# Thèse de Doctorat en Mathématiques


Par

**Aymeric Jacob de Cordemoy**


# Analyse de sensibilité et optimisation pour des problèmes de contact


École Doctorale N° 211

*Sciences Exactes et ses Applications*

**Thèse soutenue à Pau, le 5 juillet 2023**

**Unité de recherche : LMAP - UMR CNRS 5142**


## Composition du jury :

| | | |
|---|---|---|
| Présidente : | Aude RONDEPIERRE | Professeure des Universités, Université Toulouse III - Paul Sabatier |
| Rapporteurs : | Giuseppe BUTTAZZO | Professeur des Universités, Università di Pisa |
| | Aris DANIILIDIS | Professeur des Universités, Technische Universität Wien |
| Examinateurs : | Samir ADLY | Professeur des Universités, Université de Limoges |
| | Samuel AMSTUTZ | Maître de conférences HDR, Université d'Avignon |
| Directeur de thèse : | Fabien CAUBET | Maître de conférences HDR, Université de Pau et des Pays de l'Adour |
| Co-directeur de thèse : | Loïc BOURDIN | Maître de conférences HDR, Université de Limoges |

# REMERCIEMENTS

En premier lieu, je souhaite exprimer mes plus sincères remerciements à mes directeurs de thèse, Loïc Bourdin et Fabien Caubet, pour leurs encouragements et leur disponibilité pendant ces trois années de doctorat. Je remercie également les différents professeurs du laboratoire de Mathématiques de Pau, d'avoir pris le temps de répondre à mes questions, en particulier Chérif Amrouche et Daniela Capatina. Je souhaite également remercier les différents doctorants avec qui j'ai voyagé pendant les conférences (Giulio, Joyce, Salah, etc.), et merci pour tous ces bons moments passés avec vous.
Enfin, je remercie ma famille pour ses soutiens et encouragements tout au long de ce doctorat.

# TABLE OF CONTENTS

# INTRODUCTION (EN FRANÇAIS)

Cette thèse s'inscrit dans deux domaines mathématiques majeurs : la ***théorie des inéquations variationnelles*** et l'étude de problèmes de ***mécanique du contact***. Plus précisément, l'objectif de ce travail est l'analyse de sensibilité de deux problèmes principaux issus de la mécanique du contact, à savoir, ***le problème de Signorini*** modélisant le cas de contact unilatéral, et ***le problème de Tresca*** qui prend en compte des phénomènes de frottement. Ces deux problèmes seront étudiés dans deux modèles différents : le modèle scalaire (la solution devra satisfaire une équation de type Laplace) et le modèle de l'élasticité linéaire. Deux applications majeures sont alors visées : l'étude de problèmes d'***optimisation de forme*** et l'étude de problèmes de ***contrôle optimal***. Dans cette introduction nous décrivons :

(i) dans un premier paragraphe, les motivations et les objectifs de ce manuscrit, ainsi que son positionnement par rapport à la littérature ;

(ii) dans le second paragraphe, nous présentons brièvement les problèmes considérés et une courte description de certains résultats ;

(iii) dans les derniers paragraphes, nous donnons un résumé des différents chapitres de ce manuscrit, et nous détaillons la méthodologie utilisée ainsi que les résultats obtenus.

**Motivations et objectifs.** Les modèles mathématiques relatifs aux problèmes de mécanique du contact entre des corps déformables sont étudiés dans la littérature en vue de diverses applications d'ingénierie, telles que l'analyse du contact roue-sol pour un véhicule automobile, la modélisation de prothèses médicales, etc. Plus précisément, la mécanique du contact décrit la déformation des solides qui se touchent sur des parties de leurs frontières. Le plus souvent, le cadre mécanique consiste en un corps solide élastique qui est en contact avec une fondation rigide, sans la pénétrer et qui peut éventuellement glisser sur elle, ce qui provoque des frottements. Du point de vue mathématique, ces phénomènes se traduisent par différentes contraintes : les conditions de non-perméabilité prennent la forme d'inégalités sur la surface de contact appelées *conditions unilatérales de Signorini* (voir, par exemple, [75, 76]) ; le frottement survenant sur la surface de contact est généralement modélisé par la *loi de frottement de Tresca* (voir, par exemple, [51]) qui apparaît comme une condition impliquant des inégalités et des termes non lisses, et qui dépend d'un seuil d'adhérence. Enfin, ces problèmes de mécanique du contact sont généralement étudiés dans le cadre de la théorie des inéquations variationnelles, puisque les conditions unilatérales de Signorini et la loi de frottement de Tresca entraînent des conditions sur le bord contenant des inégalités.

