$$

which can reformulated on $\mathcal{H}(\Omega)$ as

$$\Lambda_\Omega(\Omega_t) = \inf_{u\in\mathcal{H}(\Omega)} \left\{ \frac{\displaystyle\int_{\Omega_t} |\nabla(u\circ\Psi_t^{-1})|^2 \,\mathrm{d}x + \alpha\int_{\partial\Omega_t} (u\circ\Psi_t^{-1})^2 \,\mathrm{d}\varsigma(x) + \beta\int_{\partial\Omega_t} |\nabla_\Gamma(u\circ\Psi_t^{-1})|^2 \,\mathrm{d}\varsigma(x)}{\displaystyle\int_{\Omega_t} (u\circ\Psi_t^{-1})^2 \,\mathrm{d}x} \right\}.$$

This corresponds to a minimization of the functional

$$G_\Omega(t,u) = \frac{\displaystyle\int_{\Omega_t} |\nabla(u\circ\Psi_t^{-1})|^2 \,\mathrm{d}x + \alpha\int_{\partial\Omega_t} (u\circ\Psi_t^{-1})^2 \,\mathrm{d}\varsigma(x) + \beta\int_{\partial\Omega_t} |\nabla_\Gamma(u\circ\Psi_t^{-1})|^2 \,\mathrm{d}\varsigma(x)}{\displaystyle\int_{\Omega_t} (u\circ\Psi_t^{-1})^2 \,\mathrm{d}x}.$$

First we verify that the assumptions of Theorem 1.1 are satisfied before proceeding to calculate an expression for $\Lambda'_\Omega(\Omega;\boldsymbol{V})$.

Existence of the semi-derivative. Let us start by assumption $(H1)$. The arguments to show that the set of minimizers of (3.7) is non-empty for each t are classical and this is based on the direct method of calculus of variations. In fact, the functional is lower semi-continuous for the weak topology on $\mathrm{H}(\Omega_t)$, since the numerator is convex and continuous for the strong topology on $\mathrm{H}(\Omega_t)$ (and therefore weakly lower semi-continuous), and since the denominator is continuous due to the compact inclusion of $\mathrm{H}^1(\Omega_t)$ into $\mathrm{L}^2(\Omega_t)$. As concerns the coercivity of the numerator for given t, it is enough to show that the numerator dominates the square of the norm or a quotient norm on $\mathcal{H}(\Omega_t)$. In the case of Dirichlet eigenvalue problem, this can be obtained from the use of Poincaré's inequality. In the case of the first non-trivial Neumann eigenvalue problem, one uses the so-called Poincaré-Wirtinger's inequality. When $\alpha > 0$ it is enough, once again, to use Poincaré's inequality. The set $X(t)$, defined in Theorem 1.1 of minimizers for $G_\Omega(t,\cdot)$ is obtained by transporting the minimizers in (3.7) to Ω by composition with Ψ_t. Therefore assumption $(H1)$ is satisfied.

Let us now check assumption $(H2)$. Since

$$\nabla(u\circ\Psi_t^{-1}) = \left((\mathrm{D}\Psi_t^{-1})^\top\nabla u\right)\circ\Psi_t^{-1} \qquad \text{and} \qquad \nabla_{\Gamma_t} = (\mathrm{I}_d - \mathbf{n}_t\otimes\mathbf{n}_t)\nabla,$$

where $\mathbf{n}_t$ the normal vector field on $\partial\Omega_t$, we have

$$G_\Omega(t,u) = \frac{1}{\displaystyle\int_{\Omega_t}(u\circ\Psi_t^{-1})^2\,\mathrm{d}x}\Bigg(\int_{\Omega_t} |\left((\mathrm{D}\Psi_t^{-1})^\top\nabla u\right)\circ\Psi_t^{-1}|^2\,\mathrm{d}x$$
$$+\int_{\partial\Omega_t}\left(\alpha(u\circ\Psi_t^{-1})^2 + \beta|(\mathrm{I}_d - \mathbf{n}_t\otimes\mathbf{n}_t)\left((\mathrm{D}\Psi_t^{-1})^\top\nabla u\right)\circ\Psi_t^{-1}|^2\,\mathrm{d}\varsigma(x)\right)\Bigg).$$

Then, by a change of variables, the Rayleigh quotient G_Ω can also be written as

$$G_\Omega(t,u) = \frac{1}{\displaystyle\int_\Omega |u|^2 j(t)\mathrm{d}x}\Bigg(\int_\Omega |(\mathrm{D}\Psi_t^{-1})^\top\nabla u|^2 j(t)\mathrm{d}x$$
$$+\alpha\int_{\partial\Omega}|u|^2\omega(t)\mathrm{d}\varsigma(x) + \beta\int_{\partial\Omega}|(\mathrm{I}_d - \mathbf{n}^t\otimes\mathbf{n}^t)(\mathrm{D}\Psi_t^{-1})^\top\nabla u|^2\omega(t)\mathrm{d}\varsigma(x)\Bigg), \quad (3.8)$$

where $j(t) = \det(\mathrm{D}\Psi_t(x))$ is the Jacobian, $\omega(t) = \det(\mathrm{D}\Psi_t(x))\|(\mathrm{D}\Psi_t^{-1})^\top(x)\mathbf{n}(x)\|$ is the surface Jacobian and $\mathbf{n}^t = \mathbf{n}_t\circ\Psi_t$. Since the deformation Ψ_t is smooth with respect to t and x, it follows that $j(t)$ and $\omega(t)$ are smooth functions of t and since $\partial\Omega$ is also smooth (at least of class $\mathcal{C}^{2,1}$), $\mathbf{n}_t$ is a smooth function too, for t small enough. Therefore we are able to conclude from the previous expression (3.8) that $G_\Omega(\cdot,u)$ is derivable for t small enough for all $u\in\mathcal{H}(\Omega)$ and this gives the hypothesis $(H2)$ of Theorem 1.1. Furthermore, the new coefficients, after differentiation are continuous with respect to t and so the continuity of $\partial_t G_\Omega(\cdot,u)$ follows by the dominated convergence theorem. This gives assumption $(H4)$.

Finally let us verify the remaining assumption $(H3)$ of Theorem 1.1. The functional $\partial_t G_\Omega(t,\cdot)$ is not necessarily lower semicontinuous for the weak topology on $\mathcal{H}(\Omega)$ but is continuous for the strong topology on $\mathcal{H}(\Omega)$. Our aim is to show that $(H3)$ holds for this topology. This will be done in a few steps. Firstly, we show that $G_\Omega(t,\cdot)$ has the Γ-limit $G_\Omega(0,\cdot)$ as $t\to 0^+$, in the weak topology on $\mathcal{H}(\Omega)$ (see Definition A.5 and Proposition A.6 in the appendix for some reminders on this notion; also refer to [7]). We use the expression (3.8) and prove the $\Gamma-\liminf$ and $\Gamma-\limsup$ inequalities as follows.

(i) Consider u^t which converges weakly to a u in $\mathcal{H}(\Omega)$. We obtain the estimate

$$G_\Omega(t,u^t) = G_\Omega(0,u^t) + (G_\Omega(t,u^t) - G_\Omega(0,u^t)) \geq G_\Omega(0,u^t) + O(t).$$

Indeed we obtain that $G_\Omega(t,u^t) - G_\Omega(0,u^t)$ is $O(t)$ (that is, goes to 0 as $t\to 0^+$) using the uniform convergence of the coefficients, as $t\to 0^+$, in the numerator and denominator of G_Ω and using the fact that any weakly convergent sequence u^t is bounded in $\mathcal{H}(\Omega)$. Then the $\Gamma-\liminf$ inequality follows from the already observed fact that $G_\Omega(0,\cdot)$ is lower semi-continuous for the weak topology on $\mathcal{H}(\Omega)$.

(ii) The $\Gamma-\limsup$ inequality is obtained by taking the constant sequence u, for any given $u\in\mathcal{H}(\Omega)$, and observing that $G_\Omega(t,u)\to G_\Omega(0,u)$ as $t\to 0^+$.

Then Theorem A.6 allows us to deduce that the minimum of $G_\Omega(t,\cdot)$ (which is $\Lambda_\Omega(\Omega_t)$) converges to the minimum of $G_\Omega(0,\cdot)$ (namely, $\Lambda_\Omega(\Omega)$). Now, using the 0-homogeneity of $G_\Omega(t,\cdot)$, for each t, consider a minimizer u^t for which the denominator in (3.8) is 1. The boundedness of $\Lambda_\Omega(\Omega_t)$ and the equi-coercivity of the numerators in (3.8) implies, by Theorem A.6, that u^t converges weakly in $\mathcal{H}(\Omega)$ to a minimizer u of $G_\Omega(0,\cdot)$. Finally we prove the strong convergence of u^t to u in $\mathcal{H}(\Omega)$ as follows:

$$\begin{aligned} C\|u^t-u\|^2_{\mathcal{H}(\Omega)} \leq \int_\Omega |\mathrm{D}\Psi_t^{-T}\nabla(u^t-u)|^2 j(t)\mathrm{d}x + \alpha\int_{\partial\Omega}|(u^t-u)|^2\omega(t)\mathrm{d}\varsigma(x)& \\ + \beta\int_{\partial\Omega}|(\mathrm{I}_d - \mathbf{n}^t\otimes\mathbf{n}^t)\mathrm{D}\Psi_t^{-T}\nabla(u^t-u)|^2\omega(t)\mathrm{d}\varsigma(x) =: \mathfrak{A}(t).& \end{aligned}$$

It remains to prove that $\mathfrak{A}(t)\to 0$ when $t\to 0^+$. We expand the quadratic expression for $\mathfrak{A}(t)$ which gives

$$\begin{aligned} \mathfrak{A}(t) &= \int_\Omega |\mathrm{D}\Psi_t^{-T}\nabla u^t|^2 j(t)\mathrm{d}x + \alpha\int_{\partial\Omega}|u^t|^2\omega(t)\mathrm{d}\varsigma(x) + \beta\int_{\partial\Omega}|(\mathrm{I}_d-\mathbf{n}^t\otimes\mathbf{n}^t)\mathrm{D}\Psi_t^{-T}\nabla u^t|^2\omega(t)\mathrm{d}\varsigma(x) \\ &\quad + \int_\Omega |\mathrm{D}\Psi_t^{-T}\nabla u|^2 j(t)\mathrm{d}x + \alpha\int_{\partial\Omega}|u|^2\omega(t)\mathrm{d}\varsigma(x) + \beta\int_{\partial\Omega}|(\mathrm{I}_d-\mathbf{n}^t\otimes\mathbf{n}^t)\mathrm{D}\Psi_t^{-T}\nabla u|^2\omega(t)\mathrm{d}\varsigma(x) \\ &\qquad - 2\left\{\int_\Omega \mathrm{D}\Psi_t^{-T}\nabla u^t\cdot\mathrm{D}\Psi_t^{-T}\nabla u\, j(t)\mathrm{d}x + \alpha\int_{\partial\Omega} 2u^t\,u\,\omega(t)\mathrm{d}\varsigma(x)\right. \\ &\qquad\quad \left. +\beta\int_{\partial\Omega}|(\mathrm{I}_d-\mathbf{n}^t\otimes\mathbf{n}^t)\mathrm{D}\Psi_t^{-T}\nabla u^t\cdot\mathrm{D}\Psi_t^{-T}\nabla u\,\omega(t)\mathrm{d}\varsigma(x)\right\} \\ &= \Lambda_\Omega(\Omega_t) + \int_\Omega |\mathrm{D}\Psi_t^{-T}\nabla u|^2 j(t)\mathrm{d}x + \alpha\int_{\partial\Omega}|u|^2\omega(t)\mathrm{d}\varsigma(x) + \beta\int_{\partial\Omega}|(\mathrm{I}_d-\mathbf{n}^t\otimes\mathbf{n}^t)\mathrm{D}\Psi_t^{-T}\nabla u|^2\omega(t)\mathrm{d}\varsigma(x) \\ &\qquad - 2\left\{\int_\Omega \mathrm{D}\Psi_t^{-T}\nabla u^t\cdot\mathrm{D}\Psi_t^{-T}\nabla u\, j(t)\mathrm{d}x + \alpha\int_{\partial\Omega} 2u^t\,u\,\omega(t)\mathrm{d}\varsigma(x)\right. \\ &\qquad\qquad \left. +\beta\int_{\partial\Omega}|(\mathrm{I}_d-\mathbf{n}^t\otimes\mathbf{n}^t)\mathrm{D}\Psi_t^{-T}\nabla u^t\cdot\mathrm{D}\Psi_t^{-T}\nabla u\,\omega(t)\mathrm{d}\varsigma(x)\right\}. \end{aligned}$$