L'optimisation de forme (resp. le contrôle optimal) est le domaine mathématique visant à trouver la forme d'un objet (resp. le contrôle d'un système) qui minimise une certaine fonctionnelle coût, tout en satisfaisant des contraintes données. Pour résoudre numériquement un problème d'optimisation de forme (resp. de contrôle optimal), une méthodologie classique est de mettre en place une méthode de descente de type gradient, nécessitant alors de calculer le gradient de la fonctionnelle coût qui dépend généralement de la solution d'un problème aux limites donné. Par conséquent, un point crucial est d'effectuer l'analyse de sensibilité du problème aux limites par rapport à des perturbations, afin de caractériser la dérivée de la solution du problème aux limites par rapport à ces perturbations (dans la suite, nous dirons plus simplement la dérivée de la solution).

Afin de traiter des problèmes d'optimisation de forme et de contrôle optimal en mécanique du contact, il est parfois nécessaire de faire l'analyse de sensibilité de problèmes aux limites décrits par des inéquations variationnelles. Jusqu'à présent, les approches utilisées dans la littérature peuvent être divisées en quatre méthodes : la différentiabilité conique, la dualisation, la pénalisation et la régularisation.

(i) La méthode de différentiabilité conique concerne les inéquations variationnelles dont les solutions sont caractérisées par des opérateurs de projection (voir, par exemple, [40, 58]). Si l'ensemble sur lequel est projeté la solution est polyédrique, alors on peut appliquer le théorème de Mignot [58] afin d'en déduire que l'opérateur de projection admet une différentielle conique, ce qui permet alors de prouver que la dérivée de la solution satisfait une inéquation variationnelle.

(ii) La méthode de dualisation concerne les inéquations variationnelles non lisses, et a été développée par J. Sokolowski et J.P. Zolésio (voir, par exemple, [77, 78, 79]). Elle consiste à décrire la paire primal/dual comme point-selle du Lagrangien associé. Cette approche offre l'avantage que la caractérisation de la solution duale implique uniquement des opérateurs de projection, et par conséquent le théorème de Mignot [58] peut être appliqué.

Cependant, ces deux méthodes résultent en des dérivées qui satisfont des inéquations variationnelles abstraites (en particulier la méthode de dualisation puisqu'elle implique des élèments du dual) et par conséquent rendent difficile une caractérisation utilisable numériquement du gradient de la fonctionnelle coût dans des problèmes d'optimisation de forme ou de contrôle optimal. Ces difficultés sont habituellement résolues dans la littérature en utilisant la méthode de pénalisation ou la méthode de régularisation.

(iii) La méthode de pénalisation, utilisée par exemple dans [18, 49] concerne, en particulier, les inéquations variationnelles avec contraintes. Elle consiste à remplacer les contraintes par une fonction de pénalisation dans le problème d'optimisation associé au modèle de mécanique étudié. Par conséquent, cela a l'avantage de décrire la condition d'optimalité par une équation variationnelle à la place d'une inéquation.

(iv) La méthode de régularisation concerne les inéquations variationnelles qui comportent des termes non différentiables, et est utilisée dans des références telles que [9, 19, 53]. Elle consiste à remplacer le terme non lisse dans le problème d'optimisation associé au modèle de mécanique étudié, par son enveloppe de Moreau. Par conséquent, comme pour la méthode de pénalisation, la condition d'optimalité est décrite par une équation variationnelle à la place d'une inéquation.

Le principal inconvénient de ces deux méthodes est qu'elles ne prennent pas en compte la caractérisation exacte des solutions et perturbent ainsi la physique d'origine des modèles de mécanique étudiés.

Cependant, elles ont l'avantage de permettre d'obtenir une expression utilisable numériquement du gradient de la fonctionnelle coût dans des problèmes d'optimisation de forme ou de contrôle optimal.