Then we use the uniform convergence of the coefficients, the weak convergence of u^t to u and the convergence of $\Lambda_\Omega(\Omega_t)$ to $\Lambda_\Omega(\Omega)$ to obtain that

$$\begin{aligned} \mathfrak{A}(t) \longrightarrow \Lambda_\Omega(\Omega) + \Lambda_\Omega(\Omega) - 2\left\{\int_\Omega |\nabla u|^2\mathrm{d}x + \alpha\int_{\partial\Omega} 2|u|^2\mathrm{d}\varsigma(x) + \beta\int_{\partial\Omega}|\nabla u|^2\mathrm{d}\varsigma(x)\right\}& \\ = \Lambda_\Omega(\Omega) + \Lambda_\Omega(\Omega) - 2\Lambda_\Omega(\Omega) = 0.& \end{aligned}$$

The existence of the semi-derivative $\Lambda'_\Omega(\Omega;\boldsymbol{V})$ follows from Theorem 1.1 since we have proved above that all the the four assumptions of the theorem are satisfied for G_Ω.

COMPUTATION OF THE SEMI-DERIVATIVE. By Theorem 1.1 we have

$$\Lambda'_{\Omega}(\Omega;\boldsymbol{V}) = \inf\{\partial_t G_\Omega(0,u); \Lambda_\Omega(\Omega) \text{ is attained at } u\}.$$

In what follows we show how to calculate $\partial_t G_\Omega(0,u)$ and establish a suitable expression for it whenever u is a normalized eigenfunction for $\Lambda_\Omega(\Omega)$. First, using the expressions (3.1)-(3.4), we get

$$\begin{aligned}\partial_t G_\Omega(0,u)|_{t=0} = &\int_{\partial\Omega} |\nabla u|^2 V_{\mathrm{n}}\ \mathrm{d}\varsigma(x) + 2\int_\Omega \nabla u\cdot\nabla(-\nabla u\cdot\boldsymbol{V})\ \mathrm{d}x\\ &+\alpha\left(\int_{\partial\Omega}\left(\mathrm{H}\,|u|^2+\partial_{\mathrm{n}}u^2\right)V_{\mathrm{n}}\ \mathrm{d}\varsigma(x)+2\int_{\partial\Omega}u\,(-\nabla u\cdot\boldsymbol{V})\ \mathrm{d}\varsigma(x)\right)\\ &+\beta\int_{\partial\Omega}\left(\mathrm{H}\,|\nabla_\Gamma u|^2+2\nabla_\Gamma u\cdot\nabla_\Gamma(\partial_{\mathrm{n}}u)-2\mathrm{D}^2 b\,\nabla_\Gamma u\cdot\nabla_\Gamma u\right)V_{\mathrm{n}}\ \mathrm{d}\varsigma(x)\\ &+2\beta\int_{\partial\Omega}\nabla_\Gamma u\cdot\left(\nabla_\Gamma(-\nabla u\cdot\boldsymbol{V})+\nabla u\cdot\nabla_\Gamma V_{\mathrm{n}}\mathbf{n}+\partial_{\mathrm{n}}u\nabla_\Gamma V_{\mathrm{n}}\right)\mathrm{d}\varsigma(x)\\ &-\Lambda_\Omega(\Omega)\left(\int_{\partial\Omega}|u|^2V_{\mathrm{n}}\ \mathrm{d}\varsigma(x)+2\int_\Omega u\,(-\nabla u\cdot\boldsymbol{V})\ \mathrm{d}x\right).\end{aligned}$$

Using $-\nabla u\cdot\boldsymbol{V}$ as a test function in (2.1), we observe that

$$\begin{aligned}\int_\Omega\nabla u\cdot\nabla(-\nabla u\cdot\boldsymbol{V})\ \mathrm{d}x+\alpha\int_{\partial\Omega}u(-\nabla u\cdot\boldsymbol{V})\ \mathrm{d}\varsigma(x)+\beta\int_{\partial\Omega}\nabla_\Gamma u\cdot\nabla_\Gamma(-\nabla u\cdot\boldsymbol{V})\ \mathrm{d}\varsigma(x)\\ =\Lambda_\Omega(\Omega)\int_\Omega u(-\nabla u\cdot\boldsymbol{V})\ \mathrm{d}x,\end{aligned}$$

Notice that the function $-\nabla u\cdot\boldsymbol{V}$ belongs to $\mathrm{H}^2(\Omega)$ and can be used as test function. Indeed the boundary $\partial\Omega$ has the $\mathcal{C}^{2,1}$ regularity and u belongs to $\mathrm{H}^3(\Omega)$ by usual elliptic a priori estimates (see [2]). Since $\mathbf{n}$ is orthogonal to $\nabla_\Gamma u$, we conclude that

$$\begin{aligned}\partial_t G_\Omega(0,u)|_{t=0} = &\int_{\partial\Omega}|\nabla u|^2V_{\mathrm{n}}\ \mathrm{d}\varsigma(x)+\alpha\int_{\partial\Omega}\left(\mathrm{H}\,|u|^2+2u\partial_{\mathrm{n}}u\right)V_{\mathrm{n}}\ \mathrm{d}\varsigma(x)\\ &+\beta\int_{\partial\Omega}\left(\mathrm{H}\,|\nabla_\Gamma u|^2+2\nabla_\Gamma u\cdot\nabla_\Gamma(\partial_{\mathrm{n}}u)-2\mathrm{D}^2b\,\nabla_\Gamma u\cdot\nabla_\Gamma u\right)V_{\mathrm{n}}\ \mathrm{d}\varsigma(x)\\ &+2\beta\int_{\partial\Omega}\nabla_\Gamma u\cdot(\partial_{\mathrm{n}}u\nabla_\Gamma V_{\mathrm{n}})\,\mathrm{d}\varsigma(x)-\Lambda_\Omega(\Omega)\int_{\partial\Omega}|u|^2V_{\mathrm{n}}\ \mathrm{d}\varsigma(x).\end{aligned}\tag{3.9}$$

By an integration by parts (which is just the tangential Stokes formula given in the appendix) in the term $2\beta\int_{\partial\Omega}\nabla_\Gamma u\cdot(\partial_{\mathrm{n}}u\nabla_\Gamma V_{\mathrm{n}})\,\mathrm{d}\varsigma(x)$ which appears in the last line of (3.9), we get

$$\begin{aligned}2\beta\int_{\partial\Omega}\nabla_\Gamma u\cdot(\partial_{\mathrm{n}}u\nabla_\Gamma V_{\mathrm{n}})\,\mathrm{d}\varsigma(x) &= -2\beta\int_{\partial\Omega}\mathrm{div}_\Gamma(\partial_{\mathrm{n}}u\nabla_\Gamma u)V_{\mathrm{n}}\ \mathrm{d}\varsigma(x)\\ &= -2\beta\int_{\partial\Omega}(\partial_{\mathrm{n}}u\,\Delta_\Gamma u+\nabla_\Gamma(\partial_{\mathrm{n}}u)\cdot\nabla_\Gamma u)V_{\mathrm{n}}\ \mathrm{d}\varsigma(x).\end{aligned}\tag{3.10}$$

Thus, inserting (3.10) in (3.9), we get

$$\begin{aligned}\partial_t G_\Omega(0,u)|_{t=0} = &\int_{\partial\Omega}|\nabla u|^2V_{\mathrm{n}}\ \mathrm{d}\varsigma(x)+\alpha\int_{\partial\Omega}\left(\mathrm{H}\,|u|^2+2u\partial_{\mathrm{n}}u\right)V_{\mathrm{n}}\ \mathrm{d}\varsigma(x)\\ &+\beta\int_{\partial\Omega}\left(\mathrm{H}\,|\nabla_\Gamma u|^2-2\mathrm{D}^2b\,\nabla_\Gamma u\cdot\nabla_\Gamma u\right)V_{\mathrm{n}}\ \mathrm{d}\varsigma(x)-2\beta\int_{\partial\Omega}\partial_{\mathrm{n}}u\,\Delta_\Gamma u\,V_{\mathrm{n}}\mathrm{d}\varsigma(x)-\Lambda_\Omega(\Omega)\int_{\partial\Omega}|u|^2V_{\mathrm{n}}\ \mathrm{d}\varsigma(x).\end{aligned}$$

Then using the boundary condition in (2.1), we obtain

$$\begin{aligned}\partial_t G_\Omega(0,u)|_{t=0} &= \int_{\partial\Omega}|\nabla u|^2V_{\mathrm{n}}\ \mathrm{d}\varsigma(x)+\alpha\int_{\partial\Omega}\left(\mathrm{H}\,|u|^2+2u\partial_{\mathrm{n}}u\right)V_{\mathrm{n}}\ \mathrm{d}\varsigma(x)\\ &\quad+\beta\int_{\partial\Omega}\left(\mathrm{H}\,|\nabla_\Gamma u|^2-2\mathrm{D}^2b\,\nabla_\Gamma u\cdot\nabla_\Gamma u\right)V_{\mathrm{n}}\ \mathrm{d}\varsigma(x)\\ &\quad+2\int_{\partial\Omega}\partial_{\mathrm{n}}u\,(-\alpha u-\partial_{\mathrm{n}}u)V_{\mathrm{n}}\mathrm{d}\varsigma(x)-\Lambda_\Omega(\Omega)\int_{\partial\Omega}|u|^2V_{\mathrm{n}}\ \mathrm{d}\varsigma(x)\\ &= \int_{\partial\Omega}\left(|\nabla_\Gamma u|^2-(\partial_{\mathrm{n}}u)^2+\alpha\mathrm{H}\,|u|^2+\beta(\mathrm{H}\mathrm{I}_d-2\mathrm{D}^2b)\,\nabla_\Gamma u\cdot\nabla_\Gamma u-\Lambda_\Omega(\Omega)|u|^2\right)V_{\mathrm{n}}\ \mathrm{d}\varsigma(x),\end{aligned}$$

which concludes the proof.