Le but du travail effectué dans cette thèse est de proposer une nouvelle méthodologie afin de faire l'analyse de sensibilité d'inéquations variationnelles non linéaires, et éventuellement non lisses, permettant de :

1. préserver la physique d'origine des problèmes de mécanique du contact, c'est-à-dire sans utiliser de procédure de pénalisation ou de régularisation ;
2. caractériser la dérivée de la solution du problème aux limites, comme étant elle-même solution d'un problème aux limites ;
3. traiter des problèmes d'optimisation de forme ou de contrôle optimal en proposant une expression utilisable numériquement du gradient d'une fonctionnelle coût.

Plus précisèment, notre méthodologie est basée sur la théorie des inéquations variationnelles et sur des outils et résultats d'analyse convexe et variationnelle, tels que l'opérateur proximal introduit par J.J. Moreau en 1965 [61], ou bien encore la notion d'épi-différentiabilité du second ordre introduite par R.T. Rockafellar en 1985 [68]. De plus, à notre connaissance, c'est la première fois que ces concepts sont appliqués dans le contexte de l'optimisation de forme et du contrôle optimal, ce qui rend ce travail nouveau et original.

**Problèmes considérés et brève description des résultats obtenus.** Dans cette thèse, nous nous focalisons sur deux problèmes aux limites particuliers issus de la mécanique du contact : un problème impliquant des conditions unilatérales de Signorini et un problème impliquant la loi de frottement de Tresca.

(i) Le premier problème, dit de Signorini, est un problème de contact sans frottement formulé pour la première fois par A. Signorini dans [75] en 1933, puis en 1959 dans [76]. En 1963, G. Fichera a prouvé dans [35] l'existence et l'unicité de la solution du problème de Signorini en minimisant la fonctionnelle énergie correspondante. La formulation faible du problème de Signorini est une inéquation variationnelle non linéaire, et la littérature est plutôt abondante sur ce problème (voir, par exemple, [10, 50]).

L'analyse de sensibilité de ce problème a déjà été étudiée dans la littérature sans utiliser de méthode de pénalisation ou de régularisation, à l'aide de la méthode de la différentiabilité conique des opérateurs de projection (voir, par exemple, [19], [57, Chapitre 5 Section 5.2 p.111] et [78, Chapitre 4 Section 4.6 p.205]). En utilisant notre nouvelle méthodologie, nous retrouvons les résultats déjà obtenus dans la littérature pour ce type de problème. En effet, si un ensemble convexe fermé non vide est polyédrique, alors, d'après le théorème de Mignot, l'opérateur de projection sur cet ensemble admet une différentielle conique, laquelle coïncide exactement avec l'opérateur proximal associé à l'épi-dérivée du second ordre de la fonction indicatrice associée. De plus, notre approche permet de caractériser la dérivée de la solution du problème de Signorini, comme étant elle-même la solution d'un problème de Signorini, ce qui est une nouveauté par rapport à la littérature. Nous appliquons alors les résultats obtenus à un problème d'optimisation de forme, et caractérisons ainsi la *dérivée matérielle directionnelle* et la *dérivée de forme directionnelle*. Enfin, en utilisant ces caractérisations, nous obtenons une expression facilement utilisable numériquement du gradient de forme de la fonctionnelle énergie correspondante (voir le résumé de la Partie III ci-dessous), ce qui n'a également pas été obtenu dans la littérature. Par conséquent, sans utiliser de méthode de pénalisation et de régularisation, ce travail peut-être vu comme un complément et une extension des précédents articles sur ce sujet.

(ii) Le second problème, dit de Tresca, est un problème de contact avec frottement (voir, par exemple, [31, Chapitre 3] ou [37, Section 1.3 Chapter 1]). La formulation faible de ce problème est une inéquation variationnelle non linéaire impliquant une fonctionnelle propre convexe semi-continue inférieurement qui est non différentiable : *la fonctionnelle de Tresca*.

L'analyse de sensibilité de ce problème a déjà été étudiée sans méthode de régularisation et de pénalisation, en utilisant la méthode de dualisation expliquée plus haut (voir, par exemple, [77, 78]), et qui fournit des résultats assez abstraits.