3.3.2 Proof of Theorem 2.3

The eigenvalue functional on the perturbed domain is

$$\Lambda_{\partial\Omega}(\Omega_t) = \inf_{v\in\mathcal{H}(\Omega_t)} \left\{ \frac{\displaystyle\int_{\Omega_t} |\nabla v|^2 \, \mathrm{d}x + \alpha \int_{\partial\Omega_t} v^2 \, \mathrm{d}\varsigma(x) + \beta \int_{\partial\Omega_t} |\nabla_\Gamma v|^2 \, \mathrm{d}\varsigma(x)}{\displaystyle\int_{\partial\Omega_t} v^2 \, \mathrm{d}\varsigma(x)} \right\}, \tag{3.11}$$

and we have the following reformulation on Ω

$$\Lambda_{\partial\Omega}(\Omega_t) = \inf_{u\in\mathcal{H}(\Omega)} \left\{ \frac{\displaystyle\int_{\Omega_t} |\nabla(u\circ\Psi_t^{-1})|^2 \, \mathrm{d}x + \alpha \int_{\partial\Omega_t} (u\circ\Psi_t^{-1})^2 \, \mathrm{d}\varsigma(x) + \beta \int_{\partial\Omega_t} |\nabla_\Gamma (u\circ\Psi_t^{-1})|^2 \, \mathrm{d}\varsigma(x)}{\displaystyle\int_{\partial\Omega_t} (u\circ\Psi_t^{-1})^2 \, \mathrm{d}\varsigma(x)} \right\}.$$

This corresponds to a minimization of the functional

$$G_{\partial\Omega}(t,u) = \frac{\displaystyle\int_{\Omega_t} |\nabla(u\circ\Psi_t^{-1})|^2 \, \mathrm{d}x + \alpha \int_{\partial\Omega_t} (u\circ\Psi_t^{-1})^2 \, \mathrm{d}\varsigma(x) + \beta \int_{\partial\Omega_t} |\nabla_\Gamma (u\circ\Psi_t^{-1})|^2 \, \mathrm{d}\varsigma(x)}{\displaystyle\int_{\partial\Omega_t} (u\circ\Psi_t^{-1})^2 \, \mathrm{d}\varsigma(x)}.$$

The arguments for verifying the assumptions of Theorem 1.1 for $\Lambda'_{\partial\Omega}(\Omega;\boldsymbol{V})$ are similar to those used in the previous subsection. We only indicate some pertinent differences.

Existence of the semi-derivative. For proving that the set of minimizers of (3.11) is non-empty for each t, the arguments are the same as in the previous case. However, for the continuity of the denominator, one requires now the compact injection of $\mathrm{H}^1(\Omega_t)$ into $\mathrm{L}^2(\partial\Omega_t)$ which also holds (see, e.g., [3, Theorem 1.1]). Then, as in the previous case, the set $X(t)$ of minimizers for $G_{\partial\Omega}(t,\cdot)$ is obtained by transporting the minimizers in (3.11) to Ω by composition with Ψ_t. Thus assumption $(H1)$ holds.

Concerning assumption $(H2)$, we first get the following expression for $G_{\partial\Omega}$

$$G_{\partial\Omega}(t,u) = \frac{1}{\displaystyle\int_{\partial\Omega} |u|^2\omega(t)\mathrm{d}\varsigma(x)} \left(\int_\Omega |(\mathrm{D}\Psi_t^{-1})^\top \nabla u|^2 j(t)\mathrm{d}x \right.$$
$$\left. + \alpha \int_{\partial\Omega} |u|^2\omega(t)\mathrm{d}\varsigma(x) + \beta \int_{\partial\Omega} |(\mathrm{I}_d - \mathbf{n}^t\otimes\mathbf{n}^t)(\mathrm{D}\Psi_t^{-1})^\top \nabla u|^2\omega(t)\mathrm{d}\varsigma(x) \right), \tag{3.12}$$

where we recall that $j(t) = \det(\mathrm{D}\Psi_t(x))$ is the Jacobian, $\omega(t) = \det(\mathrm{D}\Psi_t(x))\|(\mathrm{D}\Psi_t^{-1})^\top(x)\mathbf{n}(x)\|$ is the surface Jacobian and $\mathbf{n}^t = \mathbf{n}_t\circ\Psi_t$. Due to the smoothness of these functions in (3.12) we conclude that $G(\cdot,u)$ is derivable for all $u\in\mathcal{H}(\Omega)$ giving assumption $(H2)$.

The derivative of $G(\cdot,u)$ is obtained by deriving under the integral sign and the continuity of the ensuing coefficients leads to the assumption $(H4)$.

Finally, for assumption $(H3)$, firstly it has to be shown that $G_{\partial\Omega}(t,\cdot)$ converges to $G_{\partial\Omega}(0,\cdot)$ as $t\to 0$ in the sense of Γ-limit in the weak topology on $\mathcal{H}(\Omega)$. Then it is possible to show that there exists a sequence u^t, with $u^t\in \operatorname{argmin} G_{\partial\Omega}(t,\cdot)$, such that u^t converges strongly in $\mathcal{H}(\Omega)$ to a minimizer u of $G_\Omega(0,\cdot)$. This gives assumption $(H3)$.

The existence of the semi-derivative $\Lambda'_{\partial\Omega}(\Omega;\boldsymbol{V})$ then follows from Theorem 1.1.

Computation of the semi-derivative. We will now obtain a suitable expression for $\partial_t G_{\partial\Omega}(0,u)$ whenever u is a normalized eigenfunction for $\Lambda_{\partial\Omega}(\Omega)$ since, by the theorem,

$$\Lambda'_{\partial\Omega}(\Omega;\boldsymbol{V}) = \inf\{\partial_t G_{\partial\Omega}(0,u); \Lambda_{\partial\Omega}(\Omega) \text{ is attained at } u\}.$$

Using the expressions (3.1)-(3.4), we get

$$
\begin{aligned}
\partial_t G_{\partial\Omega}(0,u)|_{t=0} = & \int_{\partial\Omega} |\nabla u|^2 V_{\rm n}\, {\rm d}\varsigma(x) + 2\int_{\Omega} \nabla u\cdot\nabla(-\nabla u\cdot \boldsymbol{V})\,{\rm d}x \\
& +\alpha\left(\int_{\partial\Omega}\left({\rm H}\,|u|^2+\partial_{\rm n}u^2\right)V_{\rm n}\,{\rm d}\varsigma(x) + 2\int_{\partial\Omega} u\,(-\nabla u\cdot\boldsymbol{V})\,{\rm d}\varsigma(x)\right) \\
& +\beta\int_{\partial\Omega}\left({\rm H}\,|\nabla_\Gamma u|^2 + 2\nabla_\Gamma u\cdot\nabla_\Gamma(\partial_{\rm n}u) - 2{\rm D}^2 b\,\nabla_\Gamma u\cdot\nabla_\Gamma u\right)V_{\rm n}\,{\rm d}\varsigma(x) \\
& +2\beta\int_{\partial\Omega}\nabla_\Gamma u\cdot\left(\nabla_\Gamma(-\nabla u\cdot\boldsymbol{V}) + \nabla u\cdot\nabla_\Gamma V_{\rm n}\mathbf{n} + \partial_{\rm n}u\nabla_\Gamma V_{\rm n}\right){\rm d}\varsigma(x) \\
& \qquad - \Lambda_{\partial\Omega}(\Omega)\left(\int_{\partial\Omega}\left({\rm H}\,|u|^2+\partial_{\rm n}u^2\right)V_{\rm n}\,{\rm d}\varsigma(x) + 2\int_{\partial\Omega} u\,(-\nabla u\cdot\boldsymbol{V})\,{\rm d}\varsigma(x)\right).
\end{aligned}
$$

Using $-\nabla u\cdot\boldsymbol{V}$ as a test function in (2.2) we observe that

$$
\begin{aligned}
\int_{\Omega}\nabla u\cdot(-\nabla u\cdot\boldsymbol{V})\,{\rm d}x + \alpha\int_{\partial\Omega} u(-\nabla u\cdot\boldsymbol{V})\,{\rm d}\varsigma(x) + \beta\int_{\partial\Omega}\nabla_\Gamma u\cdot\nabla_\Gamma(-\nabla u\cdot\boldsymbol{V})\,{\rm d}\varsigma(x) \\
= \Lambda_{\partial\Omega}(\Omega)\int_{\partial\Omega} u(-\nabla u\cdot\boldsymbol{V})\,{\rm d}x,
\end{aligned}
$$

and then arguing as in the previous subsection while using the boundary condition in (2.2), we get

$$
\begin{aligned}
\partial_t G_{\partial\Omega}(0,u)|_{t=0} \;=\; & \int_{\partial\Omega}|\nabla u|^2 V_{\rm n}\,{\rm d}\varsigma(x) \\
& +\alpha\int_{\partial\Omega}\left({\rm H}\,|u|^2 + 2u\partial_{\rm n}u\right)V_{\rm n}\,{\rm d}\varsigma(x) + \beta\int_{\partial\Omega}\left({\rm H}\,|\nabla_\Gamma u|^2 - 2{\rm D}^2 b\,\nabla_\Gamma u\cdot\nabla_\Gamma u\right)V_{\rm n}\,{\rm d}\varsigma(x) \\
& +2\int_{\partial\Omega}\partial_{\rm n}u\,(\Lambda_{\partial\Omega}(\Omega)u - \alpha u - \partial_{\rm n}u)V_{\rm n}{\rm d}\varsigma(x) - \Lambda_{\partial\Omega}(\Omega)\int_{\partial\Omega}\left({\rm H}\,|u|^2 + 2u\partial_{\rm n}u\right)V_{\rm n}\,{\rm d}\varsigma(x). \\
=\; & \int_{\partial\Omega}\left(|\nabla_\Gamma u|^2 - (\partial_{\rm n}u)^2 + \alpha{\rm H}\,|u|^2 + \beta({\rm H}{\rm I}_d - 2{\rm D}^2 b)\,\nabla_\Gamma u\cdot\nabla_\Gamma u - \Lambda_{\partial\Omega}(\Omega){\rm H}|u|^2\right)V_{\rm n}\,{\rm d}\varsigma(x),
\end{aligned}
$$

which concludes the proof.

3.4 Shape derivatives for the eigenvalue problems for composites

3.4.1 Proof of Theorem 2.5

The perturbed problem in this case reads

$$
\mathfrak{M}_\Omega(\Omega_t) = \inf_{v\in\mathcal{H}(\Omega_t)}\left\{\frac{\displaystyle\int_{\Omega_t}\sigma_t(x)\,|\nabla v|^2\,{\rm d}x + \alpha\int_{\partial\Omega_t} v^2\,{\rm d}\varsigma(x) + \beta\int_{\partial\Omega_t}|\nabla_\Gamma v|^2\,{\rm d}\varsigma(x)}{\displaystyle\int_{\Omega_t}\rho_t\,|v|^2\,{\rm d}x}\right\},
$$

where $\sigma_t := \sigma_1\chi_{\Omega_{1,t}} + \sigma_2\chi_{\Omega_{2,t}}$ and $\rho_t := \rho_1\chi_{\Omega_{1,t}} + \rho_2\chi_{\Omega_{2,t}}$ with $\Omega_{1,t} := \Psi_t(\Omega_1)$ and $\Omega_{2,t} := \Psi_t(\Omega_2)$. The above can be formulated as

$$
\mathfrak{M}_\Omega(\Omega_t) = \inf_{u\in\mathcal{H}(\Omega)} G_\Omega(t,u),
$$

with the corresponded functional defined by

$$
\begin{aligned}
G_\Omega(t,u) = \frac{1}{\displaystyle\int_{\Omega_t}\rho_t\left|(u\circ\Psi_t^{-1})\right|^2{\rm d}x}\Bigg(&\int_{\Omega_t}\sigma_t(x)\left|\nabla(u\circ\Psi_t^{-1})\right|^2{\rm d}x \\
& + \alpha\int_{\partial\Omega_t}\left|(u\circ\Psi_t^{-1})\right|^2\,{\rm d}\varsigma(x) + \beta\int_{\partial\Omega_t}|\nabla_\Gamma(u\circ\Psi_t^{-1})|^2\,{\rm d}\varsigma(x)\Bigg). \quad (3.13)
\end{aligned}
$$

The existence of the semi-derivative $\mathfrak{M}'_\Omega(\Omega;\boldsymbol{V})$ will follow from Theorem 1.1 since it can be shown, similarly as in Subsection 3.3.1, that, for G_Ω given by (3.13), the hypotheses of the theorem are satisfied. Thus, as before, we only need to get a suitable expression for $\partial_t G_\Omega(0,u)$ whenever u is a normalized eigenfunction for $\mathfrak{M}_\Omega(\Omega)$ since, by the theorem,

$$\mathfrak{M}'_\Omega(\Omega;\boldsymbol{V}) = \inf\{\partial_t G_\Omega(0,u); \mathfrak{M}_\Omega(\Omega) \text{ is attained at } u\}.$$

Using the expressions (3.5) and (3.6), and the previous formulae (3.3) and (3.4), we get

$$\begin{aligned}
\partial_t G_\Omega(0,u)|_{t=0} = &\int_{\partial\Omega_1} [\sigma|\nabla u|^2] V_{\rm n}\, {\rm d}\varsigma(x) + 2\int_{\Omega_1} \sigma_1 \nabla u\cdot\nabla(-\nabla u\cdot\boldsymbol{V})\,{\rm d}x\\
&+ \int_{\partial\Omega} \sigma_2|\nabla u|^2 V_{\rm n}\,{\rm d}\varsigma(x) + 2\int_{\Omega_2}\sigma_2\nabla u\cdot\nabla(-\nabla u\cdot\boldsymbol{V})\,{\rm d}x\\
&+ \alpha\left(\int_{\partial\Omega}\left({\rm H}\,|u|^2 + \partial_{\rm n}u^2\right)V_{\rm n}\,{\rm d}\varsigma(x) + 2\int_{\partial\Omega} u\,(-\nabla u\cdot\boldsymbol{V})\,{\rm d}\varsigma(x)\right)\\
&+\beta\Bigg(\int_{\partial\Omega}\left({\rm H}\,|\nabla_\Gamma u|^2 + 2\nabla_\Gamma u\cdot\nabla_\Gamma(\partial_{\rm n}u) - 2{\rm D}^2 b\,\nabla_\Gamma u\cdot\nabla_\Gamma u\right)V_{\rm n}\,{\rm d}\varsigma(x)\\
&+2\int_{\partial\Omega}\nabla_\Gamma u\cdot(\nabla_\Gamma(-\nabla u\cdot\boldsymbol{V}) + \nabla u\cdot\nabla_\Gamma V_{\rm n}\mathbf{n} + \partial_{\rm n}u\nabla_\Gamma V_{\rm n})\,{\rm d}\varsigma(x)\Bigg)\\
&-\mathfrak{M}_\Omega\Bigg(\int_{\partial\Omega_1}[\rho]|u|^2V_{\rm n}\,{\rm d}\varsigma(x) + 2\int_{\Omega_1}\rho_1 u\,(-\nabla u\cdot\boldsymbol{V})\,{\rm d}x\\
&\qquad+\int_{\partial\Omega}\rho_2|u|^2V_{\rm n}\,{\rm d}\varsigma(x) + 2\int_{\Omega_2}\rho_2 u\,(-\nabla u\cdot\boldsymbol{V})\,{\rm d}x\Bigg).
\end{aligned}$$

Notice that the eigenmode u does not belong to ${\rm H}^3(\Omega)$ due to the jumps of the interface. Therefore, the function $-\nabla u\cdot\boldsymbol{V}$ does not belong anymore to ${\rm H}^2(\Omega)$ (and not even in ${\rm H}^1(\Omega)$) and hence cannot be used as test function directly. However, its restriction to each Ω_i for $i=1,2$ belongs to ${\rm H}^3(\Omega_i)$ thanks to the regularity assumptions on both the outer boundary and the interface. Multiplying (2.5) by $-\nabla u\cdot\boldsymbol{V}$ in each Ω_i and integrating by parts in Ω_i for $i=1,2$, while noticing that the flux has to be continuous which implies

$$[\sigma\partial_{\rm n}u\,(-\nabla u\cdot\boldsymbol{V})] = \sigma\partial_{\rm n}u\,[(\nabla u\cdot\boldsymbol{V})] \;= \sigma\partial_{\rm n}u\,[\partial_{\rm n}u]\,V_{\rm n},$$

we observe that

$$\begin{aligned}
0 = &\int_{\Omega_1}\sigma_1\nabla u\cdot\nabla(-\nabla u\cdot\boldsymbol{V})\,{\rm d}x + \int_{\Omega_2}\sigma_2\nabla u\cdot\nabla(-\nabla u\cdot\boldsymbol{V})\,{\rm d}x + \int_{\partial\Omega_1}\left[\sigma(\partial_{\rm n}u)^2\right]V_{\rm n}\,{\rm d}\varsigma(x)\\
&+\alpha\int_{\partial\Omega}u\,(-\nabla u\cdot\boldsymbol{V}) + \beta\int_{\partial\Omega}\nabla_\Gamma u\cdot\nabla_\Gamma(-\nabla u\cdot\boldsymbol{V}) - \mathfrak{M}_\Omega\left(\int_{\Omega_1}\rho_1 u\,(-\nabla u\cdot\boldsymbol{V})\,{\rm d}x + \int_{\Omega_2}\rho_2 u\,(-\nabla u\cdot\boldsymbol{V})\,{\rm d}x\right).
\end{aligned}$$

Using the above we get

$$\begin{aligned}
\partial_t G_\Omega(0,u)|_{t=0} = &\int_{\partial\Omega_1}[\sigma|\nabla u|^2]V_{\rm n}\,{\rm d}\varsigma(x) + \int_{\partial\Omega}\sigma_2|\nabla u|^2V_{\rm n}\,{\rm d}\varsigma(x) - 2\int_{\partial\Omega_1}\left[\sigma(\partial_{\rm n}u)^2\right]V_{\rm n}\,{\rm d}\varsigma(x)\\
&+\alpha\int_{\partial\Omega}\left({\rm H}\,|u|^2 + \partial_{\rm n}u^2\right)V_{\rm n}\,{\rm d}\varsigma(x)\\
&+\beta\Bigg(\int_{\partial\Omega}\left({\rm H}\,|\nabla_\Gamma u|^2 + 2\nabla_\Gamma u\cdot\nabla_\Gamma(\partial_{\rm n}u) - 2{\rm D}^2 b\,\nabla_\Gamma u\cdot\nabla_\Gamma u\right)V_{\rm n}\,{\rm d}\varsigma(x)\\
&+2\int_{\partial\Omega}\nabla_\Gamma u\cdot(\nabla u\cdot\nabla_\Gamma V_{\rm n}\mathbf{n} + \partial_{\rm n}u\nabla_\Gamma V_{\rm n})\,{\rm d}\varsigma(x)\Bigg)\\
&-\mathfrak{M}_\Omega\left(\int_{\partial\Omega_1}[\rho]|u|^2V_{\rm n}\,{\rm d}\varsigma(x) + \int_{\partial\Omega}\rho_2|u|^2V_{\rm n}\,{\rm d}\varsigma(x)\right).
\end{aligned}$$

Using the facts that $|\nabla u|^2 = |\nabla_\Gamma u|^2 + |\partial_n u|^2$ and that both $\nabla_\Gamma u$ and u have a continuous trace on $\partial\Omega_1$, the above may be written as

$$\begin{aligned}\partial_t G_\Omega(0,u)|_{t=0} = &\int_{\partial\Omega_1} \left([\sigma]\,|\nabla_\Gamma u|^2 - \left[\sigma(\partial_n u)^2\right]\right) V_n \,\mathrm{d}\varsigma(x) + \int_{\partial\Omega} \left(\sigma_2|\nabla u|^2 + \alpha\left(\mathrm{H}\,|u|^2 + \partial_n u^2\right)\right) V_n \,\mathrm{d}\varsigma(x)\\ &+\beta\Bigg(\int_{\partial\Omega} \left(\mathrm{H}\,|\nabla_\Gamma u|^2 + 2\nabla_\Gamma u\cdot\nabla_\Gamma(\partial_n u) - 2\mathrm{D}^2 b\,\nabla_\Gamma u\cdot\nabla_\Gamma u\right) V_n \,\mathrm{d}\varsigma(x)\\ &\qquad + 2\int_{\partial\Omega} \nabla_\Gamma u\cdot\left(\nabla u\cdot\nabla_\Gamma V_n \mathbf{n} + \partial_n u \nabla_\Gamma V_n\right)\mathrm{d}\varsigma(x)\Bigg)\\ &\qquad\qquad - \mathfrak{M}_\Omega\left(\int_{\partial\Omega_1} [\rho]\,|u|^2 V_n \,\mathrm{d}\varsigma(x) + \int_{\partial\Omega} \rho_2|u|^2 V_n \,\mathrm{d}\varsigma(x)\right).\end{aligned}$$

By an integration by parts in the term $2\beta\int_{\partial\Omega}\nabla_\Gamma u\cdot(\partial_n u\nabla_\Gamma V_n)\,\mathrm{d}\varsigma(x)$ which appears in the above equality, we get

$$\begin{aligned}2\beta\int_{\partial\Omega}\nabla_\Gamma u\cdot(\partial_n u\nabla_\Gamma V_n)\,\mathrm{d}\varsigma(x) &= -2\beta\int_{\partial\Omega}\mathrm{div}_\Gamma(\partial_n u\nabla_\Gamma u)V_n \,\mathrm{d}\varsigma(x)\\ &= -2\beta\int_{\partial\Omega}(\partial_n u\,\Delta_\Gamma u + \nabla_\Gamma(\partial_n u)\cdot\nabla_\Gamma u)V_n \,\mathrm{d}\varsigma(x).\end{aligned}$$

Hence, using the fact that $\nabla_\Gamma u$ is orthogonal to $\mathbf{n}$, we obtain

$$\begin{aligned}\partial_t G_\Omega(0,u)|_{t=0} = &\int_{\partial\Omega_1} [\sigma]\,|\nabla_\Gamma u|^2 V_n \,\mathrm{d}\varsigma(x) - \int_{\partial\Omega_1}\left[\sigma(\partial_n u)^2\right] V_n \,\mathrm{d}\varsigma(x)\\ &+\int_{\partial\Omega}\sigma_2|\nabla u|^2 V_n \,\mathrm{d}\varsigma(x) + \alpha\int_{\partial\Omega}\left(\mathrm{H}\,|u|^2 + \partial_n u^2\right) V_n \,\mathrm{d}\varsigma(x)\\ &+\beta\Bigg(\int_{\partial\Omega} \left(\mathrm{H}\,|\nabla_\Gamma u|^2 + 2\nabla_\Gamma u\cdot\nabla_\Gamma(\partial_n u) - 2\mathrm{D}^2 b\,\nabla_\Gamma u\cdot\nabla_\Gamma u\right) V_n \,\mathrm{d}\varsigma(x)\\ &\qquad - 2\int_{\partial\Omega}(\partial_n u\,\Delta_\Gamma u + \nabla_\Gamma(\partial_n u)\cdot\nabla_\Gamma u)V_n \,\mathrm{d}\varsigma(x)\Bigg)\\ &\qquad\qquad -\mathfrak{M}_\Omega\left(\int_{\partial\Omega_1} [\rho]\,|u|^2 V_n \,\mathrm{d}\varsigma(x) + \int_{\partial\Omega} \rho_2|u|^2 V_n \,\mathrm{d}\varsigma(x)\right).\end{aligned}$$

We conclude using the boundary condition $-\beta\Delta_\Gamma u + \alpha u + \partial_n u = 0$ on $\partial\Omega$ and the fact that $\partial_n u^2 = 2u\partial_n u$.