Pour mener à bien l'étude de ce problème à l'aide de notre méthodologie, nous devons perturber la loi de frottement de Tresca, et par conséquent la fonctionnelle de Tresca associée. Nous devons alors utiliser les récents travaux issus de [2], dans lequel la notion d'épi-différentiabilitié du second ordre a été adaptée à des fonctions convexes dépendant d'un paramètre. Nous sommes alors capables de faire l'analyse de sensibilité de ce problème et de caractériser la dérivée de la solution de ce problème, comme étant elle-même la solution d'un problème de Signorini, ce qui est nouveau dans la littérature. Cela n'avait pas été obtenu avec la méthode de dualisation puisque des élèments du dual intervenaient, et par conséquent rendaient la caractérisation des dérivées abstraite. De plus, nous pouvons alors appliquer les résultats obtenus à des problèmes d'optimisation de forme et de contrôle optimal, ce qui nous permet, en particulier, d'obtenir une caractérisation de la dérivée matérielle et de la dérivée de forme, lesquelles nous permettent ensuite d'obtenir une expression facilement utilisable numériquement du gradient de forme de la fonctionnelle énergie correspondante (voir le résumé de la Partie III ci-dessous), ce qui là aussi, est nouveau dans la littérature.

Mentionnons également que dans les problèmes d'optimisation de forme (resp. de contrôle optimal) que nous avons considérés, nous arrivons à calculer la différentielle de la fonctionnelle énergie (resp. de la fonctionnelle coût définie par (10.2)) correspondante et à caractériser son gradient à l'aide d'une égalité, alors que les problèmes de mécanique du contact étudiés sont des inéquations variationnelles non linéaires, voir non lisses.

**Résumé Partie I.** La partie I de cette thèse se focalise sur tous les outils et résultats dont on a besoin dans ce manuscrit. Nous commençons avec le Chapitre 1 qui est un rappel sur le modèle de l'élasticité linéaire. Dans le chapitre 2, nous rappelons des résultats de géométrie différentielle, d'analyse fonctionnelle et de théorie de la capacité, tels que l'opérateur de Laplace-Beltrami ou bien encore la formule de la divergence. Le Chapitre 3 concerne l'analyse convexe et variationnelle, et comporte les outils et théorèmes les plus importants de notre méthodologie. En particulier, on y trouve la définition de l'opérateur proximal et la notion d'épi-différentiabilité du second ordre. Enfin, cette partie se termine avec le Chapitre 4 dans lequel nous présentons les principaux problèmes aux limites considérés tout au long de ce manuscrit.

**Résumé Partie II.** Cette partie est le cœur de ce manuscrit dans laquelle nous effectuons l'analyse de sensibilité de problèmes aux limites impliquant des conditions unilatérales de Signorini et la loi de frottement de Tresca. Dans toute cette partie, nous considérons $\Omega$ un ensemble ouvert non vide borné connexe lipschitzien de $\mathbb{R}^d$, où $d \in \mathbb{N}^*$, avec sa frontière notée $\Gamma := \partial\Omega$, et nous notons $\mathrm{n}$ le vecteur normal unitaire extérieur à $\Gamma$. Nous commençons par le Chapitre 5, dans lequel nous nous plaçons dans un modèle simplifié dit scalaire, c'est-à-dire que l'on considère une équation de type Laplace et que les champs de déplacements considérés (voir le Chapitre 1) sont dans $\mathrm{H}^1(\Omega)$. Nous étudions dans ce modèle, entre autres :

(i) L'analyse de sensibilité, par rapport au terme source et aux conditions sur le bord, du problème de Signorini scalaire

$$\begin{cases} -\Delta u_t + u_t = f_t & \text{dans } \Omega, \\ u_t \leq 0,\, \partial_{\mathrm{n}} u_t \leq g_t \text{ et } u_t\,(\partial_{\mathrm{n}} u_t - g_t) = 0 & \text{sur } \Gamma, \end{cases}$$

où $f_t \in \mathrm{L}^2(\Omega)$, $g_t \in \mathrm{L}^2(\Gamma)$, pour tout $t \geq 0$. L'unique solution faible $u_t \in \mathrm{H}^1(\Omega)$ de ce problème, qui appartient au cône convexe fermé non vide $\mathcal{K}_0^1(\Omega) := \{v \in \mathrm{H}^1(\Omega) \mid v \leq 0 \ p.p. \text{ sur } \Gamma\}$ (où *p.p.* signifie presque partout), satisfait l'inéquation variationnelle