3.4.2 Proof of Theorem 2.6

We will now calculate the sensitivity of $\mathfrak{M}_{\partial\Omega}(\Omega)$ with respect to variations of the domain Ω and of the interface Γ. The perturbed problem then reads

$$\mathfrak{M}_{\partial\Omega}(\Omega_t) = \inf_{v\in\mathcal{H}(\Omega_t)}\left\{\frac{\int_{\Omega_t}\sigma_t(x)\,|\nabla v|^2\,\mathrm{d}x + \alpha\int_{\partial\Omega_t} v^2\,\mathrm{d}\varsigma(x) + \beta\int_{\partial\Omega_t}|\nabla_\Gamma v|^2\,\mathrm{d}\varsigma(x)}{\int_{\partial\Omega_t}|v|^2\,\mathrm{d}\varsigma(x)}\right\},$$

where $\sigma_t := \sigma_1\chi_{\Omega_{1,t}} + \sigma_2\chi_{\Omega_{2,t}}$ with $\Omega_{1,t} := \Psi_t(\Omega_1)$ and $\Omega_{2,t} := \Psi_t(\Omega_2)$. The above can be formulated as

$$\mathfrak{M}_{\partial\Omega}(\Omega_t) = \inf_{u\in\mathcal{H}(\Omega)} G_{\partial\Omega}(t,u),$$

with the corresponded functional defined by

$$G_{\partial\Omega}(t,u) = \frac{\int_{\Omega_t}\sigma_t(x)\left|\nabla(u\circ\Psi_t^{-1})\right|^2\mathrm{d}x + \alpha\int_{\partial\Omega_t}\left|(u\circ\Psi_t^{-1})\right|^2\,\mathrm{d}\varsigma(x) + \beta\int_{\partial\Omega_t}|\nabla_\Gamma(u\circ\Psi_t^{-1})|^2\,\mathrm{d}\varsigma(x)}{\int_{\partial\Omega_t}\left|(u\circ\Psi_t^{-1})\right|^2\mathrm{d}\varsigma(x)}. \tag{3.14}$$

The existence of the semi-derivative $\mathfrak{M}'_{\partial\Omega}(\Omega;\boldsymbol{V})$ will follow from Theorem 1.1 since it can be shown, as outlined in Subsection 3.3.2, that, for $G_{\partial\Omega}$ given by (3.14), the assumptions of the theorem are satisfied. Thus, as before, we only need to get a suitable expression for $\partial_t G_{\partial\Omega}(0,u)$ whenever u is a normalized eigenfunction for $\mathfrak{M}_{\partial\Omega}(\Omega)$ since, by the theorem,

$$\mathfrak{M}'_{\partial\Omega}(\Omega;\boldsymbol{V}) = \inf\{\partial_t G(0,u); \mathfrak{M}_{\partial\Omega}(\Omega) \text{ is attained at } u\}.$$

Using the expressions in the previous formulae (3.3), (3.4) and (3.5), we get

$$\begin{aligned}
\partial_t G_{\partial\Omega}(0,u)|_{t=0} = & \int_{\partial\Omega_1} [\sigma|\nabla u|^2] V_{\mathrm{n}} \,\mathrm{d}\varsigma(x) + 2\int_{\Omega_1} \sigma_1 \nabla u\cdot\nabla(-\nabla u\cdot\boldsymbol{V})\,\mathrm{d}x \\
& + \int_{\partial\Omega} \sigma_2|\nabla u|^2 V_{\mathrm{n}}\,\mathrm{d}\varsigma(x) + 2\int_{\Omega_2}\sigma_2\nabla u\cdot\nabla(-\nabla u\cdot\boldsymbol{V})\,\mathrm{d}x \\
& + \alpha\left(\int_{\partial\Omega}\left(\mathrm{H}\,|u|^2+\partial_{\mathrm{n}}u^2\right)V_{\mathrm{n}}\,\mathrm{d}\varsigma(x) + 2\int_{\partial\Omega} u\,(-\nabla u\cdot\boldsymbol{V})\,\mathrm{d}\varsigma(x)\right) \\
& + \beta\Bigg(\int_{\partial\Omega}\left(\mathrm{H}\,|\nabla_\Gamma u|^2 + 2\nabla_\Gamma u\cdot\nabla_\Gamma(\partial_{\mathrm{n}}u) - 2\mathrm{D}^2 b\,\nabla_\Gamma u\cdot\nabla_\Gamma u\right)V_{\mathrm{n}}\,\mathrm{d}\varsigma(x) \\
& + 2\int_{\partial\Omega}\nabla_\Gamma u\cdot\left(\nabla_\Gamma(-\nabla u\cdot\boldsymbol{V}) + \nabla u\cdot\nabla_\Gamma V_{\mathrm{n}}\mathbf{n} + \partial_{\mathrm{n}}u\nabla_\Gamma V_{\mathrm{n}}\right)\mathrm{d}\varsigma(x)\Bigg) \\
& - \mathfrak{M}_{\partial\Omega}(\Omega)\left(\int_{\partial\Omega}\left(\mathrm{H}\,|u|^2+\partial_{\mathrm{n}}u^2\right)V_{\mathrm{n}}\,\mathrm{d}\varsigma(x) + 2\int_{\partial\Omega}u\,(-\nabla u\cdot\boldsymbol{V})\,\mathrm{d}\varsigma(x)\right).
\end{aligned}$$

Multiplying (2.7) by $-\nabla u\cdot\boldsymbol{V}$ in each Ω_i and integrating by parts in Ω_i for $i=1,2$, then noticing that the jump conditions impose that

$$-\int_{\partial\Omega_1}[\sigma\partial_{\mathrm{n}}u\,(-\nabla u\cdot\boldsymbol{V})]\,\mathrm{d}\varsigma(x) = \int_{\partial\Omega_1}\sigma\partial_{\mathrm{n}}u\,[(\nabla u\cdot\boldsymbol{V})]\,\mathrm{d}\varsigma(x) = \int_{\partial\Omega_1}\sigma\partial_{\mathrm{n}}u\,[\partial_{\mathrm{n}}u]\,V_{\mathrm{n}}\,\mathrm{d}\varsigma(x),$$

we observe that

$$\begin{aligned}
0 = & \int_{\Omega_1}\sigma_1\nabla u\cdot\nabla(-\nabla u\cdot\boldsymbol{V})\,\mathrm{d}x + \int_{\Omega_2}\sigma_2\nabla u\cdot\nabla(-\nabla u\cdot\boldsymbol{V})\,\mathrm{d}x + \int_{\partial\Omega_1}\left[\sigma(\partial_{\mathrm{n}}u)^2\right]V_{\mathrm{n}}\,\mathrm{d}\varsigma(x) \\
& + \alpha\int_{\partial\Omega}u\,(-\nabla u\cdot\boldsymbol{V}) + \beta\int_{\partial\Omega}\nabla_\Gamma u\cdot\nabla_\Gamma(-\nabla u\cdot\boldsymbol{V}) - \mathfrak{M}_{\partial\Omega}(\Omega)\int_{\partial\Omega}u\,(-\nabla u\cdot\boldsymbol{V})\,\mathrm{d}\varsigma(x).
\end{aligned}$$

Using the above, we get

$$\begin{aligned}
\partial_t G_{\partial\Omega}(0,u)|_{t=0} = & \int_{\partial\Omega_1}\left([\sigma]\,|\nabla_\Gamma u|^2 - \left[\sigma(\partial_{\mathrm{n}}u)^2\right]\right)V_{\mathrm{n}}\,\mathrm{d}\varsigma(x) + \int_{\partial\Omega}\left(\sigma_2|\nabla u|^2 + \alpha\left(\mathrm{H}\,|u|^2+\partial_{\mathrm{n}}u^2\right)\right)V_{\mathrm{n}}\,\mathrm{d}\varsigma(x) \\
& + \beta\Bigg(\int_{\partial\Omega}\left(\mathrm{H}\,|\nabla_\Gamma u|^2 + 2\nabla_\Gamma u\cdot\nabla_\Gamma(\partial_{\mathrm{n}}u) - 2\mathrm{D}^2 b\,\nabla_\Gamma u\cdot\nabla_\Gamma u\right)V_{\mathrm{n}}\,\mathrm{d}\varsigma(x) \\
& + 2\int_{\partial\Omega}\nabla_\Gamma u\cdot\left(\nabla u\cdot\nabla_\Gamma V_{\mathrm{n}}\mathbf{n} + \partial_{\mathrm{n}}u\nabla_\Gamma V_{\mathrm{n}}\right)\mathrm{d}\varsigma(x)\Bigg) - \mathfrak{M}_{\partial\Omega}(\Omega)\int_{\partial\Omega}\left(\mathrm{H}\,|u|^2+\partial_{\mathrm{n}}u^2\right)V_{\mathrm{n}}\,\mathrm{d}\varsigma(x).
\end{aligned}$$

Then we conclude similarly as in the proof of Theorem 2.5.

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A Appendix

A.1 Technical results

The purpose of this subsection is to recall some auxiliary results or notions used in the calculations of the shape sensitivity. We recall here some versions in relation to the perturbation of identity from [21]. For the corresponding versions involving velocity fields we refer to [10, Chapter 9].

Classical shape derivative formulæ.

Lemma A.1. *Let $\delta > 0$. Let a vector field $\boldsymbol{V} \in \mathbf{W}^{1,\infty}(\mathbb{R}^d)$ and let*

$$\Psi \ : \ t \in [0,\delta) \mapsto \Psi_t = \mathbf{I} + t\boldsymbol{V} \in \mathbf{W}^{1,\infty}(\mathbb{R}^d).$$

Let a bounded Lipschitz open set Ω in $\mathbb{R}^d$ and let $\Omega_t := \Psi_t(\Omega)$ for all $t \in [0,\delta)$. We consider a function f such that $t \in [0,\delta) \mapsto f(t) \in \mathrm{L}^1(\mathbb{R}^d)$ is differentiable at 0 with $f(0) \in \mathrm{W}^{1,1}(\mathbb{R}^d)$. Then the function

$$t \in [0,\delta) \mapsto \mathcal{F}(t) = \int_{\Omega_t} f(t,x) \ \mathrm{d}x$$

is differentiable at 0 (we say that $\mathcal{F}$ admits a semi-derivative) and we have

$$\mathcal{F}'(0) = \int_{\partial\Omega} f(0,x) V_{\mathrm{n}} \ \mathrm{d}\varsigma(x) + \int_{\Omega} f'(0,x) \ \mathrm{d}x,$$

where $V_{\mathrm{n}} = \boldsymbol{V} \cdot \mathbf{n}$.

Lemma A.2. *Let $\delta > 0$. Let a vector field $\boldsymbol{V} \in \boldsymbol{C}^{1,\infty}(\mathbb{R}^d)$ and let*

$$\Psi \ : \ t \in [0,\delta) \mapsto \Psi_t = \mathbf{I} + t\boldsymbol{V} \in \boldsymbol{C}^{1,\infty}(\mathbb{R}^d).$$

Let a bounded open set Ω in $\mathbb{R}^d$ of classe $\mathcal{C}^2$ and let $\Omega_t := \Psi_t(\Omega)$ for all $t \in [0,\delta)$. We consider a function g such that $t \in [0,\delta) \mapsto g(t) \circ \Psi_t \in \mathrm{W}^{1,1}(\Omega)$ is differentiable at 0 with $g(0) \in \mathrm{W}^{2,1}(\Omega)$. Then the function

$$t \in [0,\delta) \mapsto \mathcal{G}(t) = \int_{\partial\Omega_t} g(t,x) \ \mathrm{d}x$$

is differentiable at 0 (we say that $\mathcal{G}$ admits a semi-derivative), the function $t \in [0,\delta) \mapsto g(t)|_{\omega} \in \mathrm{W}^{1,1}(\omega)$ is differentiable at 0 for all open set $\omega \subset \overline{\omega} \subset \Omega$ and the derivative $g'(0)$ belongs to $W^{1,1}(\Omega)$ and we have

$$\mathcal{G}'(0) = \int_{\partial\Omega} \left(g'(0,x) + (\mathrm{H}\, g(0,x) + \partial_{\mathrm{n}} g)\, V_{\mathrm{n}}\right) \mathrm{d}\varsigma(x),$$

where $V_{\mathrm{n}} = \boldsymbol{V} \cdot \mathbf{n}$ and where H is the mean curvature function on $\partial\Omega$.

Some tangential calculus. The following lemma are formulated in terms of an intrinsic tangential calculus developed by Delfour and Zolésio [12, 13] (see also [10, Chapter 9, Section 5]) for modelling thin shells.