$$\int_\Omega \nabla u_t \cdot \nabla(v - u_t) + \int_\Omega u_t\,(v-u_t) \geq \int_\Omega f_t\,(v-u_t) + \int_\Gamma g_t\,(v-u_t)\,, \qquad \forall v \in \mathcal{K}_0^1(\Omega),$$

et est donnée par $u_t = \mathrm{proj}_{\mathcal{K}_0^1(\Omega)}(F_t)$, où $F_t \in \mathrm{H}^1(\Omega)$ est l'unique solution d'un certain problème de Neumann scalaire (voir Sous-section 5.1.1 pour les détails), et où $\mathrm{proj}_{\mathcal{K}_0^1(\Omega)}(F_t)$ est l'opérateur de projection sur $\mathcal{K}_0^1(\Omega)$. Afin d'utiliser notre méthodologie, nous utilisons l'opérateur proximal pour caractériser $u_t$, c'est-à-dire (voir Remark 3.1.11)

$$u_t = \mathrm{prox}_{\iota_{\mathcal{K}_0^1(\Omega)}}(F_t),$$

où $\mathrm{prox}_{\iota_{\mathcal{K}_0^1(\Omega)}}$ est l'opérateur proximal associé à la fonction indicatrice de Signorini $\iota_{\mathcal{K}_0^1(\Omega)}$. Nous prouvons alors, en utilisant notamment l'épi-différentiabilité du second ordre (voir Définition 3.2.5) de la fonction indicatrice de Signorini $\iota_{\mathcal{K}_0^1(\Omega)}$, que sous certaines hypothèses (voir Corollaire 5.1.5), l'application $t \in \mathbb{R}_+ \mapsto u_t \in \mathrm{H}^1(\Omega)$ est dérivable en $t = 0$, et que sa dérivée $u_0' \in \mathrm{H}^1(\Omega)$ est l'unique solution faible du problème de Signorini scalaire

$$\begin{cases} -\Delta u_0' + u_0' = f_0' & \text{dans } \Omega, \\ \partial_{\mathrm{n}} u_0' = g_0' & \text{sur } \Gamma_{\mathrm{N}}^{u_0,g_0}, \\ u_0' = 0 & \text{sur } \Gamma_{\mathrm{D}}^{u_0,g_0}, \\ u_0' \leq 0,\, \partial_{\mathrm{n}} u_0' \leq g_0' \text{ et } u_0'\,(\partial_{\mathrm{n}} u_0' - g_0') = 0 & \text{sur } \Gamma_{\mathrm{S-}}^{u_0,g_0}, \end{cases}$$

où $f_0' \in \mathrm{L}^2(\Omega)$ (resp. $g_0' \in \mathrm{L}^2(\Gamma)$) est la dérivée en $t = 0$ de l'application $t \in \mathbb{R}_+ \mapsto f_t \in \mathrm{L}^2(\Omega)$

(resp. $t \in \mathbb{R}_+ \mapsto g_t \in \mathrm{L}^2(\Gamma)$), et où $\Gamma$ se décompose, à un ensemble de mesure nulle près, en $\Gamma_{\mathrm{N}}^{u_0,g_0} \cup \Gamma_{\mathrm{D}}^{u_0,g_0} \cup \Gamma_{\mathrm{S-}}^{u_0,g_0}$, avec

$$\begin{aligned}
\Gamma_{\mathrm{N}}^{u_0,g_0} &:= \{s \in \Gamma \mid u_0(s) \neq 0\}, \\
\Gamma_{\mathrm{D}}^{u_0,g_0} &:= \{s \in \Gamma \mid u_0(s) = 0 \text{ et } \partial_{\mathrm{n}} u_0(s) < g_0(s)\}, \\
\Gamma_{\mathrm{S-}}^{u_0,g_0} &:= \{s \in \Gamma \mid u_0(s) = 0 \text{ et } \partial_{\mathrm{n}} u_0(s) = g_0(s)\}.
\end{aligned}$$