An important role is played by the *signed distance* to the boundary $\partial\Omega$ defined by

$$b(x) := \begin{cases} \mathrm{d}(x,\partial\Omega), & \text{if } x \in \Omega, \\ -\mathrm{d}(x,\partial\Omega), & \text{if } x \in \mathbb{R}^d \setminus \overline{\Omega}. \end{cases}$$

Indeed, if $\partial\Omega$ is smooth then we have $Db = \mathbf{n}$ is the outward unit normal, $D^2 b$ is second fundamental form on the boundary. If $\Pi_d = \mathrm{I}_d - \mathbf{n} \otimes \mathbf{n}$ is the projection on the tangent plane, then $\nabla_\Gamma u = \Pi_d \nabla u$ is the tangential gradient of a scalar function in $\mathrm{H}^1(\Omega)$. For any smooth vector field V on $\Gamma = \partial\Omega$, the tangential divergence is given by $\mathrm{div}_\Gamma V := \mathrm{tr}(DV\,\Pi_d)$ and the *mean curvature* at any point on $\partial\Omega$ is given by

$$\mathrm{H} := \mathrm{div}_\Gamma\, \mathbf{n}.$$

The Laplace-Beltrami operator is defined by

$$\Delta_\Gamma u = \mathrm{div}_\Gamma(\nabla_\Gamma u)$$

for every sufficiently smooth function u on Γ. Keep in mind also that $\mathrm{D}^2 b\, \mathbf{n} = 0$ and the tangential Stoke formula

$$\int_\Gamma \mathrm{div}_\Gamma V \; dS = \int_\Gamma H \; V \cdot \mathbf{n} \; dS$$

while working with the intrinsic tangential calculus.

Lemma A.3. *Given a bounded open set Ω in $\mathbb{R}^d$ of class $\mathcal{C}^2$ and $u \in \mathrm{H}^2(\mathbb{R}^d)$, we have*

$$\partial_\mathrm{n} |\nabla_\Gamma u|^2 = 2\nabla_\Gamma u \cdot (\nabla_\Gamma(\partial_\mathrm{n} u) - \mathrm{D}^2 b \, \nabla_\Gamma u),$$

where b is the signed distance to the boundary $\partial\Omega$.

Proof. From that $\mathrm{D}^2 bn = 0$ it follows that $\partial_\mathrm{n} \Pi_d = 0$ and that $\Pi_d \mathrm{D}^2 b = \mathrm{D}^2 b \Pi_d = \mathrm{D}^2 b$. After recalling that $\nabla_\Gamma u = \Pi_d \nabla u$, we have

$$\partial_\mathrm{n}(\nabla_\Gamma u) = \partial_\mathrm{n}(\Pi_d \nabla u) = \Pi_d \partial_\mathrm{n}(\nabla u) \qquad \text{and} \qquad \partial_\mathrm{n}(\nabla u) = \mathrm{D}^2 u \, \mathbf{n}.$$

Thus $\nabla(\partial_\mathrm{n} u) = \nabla(\nabla u \cdot \mathbf{n}) = \mathrm{D}^2 u \, \mathbf{n} + \nabla \mathbf{n} \, \nabla u = \partial_\mathrm{n}(\nabla u) + \mathrm{D}^2 b \, \nabla u$. Hence we obtain

$$\nabla_\Gamma(\partial_\mathrm{n} u) = \Pi_d \nabla(\partial_\mathrm{n} u) = \Pi_d \partial_\mathrm{n}(\nabla u) + \Pi_d \mathrm{D}^2 b \nabla \boldsymbol{u} = \partial_\mathrm{n}(\nabla_\Gamma u) + \mathrm{D}^2 b \nabla u.$$

Therefore, we obtain the result since $\partial_\mathrm{n}\left(|\nabla_\Gamma u|^2\right) = 2\partial_\mathrm{n}\left(\nabla_\Gamma u\right) \cdot \nabla_\Gamma u$. □

Lemma A.4. *Given a bounded open set Ω in $\mathbb{R}^d$ of class $\mathcal{C}^2$, $\boldsymbol{V}$ in $\boldsymbol{\mathcal{C}}^1(\mathbb{R}^d; \mathbb{R}^d)$ and $u \in \mathrm{H}^2(\mathbb{R}^d)$, we have*

$$\partial_t \big(\nabla_{\Gamma_t}(u \circ \Psi_t^{-1})\big)\,|_{t=0} = (\nabla_\Gamma(-\nabla u \cdot \boldsymbol{V}) + \nabla u \cdot \nabla_\Gamma V_\mathrm{n} \, \mathbf{n} + \partial_\mathrm{n} u \, \nabla_\Gamma V_\mathrm{n})$$

where V_n is the normal component $\boldsymbol{V} \cdot \mathbf{n}$ of the vector field $\boldsymbol{V}$.

Proof. We first recall that, since $\partial_t \mathbf{n}_{t\,|t=0} = -\nabla_\Gamma V_\mathrm{n}$, we have $\partial_t \Pi_{d\,|t=0} = \mathbf{n} \otimes \nabla_\Gamma V_\mathrm{n} + \nabla_\Gamma V_\mathrm{n} \otimes \mathbf{n}$. Hence we obtain the result noticing that $\nabla_\Gamma u = \Pi_d \nabla u$. □

Some reminders on Γ-convergence. For the convenience of the reader, we recall the definition and the main property of the Γ-convergence. For further details we refer to the book of G. Dal Maso [7].

Definition A.5. *(Sequential Γ-convergence) A family of functionals $\{F_t\}_{t>0}$ defined on a topological space X is said to be sequentially Γ-convergent to a functional F as $t \to 0^+$ if the two following statements hold:*

(i) $\Gamma-\liminf$ inequality: *for every sequence $\{x_t\}$ converging to $x \in X$, we have*

$$\liminf_{t \to 0^+} F_t(x_t) \geq F(x);$$

(ii) $\Gamma-\limsup$ inequality: *for every $x \in X$ there exists a sequence $\{x_t\}$ converging to x such that*

$$\limsup_{t \to 0^+} F_t(x_t) \leq F(x).$$

When Properties (i) *and* (ii) *are satisfied, we write $F = \Gamma-\lim\limits_{t \to 0^+} F_t$.*

Proposition A.6. *Let $F_t : X \to \mathbb{R}$ be a sequence of functionals on a topological space such that:*

(i) $F = \Gamma-\lim\limits_{t \to 0^+} F_t$;

(ii) $\sup_t F_t(x_t) < +\infty \;\Rightarrow\; \{x_t\}$ *is sequentially relatively compact in X.*

Then we have the convergence: $\inf F_t \to \inf F$ *as $t \to 0^+$ and, every cluster point of a minimizing sequence $\{x_t\}$ (i.e. such that $F_t(x_t) = \inf\limits_{x \in X} F_t(x)$) achieves the minimum of F.*

A.2 Second proof of Theorem 2.5 in the Dirichlet case using the material derivative

In this section, we shall recalculate the expression for $\mathfrak{M}'_\Omega(\Omega;\boldsymbol{V})$ obtained in Theorem 2.5, while considering the particular case of Dirichlet boundary condition on $\partial\Omega$, based on the material derivative approach. We omit the proof of the existence of the material derivative which is a direct adaptation of the existing works (see for example [8]). It can be made out from the following calculations that those based on the classical material derivative method are much more tedious as compared to the calculations obtained in the previous sections.

A.2.1 First characterization with the material derivative

We use the notations for the problem on the perturbed domain given in the beginning of subsection 3.4.1. Let u_t be a normalized eigenfunctions for $\mathfrak{M}_\Omega(\Omega_t)$. We set $u^t = u_t \circ \Psi_t$. The existence of the shape derivative and of the material derivative of u, and of the shape derivative $\mathfrak{M}'_\Omega(\Omega;\boldsymbol{V})$ are assumed to begin with and we will perform calculations using them.

The shape derivative $\mathfrak{M}'_\Omega(\Omega;\boldsymbol{V})$ can be obtained by deriving the Rayleigh quotient on Ω_t evaluated at a normalized eigenfunction u^t. In view of the normalization condition $\displaystyle\int_{\Omega_t} \rho_t \left|u_t\right|^2 \mathrm{d}x = 1$, it is enough to derive $\displaystyle\int_{\Omega_t} \sigma_t(x)\left|\nabla u_t\right|^2 \mathrm{d}x$ for which we use the Hadamard's formula. So, arguing similarly as in Proposition 3.5, we obtain,

$$\mathfrak{M}'_\Omega(\Omega;\boldsymbol{V}) = 2\int_\Omega \sigma(x)\nabla u'\cdot\nabla u\,\mathrm{d}x + \int_\Gamma \left[\sigma(x)\left|\nabla u\right|^2\right] V_{\mathrm{n}}\,\mathrm{d}\varsigma(x) + \int_{\partial\Omega} \sigma_2\left|\nabla u\right|^2 V_{\mathrm{n}}\,\mathrm{d}\varsigma(x). \tag{A.1}$$

This does not give a boundary expression of the shape derivative and also involves u' which has to be characterized through a boundary value problem. This involves several difficulties and so, classically, one takes the route through the material derivative (cf. [8]).

We have the following variational formulation for the perturbed problem on Ω_t

$$\int_{\Omega_t} \sigma_t(x)\nabla u_t\cdot\nabla\varphi_t\,\mathrm{d}x = \mathfrak{M}_\Omega(\Omega_t)\int_{\Omega_t}\rho_t u_t\varphi_t\,\mathrm{d}x, \qquad \text{for all } \varphi_t\in \mathrm{H}^1_0(\Omega_t).$$

So, for any $\varphi\in\mathrm{H}^1_0(\Omega)$, by choosing $\varphi_t := \varphi\circ\Psi_t^{-1}$ and then making a change of variables in the variational problem $y=\Psi_t(x)$ we have

$$\int_\Omega \sigma(x)\left(A(t)\nabla u^t\right)\cdot\nabla\varphi\,\mathrm{d}x = \mathfrak{M}_\Omega(\Omega_t)\int_\Omega \rho\, u^t\varphi j(t)\,\mathrm{d}x,$$

by noticing that $\sigma_t(\Psi_t(x)) = \sigma(x)$ and $\rho_t(\Psi_t(x)) = \rho(x)$. We set $A(t) := j(t)\mathrm{D}\Psi_t^{-1}\left(\mathrm{D}\Psi_t^{-1}\right)^\top$ while recalling that $j(t) = \det(\mathrm{D}\Psi_t)$. Deriving the equation with respect to t at $t=0$ under the integral sign we obtain that,

$$\int_\Omega \sigma(x)\left(\nabla\dot{u} + A'(0)\nabla u\right)\cdot\nabla\varphi\,\mathrm{d}x = \int_\Omega \rho\big(\mathfrak{M}'_\Omega(\Omega;\boldsymbol{V})u + \mathfrak{M}_\Omega(\Omega)\left(j'(0)u+\dot{u}\right)\big)\varphi\,\mathrm{d}x. \tag{A.2}$$

Using the Hadamard's formula given in Lemma A.1 on the normalization condition

$$1 = \int_{\Omega_t}\rho_t\left|u_t\right|^2\,\mathrm{d}x = \int_{\Omega_{1,t}}\rho_1\left|u_t\right|^2\,\mathrm{d}x + \int_{\Omega_{2,t}}\rho_2\left|u_t\right|^2\,\mathrm{d}x,$$

we obtain

$$\begin{aligned}0 = \int_\Omega 2\rho\, u'u\,\mathrm{d}x + \int_{\partial\Omega}\rho\,\left|u\right|^2 V_{\mathrm{n}}\,\mathrm{d}\varsigma(x) + (\rho_1-\rho_2)\int_\Gamma \left|u\right|^2 V_{\mathrm{n}}\,\mathrm{d}\varsigma(x)&\\ = 2\int_\Omega \rho\, u'u\,\mathrm{d}x + \int_\Gamma [\rho]\left|u\right|^2 V_{\mathrm{n}}\,\mathrm{d}\varsigma(x),&\end{aligned} \tag{A.3}$$

since $u=0$ on $\partial\Omega$.

A.2.2 Rewriting of some terms

Let $\varphi \in \mathrm{H}^1_0(\Omega)$. Since $A'(0) = (\operatorname{div} \boldsymbol{V})\,\mathrm{I} - \left(\mathrm{D}\boldsymbol{V} + \mathrm{D}\boldsymbol{V}^\top\right)$ (see, e.g., [26, Lemma 2.31]), we have in both Ω_i, $i = 1, 2$, where σ is constant,

$$\begin{aligned} \sigma(x)A'(0)\nabla u \cdot \nabla\varphi &= \sigma(x)\,(\operatorname{div} \boldsymbol{V})\,\nabla u \cdot \nabla\varphi - \sigma(x)\left(\mathrm{D}\boldsymbol{V} + \mathrm{D}\boldsymbol{V}^\top\right)\nabla u \cdot \nabla\varphi \\ &= \operatorname{div}\Big((\sigma(x)\nabla u \cdot \nabla\varphi)\,\boldsymbol{V}\Big) - \nabla\left(\sigma(x)\nabla u \cdot \nabla\varphi\right) \cdot \boldsymbol{V} - \sigma(x)\left(\mathrm{D}\boldsymbol{V} + \mathrm{D}\boldsymbol{V}^\top\right)\nabla u \cdot \nabla\varphi \\ &= \operatorname{div}\Big((\sigma(x)\nabla u \cdot \nabla\varphi)\,\boldsymbol{V}\Big) - \sigma(x)\nabla\left(\nabla u \cdot \boldsymbol{V}\right) \cdot \nabla\varphi - \sigma(x)\nabla u \cdot \nabla(\nabla\varphi \cdot \boldsymbol{V}). \end{aligned}$$

Moreover

$$\int_\Omega \operatorname{div}\Big((\sigma(x)\nabla u \cdot \nabla\varphi)\,\boldsymbol{V}\Big)\,\mathrm{d}x = \int_{\partial\Omega} (\sigma(x)\nabla u \cdot \nabla\varphi)\,V_\mathrm{n}\,\mathrm{d}\varsigma(x) + \int_\Gamma [\sigma(x)\nabla u \cdot \nabla\varphi]\,V_\mathrm{n}\,\mathrm{d}\varsigma(x),$$

and

$$\begin{aligned} &\int_\Omega \sigma(x)\nabla u \cdot \nabla(\nabla\varphi \cdot \boldsymbol{V})\,\mathrm{d}x \\ &\qquad = -\int_\Omega \operatorname{div}\sigma(x)\nabla u \nabla\varphi \cdot \boldsymbol{V}\,\mathrm{d}x + \int_{\partial\Omega} \sigma(x)\partial_\mathrm{n} u \nabla\varphi \cdot \boldsymbol{V}\,\mathrm{d}\varsigma(x) + \int_\Gamma \left[\sigma(x)\partial_\mathrm{n} u \nabla\varphi \cdot \boldsymbol{V}\right]\,\mathrm{d}\varsigma(x) \\ &\qquad = \mathfrak{M}_\Omega(\Omega)\int_\Omega \rho\, u \nabla\varphi \cdot \boldsymbol{V}\,\mathrm{d}x + \int_{\partial\Omega} \sigma(x)\partial_\mathrm{n} u \nabla\varphi \cdot \boldsymbol{V}\,\mathrm{d}\varsigma(x) + \int_\Gamma \left[\sigma(x)\partial_\mathrm{n} u \nabla\varphi \cdot \boldsymbol{V}\right]\,\mathrm{d}\varsigma(x). \end{aligned}$$

Thus

$$\begin{aligned} \int_\Omega \sigma(x)A'(0)\nabla u \cdot \nabla\varphi\,\mathrm{d}x = \int_{\partial\Omega} &(\sigma(x)\nabla u \cdot \nabla\varphi)\,V_\mathrm{n}\,\mathrm{d}\varsigma(x) - \int_{\partial\Omega} \sigma(x)\partial_\mathrm{n} u \nabla\varphi \cdot \boldsymbol{V}\,\mathrm{d}\varsigma(x) \\ &+ \int_\Gamma [\sigma(x)\nabla u \cdot \nabla\varphi]\,V_\mathrm{n}\,\mathrm{d}\varsigma(x) - \int_\Gamma \left[\sigma(x)\partial_\mathrm{n} u \nabla\varphi \cdot \boldsymbol{V}\right]\,\mathrm{d}\varsigma(x) \\ &\qquad - \int_\Omega \sigma(x)\nabla\left(\nabla u \cdot \boldsymbol{V}\right) \cdot \nabla\varphi\,\mathrm{d}x - \mathfrak{M}_\Omega(\Omega)\int_\Omega \rho\, u \nabla\varphi \cdot \boldsymbol{V}\,\mathrm{d}x. \end{aligned}$$

Using this equality in (A.2) and since $j'(0) = \operatorname{div} \boldsymbol{V}$ (see, e.g., [26, Lemma 2.31]), we obtain

$$\begin{aligned} \int_\Omega \sigma(x)\nabla\dot{u} \cdot \nabla\varphi\,\mathrm{d}x &+ \int_{\partial\Omega} (\sigma(x)\nabla u \cdot \nabla\varphi)\,V_\mathrm{n}\,\mathrm{d}\varsigma(x) - \int_{\partial\Omega} \sigma(x)\partial_\mathrm{n} u \nabla\varphi \cdot \boldsymbol{V}\,\mathrm{d}\varsigma(x) \\ &+ \int_\Gamma [\sigma(x)\nabla u \cdot \nabla\varphi]\,V_\mathrm{n}\,\mathrm{d}\varsigma(x) - \int_\Gamma \left[\sigma(x)\partial_\mathrm{n} u \nabla\varphi \cdot \boldsymbol{V}\right]\,\mathrm{d}\varsigma(x) - \int_\Omega \sigma(x)\nabla\left(\nabla u \cdot \boldsymbol{V}\right) \cdot \nabla\varphi\,\mathrm{d}x \\ &- \mathfrak{M}_\Omega(\Omega)\int_\Omega \rho\, u \nabla\varphi \cdot \boldsymbol{V}\,\mathrm{d}x = \int_\Omega \rho\left(\mathfrak{M}'_\Omega(\Omega; \boldsymbol{V})u\varphi + \mathfrak{M}_\Omega(\Omega)\left((\operatorname{div}\boldsymbol{V})u + \dot{u}\right)\varphi\right)\,\mathrm{d}x. \end{aligned}$$

Hence

$$\begin{aligned} \int_\Omega \sigma(x)\left(\nabla\dot{u} - \nabla\left(\nabla u \cdot \boldsymbol{V}\right)\right) \cdot \nabla\varphi\,\mathrm{d}x &+ \int_{\partial\Omega} (\sigma(x)\nabla u \cdot \nabla\varphi)\,V_\mathrm{n}\,\mathrm{d}\varsigma(x) - \int_{\partial\Omega} \sigma(x)\partial_\mathrm{n} u \nabla\varphi \cdot \boldsymbol{V}\,\mathrm{d}\varsigma(x) \\ &= -\int_\Gamma [\sigma(x)\nabla u \cdot \nabla\varphi]\,V_\mathrm{n}\,\mathrm{d}\varsigma(x) + \int_\Gamma \left[\sigma(x)\partial_\mathrm{n} u \nabla\varphi \cdot \boldsymbol{V}\right]\,\mathrm{d}\varsigma(x) \\ &+ \mathfrak{M}'_\Omega(\Omega; \boldsymbol{V})\int_\Omega \rho\, u\varphi\,\mathrm{d}x + \mathfrak{M}_\Omega(\Omega)\int_\Omega \rho\,\dot{u}\varphi\,\mathrm{d}x + \mathfrak{M}_\Omega(\Omega)\int_\Omega \rho\left((\operatorname{div}\boldsymbol{V})\varphi + \nabla\varphi \cdot \boldsymbol{V}\right)u\,\mathrm{d}x. \end{aligned}$$

Moreover, using the fact that $u = 0$ and $\varphi = 0$ on $\partial\Omega$,

$$\begin{aligned} \int_\Omega \rho\left((\operatorname{div}\boldsymbol{V})\varphi + \nabla\varphi \cdot \boldsymbol{V}\right)u\,\mathrm{d}x &= \int_\Omega \rho \operatorname{div}\left(\varphi \cdot \boldsymbol{V}\right)u\,\mathrm{d}x \\ &= \int_{\partial\Omega} \rho\varphi V_\mathrm{n} u\,\mathrm{d}\varsigma(x) + (\rho_1 - \rho_2)\int_\Gamma \varphi V_\mathrm{n} u\,\mathrm{d}\varsigma(x) - \int_\Omega \rho\varphi\boldsymbol{V} \cdot \nabla u\,\mathrm{d}x \\ &= (\rho_1 - \rho_2)\int_\Gamma (\varphi \otimes \boldsymbol{V})\,\mathbf{n} \cdot u\,\mathrm{d}x - \int_\Omega (\varphi \otimes \boldsymbol{V}) \cdot \nabla u\,\mathrm{d}x = \int_\Gamma [\rho]\,\varphi V_\mathrm{n} u\,\mathrm{d}\varsigma(x) - \int_\Omega (\nabla u \cdot \boldsymbol{V})\,\varphi\,\mathrm{d}x, \end{aligned}$$

and since $\boldsymbol{V} = V_{\mathrm{n}}\mathbf{n}$, we have, on Γ,

$$-\left[(\sigma(x)\nabla u \cdot \nabla\varphi)\right] V_{\mathrm{n}} + \left[\sigma(x)\partial_{\mathrm{n}} u \nabla\varphi \cdot \boldsymbol{V}\right] = -\left[\sigma(x)\right] \nabla_\Gamma(u) \cdot \nabla_\Gamma(\varphi) V_{\mathrm{n}}.$$

Notice that we have used here the fact that u has no jump on Γ and so do $\nabla_\Gamma u$. Then

$$\begin{aligned}\int_\Omega \sigma(x) \left(\nabla \dot{u} - \nabla\left(\nabla u \cdot \boldsymbol{V}\right)\right) \cdot \nabla\varphi \,\mathrm{d}x \\ = -\int_\Gamma \left[\sigma(x)\right] \nabla_\Gamma(u) \cdot \nabla_\Gamma(\varphi) V_{\mathrm{n}} \,\mathrm{d}\varsigma(x) + \mathfrak{M}'_\Omega(\Omega; \boldsymbol{V}) \int_\Omega \rho\, u\varphi \,\mathrm{d}x \\ + \mathfrak{M}_\Omega(\Omega) \int_\Omega \rho\, \dot{u}\varphi \,\mathrm{d}x + \mathfrak{M}_\Omega(\Omega) \int_\Gamma \left[\rho\right] \varphi V_{\mathrm{n}} u \,\mathrm{d}\varsigma(x) - \mathfrak{M}_\Omega(\Omega) \int_\Omega \rho \left(\nabla u \cdot \boldsymbol{V}\right) \varphi \,\mathrm{d}x.\end{aligned}$$

A.2.3 Conclusion: characterization with the shape derivative

Since $u' = \dot{u} - \nabla u \cdot \boldsymbol{V}$, we deduce from the above equality that for all $\varphi \in \mathrm{H}^1_0(\Omega)$,

$$\begin{aligned}\int_\Omega \sigma(x) \nabla u' \cdot \nabla\varphi \,\mathrm{d}x = -\int_\Gamma \left[\sigma(x)\right] \nabla_\Gamma(u) \cdot \nabla_\Gamma(\varphi) V_{\mathrm{n}} \,\mathrm{d}\varsigma(x) \\ + \mathfrak{M}'_\Omega(\Omega; \boldsymbol{V}) \int_\Omega \rho\, u\varphi \,\mathrm{d}x + \mathfrak{M}_\Omega(\Omega) \int_\Omega \rho\, u'\varphi \,\mathrm{d}x + \mathfrak{M}_\Omega(\Omega) \int_\Gamma \left[\rho\right] \varphi V_{\mathrm{n}} u \,\mathrm{d}\varsigma(x). \quad \text{(A.4)}\end{aligned}$$

Thus, taking $\varphi = u$ and using the normalization conditions $\int_\Omega \rho \left|u\right|^2 = 1$ and (A.3),

$$\begin{aligned}\int_\Omega \sigma(x) \nabla u' \cdot \nabla u \,\mathrm{d}x = -\int_\Gamma \left[\sigma(x)\right] \left|\nabla_\Gamma(u)\right|^2 V_{\mathrm{n}} \,\mathrm{d}\varsigma(x) + \mathfrak{M}'_\Omega(\Omega; \boldsymbol{V}) \\ - \frac{1}{2}\mathfrak{M}_\Omega(\Omega) \int_\Gamma \left[\rho\right] \left|u\right|^2 V_{\mathrm{n}} \,\mathrm{d}\varsigma(x) + \mathfrak{M}_\Omega(\Omega) \int_\Gamma \left[\rho\right] u V_{\mathrm{n}} u \,\mathrm{d}\varsigma(x).\end{aligned}$$

Then, using (A.1), we can eliminate the volume term $\int_\Omega \sigma(x) \nabla u' \cdot \nabla u$ to obtain

$$\begin{aligned}\mathfrak{M}'_\Omega(\Omega; \boldsymbol{V}) = 2\int_\Gamma \left[\sigma(x)\right] \left|\nabla_\Gamma(u)\right|^2 V_{\mathrm{n}} \,\mathrm{d}\varsigma(x) - \int_\Gamma \left[\sigma(x) \left|\nabla u\right|^2\right] V_{\mathrm{n}} \,\mathrm{d}\varsigma(x) \\ - \mathfrak{M}_\Omega(\Omega) \int_\Gamma \left[\rho\right] \left|u\right|^2 V_{\mathrm{n}} \,\mathrm{d}\varsigma(x) - \int_{\partial\Omega} \sigma_2 \left|\nabla u\right|^2 V_{\mathrm{n}} \,\mathrm{d}\varsigma(x).\end{aligned}$$

Finally, we obtain

$$\begin{aligned}\mathfrak{M}'_\Omega(\Omega; \boldsymbol{V}) = \int_\Gamma \left[\sigma(x)\right] \left|\nabla_\Gamma(u)\right|^2 V_{\mathrm{n}} \,\mathrm{d}\varsigma(x) - \int_\Gamma \left[\sigma(x) \left|\partial_{\mathrm{n}} u\right|^2\right] V_{\mathrm{n}} \,\mathrm{d}\varsigma(x) \\ - \mathfrak{M}_\Omega(\Omega) \int_\Gamma \left[\rho\right] \left|u\right|^2 V_{\mathrm{n}} \,\mathrm{d}\varsigma(x) - \int_{\partial\Omega} \sigma_2 \left|\nabla u\right|^2 V_{\mathrm{n}} \,\mathrm{d}\varsigma(x),\end{aligned}$$

using the facts that $[\nabla_\Gamma u] = 0$ since $[u] = 0$ on Γ and $\left|\nabla_\Gamma(u)\right|^2 = \left|\nabla_\Gamma u\right|^2 + \left|\partial_{\mathrm{n}} u\right|^2$. This concludes the proof of Theorem 2.5 in the case of Dirichlet boundary conditions.

Remark A.7. *The method exposed in this appendix use the classical shape derivative approach. Notice that recently, A. Laurain and K. Sturm present in [22] another approach based on a pure material derivative approach which permits to obtain the same result faster. Indeed, in this very particular case of Dirichlet boundary conditions, we can proceed as follows.*

Taking $\dot{u} \in \mathrm{H}^1_0(\Omega)$ as a test function in the variational formulation of the problem in Ω, we have

$$\int_\Omega \sigma(x) \nabla u \cdot \nabla \dot{u} = \mathfrak{M}_\Omega(\Omega) \int_\Omega \rho u \dot{u}.$$

Then, taking $\varphi = u$ *in Equation* (A.2)*, using the previous equality and the fact that* $\int_\Omega \rho u^2 = 1$ *and that* $j'(0) = \mathrm{div}\boldsymbol{V}$*, we obtain*

$$\mathfrak{M}'_\Omega(\Omega;\boldsymbol{V}) = \int_\Omega \sigma(x)A'(0)\,|\nabla u|^2\,\mathrm{d}x - \mathfrak{M}_\Omega(\Omega)\int_\Omega \rho u^2 \mathrm{div}\boldsymbol{V}\mathrm{d}x.$$

Notice that

$$\mathrm{div}\boldsymbol{V} = \mathrm{I}_d : \mathrm{D}\boldsymbol{V} \qquad \text{and} \qquad \mathrm{D}\boldsymbol{V}\nabla u\cdot\nabla u = \mathrm{D}\boldsymbol{V} : (\nabla u\otimes\nabla u).$$

Thus, using the fact that $A'(0) = (\mathrm{div}\,\boldsymbol{V})\,\mathrm{I} - \left(\mathrm{D}\boldsymbol{V} + \mathrm{D}\boldsymbol{V}^\top\right)$*, this leads*

$$\mathfrak{M}'_\Omega(\Omega;\boldsymbol{V}) = \int_\Omega \left(\left(\sigma(x)\,|\nabla u|^2 - \mathfrak{M}_\Omega(\Omega)\rho u^2\right)\mathrm{I}_d - 2\sigma(x)\nabla u\otimes\nabla u\right) : \mathrm{D}\boldsymbol{V}\mathrm{d}x.$$

and, from [22], one can then prove that

$$\mathfrak{M}'_\Omega(\Omega;\boldsymbol{V}) = \int_{\partial\Omega} \left(\left(\sigma(x)\,|\nabla u|^2 - \mathfrak{M}_\Omega(\Omega)\rho u^2\right)\mathrm{I}_d - 2\sigma(x)\nabla u\otimes\nabla u\right)\mathbf{n}\cdot\mathbf{n}\,V_\mathrm{n}\mathrm{d}\varsigma(x)$$
$$+\int_\Gamma \left[\left(\sigma(x)\,|\nabla u|^2 - \mathfrak{M}_\Omega(\Omega)\rho u^2\right)\mathrm{I}_d - 2\sigma(x)\nabla u\otimes\nabla u\right]\mathbf{n}\cdot\mathbf{n}\,V_\mathrm{n}\mathrm{d}\varsigma(x).$$

Therefore we obtain the announced result since

$$\left(\left(\sigma(x)\,|\nabla u|^2 - \mathfrak{M}_\Omega(\Omega)\rho u^2\right)\mathrm{I}_d - 2\sigma(x)\nabla u\otimes\nabla u\right)\mathbf{n}\cdot\mathbf{n} = \sigma(x)\,|\nabla u|^2 - \mathfrak{M}_\Omega(\Omega)\rho u^2 - 2\sigma(x)\,|\partial_\mathrm{n} u|^2 = -\sigma_2\,|\nabla u|^2$$

on $\partial\Omega$ *(where we have the homogeneous Dirichlet boundary condition* $u = 0$*), and since*

$$\begin{aligned}\left[\left(\sigma(x)\,|\nabla u|^2 - \mathfrak{M}_\Omega(\Omega)\rho u^2\right)\mathrm{I}_d - 2\sigma(x)\nabla u\otimes\nabla u\right]\mathbf{n}\cdot\mathbf{n} &= \left[\sigma(x)\,|\nabla u|^2\right] - \mathfrak{M}_\Omega(\Omega)\,[\rho]\,u^2 - 2\left[\sigma(x)\,|\partial_\mathrm{n} u|^2\right]\\ &= [\sigma(x)]\left[|\nabla_\Gamma u|^2\right] - \mathfrak{M}_\Omega(\Omega)\,[\rho]\,u^2 - \left[\sigma(x)\,|\partial_\mathrm{n} u|^2\right]\end{aligned}$$

on Γ.

Once again, it is very important to notice that these methods are based on the existence of material or shape derivatives, which is not necessarily true for multiple eigenvalue. The approach that we present in this paper is uniform and permits to deal with several boundary conditions, without assuming the simplicity of the eigenvalue, using the notion of semi-derivative.

A.2.4 Characterization of the shape derivative as the solution of a transmission problem

We conclude this section noticing that we can, classically, characterize the shape derivative of the initial problem as the solution of a transmission problem. Indeed, from (A.4), we obtain that, for all $\varphi \in \mathrm{H}^1_0(\Omega)$,

$$-\int_\Omega \mathrm{div}\,(\sigma(x)\nabla u')\varphi\,\mathrm{d}x + \int_\Gamma [\sigma\partial_\mathrm{n} u']\,\varphi\,\mathrm{d}\varsigma(x)$$
$$= \int_\Gamma \mathrm{div}_\Gamma\left([\sigma]\,\nabla_\Gamma u V_\mathrm{n}\right)\varphi\,\mathrm{d}\varsigma(x) + \mathfrak{M}'_\Omega(\Omega,\boldsymbol{V})\int_\Omega \rho\,u\varphi\,\mathrm{d}x + \mathfrak{M}_\Omega(\Omega)\int_\Omega \rho\,u'\varphi\,\mathrm{d}x + \mathfrak{M}_\Omega(\Omega)\int_\Gamma [\rho]\,\varphi V_\mathrm{n} u\,\mathrm{d}\varsigma(x),$$

and then

$$[\sigma\partial_\mathrm{n} u'] = \mathrm{div}_\Gamma\left([\sigma]\,\nabla_\Gamma u V_\mathrm{n}\right) + \mathfrak{M}_\Omega(\Omega)\,[\rho]\,uV_\mathrm{n} = [\sigma]\,\mathrm{div}_\Gamma\left(\nabla_\Gamma u V_\mathrm{n}\right) + \mathfrak{M}_\Omega(\Omega)\,[\rho]\,uV_\mathrm{n}.$$

Moreover, since $u_1 = u_2$ on Γ, we have

$$u'_1 - u'_2 = (\nabla u_2 - \nabla u_1)\cdot\boldsymbol{V} = -\left[\partial_\mathrm{n} u\right]V_\mathrm{n},$$

the last equality being obtained using the fact that $\nabla_\Gamma u_1 = \nabla_\Gamma u_2$. Hence

$$[u] = -\left[\partial_\mathrm{n} u\right]V_\mathrm{n}.$$

Finally we classically have

$$u' = -\partial_\mathrm{n} u V_\mathrm{n} \text{ on } \partial\Omega.$$

Hence we obtain that the shape derivative u' is solution of

$$\begin{cases} -\mathrm{div}\,(\sigma(x)\nabla u') = \mathfrak{M}_\Omega(\Omega)\rho u' + \mathfrak{M}'_\Omega(\Omega,\boldsymbol{V})\rho u & \text{in } (\Omega\backslash\overline{\omega})\cup\omega,\\ [u] = -\left[\partial_\mathrm{n} u\right]V_\mathrm{n} & \text{on } \Gamma,\\ [\sigma\partial_\mathrm{n} u'] = [\sigma]\,\mathrm{div}_\Gamma\left(\nabla_\Gamma u V_\mathrm{n}\right) + \mathfrak{M}_\Omega(\Omega)\,[\rho]\,uV_\mathrm{n} & \text{on } \Gamma,\\ u' = -\partial_\mathrm{n} u V_\mathrm{n} & \text{on } \partial\Omega.\end{cases}$$

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