(ii) L'analyse de sensibilité, par rapport au terme source et aux conditions sur le bord, du problème de Tresca scalaire :

$$\begin{cases} -\Delta u_t + u_t = f_t & \text{dans } \Omega, \\ |\partial_{\mathrm{n}} u_t| \leq g_t \text{ et } u_t \partial_{\mathrm{n}} u_t + g_t |u_t| = 0 & \text{sur } \Gamma, \end{cases} \tag{1}$$

où $f_t \in \mathrm{L}^2(\Omega)$, $g_t \in \mathrm{L}^2(\Gamma)$, avec $g_t > 0$ *p.p.* sur $\Gamma$, pour tout $t \geq 0$. L'unique solution faible $u_t \in \mathrm{H}^1(\Omega)$ de ce problème satisfait l'inéquation variationnelle

$$\int_\Omega \nabla u_t \cdot \nabla (v - u_t) + \int_\Omega u_t \, (v - u_t) + \int_\Gamma g_t \, |v| - \int_\Gamma g_t \, |u_t| \geq \int_\Omega f_t \, (v - u_t), \qquad \forall v \in \mathrm{H}^1(\Omega),$$

et est donnée par

$$u_t = \mathrm{prox}_{\Phi(t,\cdot)}(F_t),$$

où $F_t \in \mathrm{H}^1(\Omega)$ est l'unique solution d'un certain problème de Neumann scalaire (voir Sous-section 5.2.2 pour plus de détails) et où $\mathrm{prox}_{\Phi(t,\cdot)}$ est l'opérateur proximal associé à la fonctionnelle de Tresca dépendant d'un paramètre $\Phi(t, \cdot)$ définie par

$$\begin{array}{rccl} \Phi(t,\cdot) : & \mathrm{H}^1(\Omega) & \longrightarrow & \mathbb{R} \\ & v & \longmapsto & \Phi(t,v) := \displaystyle\int_\Gamma g_t |v|. \end{array}$$

Puisque la fonctionnelle $\Phi$ dépend d'un paramètre, nous utilisons la notion plus générale d'épi-différentiabilité du second ordre dépendant d'un paramètre (voir Définition 3.2.11) afin de prouver que, sous certaines hypothèses (voir Théorème 5.2.12), l'application $t \in \mathbb{R}_+ \mapsto u_t \in \mathrm{H}^1(\Omega)$ est dérivable en $t = 0$, et que sa dérivée $u_0' \in \mathrm{H}^1(\Omega)$ est l'unique solution faible du problème de Signorini scalaire

$$\begin{cases} -\Delta u_0' + u_0' = f_0' & \text{dans } \Omega, \\ \partial_{\mathrm{n}} u_0' = g_0' \frac{\partial_{\mathrm{n}} u_0}{g_0} & \text{sur } \Gamma_{\mathrm{N}}^{u_0,g_0} \\ u_0' = 0 & \text{sur } \Gamma_{\mathrm{D}}^{u_0,g_0}, \\ u_0' \leq 0,\ \partial_{\mathrm{n}} u_0' \leq g_0' \frac{\partial_{\mathrm{n}} u_0}{g_0} \text{ et } u_0' \left( \partial_{\mathrm{n}} u_0' - g_0' \frac{\partial_{\mathrm{n}} u_0}{g_0} \right) = 0 & \text{sur } \Gamma_{\mathrm{S-}}^{u_0,g_0}, \\ u_0' \geq 0,\ \partial_{\mathrm{n}} u_0' \geq g_0' \frac{\partial_{\mathrm{n}} u_0}{g_0} \text{ et } u_0' \left( \partial_{\mathrm{n}} u_0' - g_0' \frac{\partial_{\mathrm{n}} u_0}{g_0} \right) = 0 & \text{sur } \Gamma_{\mathrm{S+}}^{u_0,g_0}, \end{cases} \tag{2}$$

où $f_0' \in \mathrm{L}^2(\Omega)$ est la dérivée en $t = 0$ de l'application $t \in \mathbb{R}_+ \mapsto f_t \in \mathrm{L}^2(\Omega)$, $g_0' \in \mathrm{L}^2(\Gamma)$ est la fonction définie, pour presque tout $s \in \Gamma$, par $g_0'(s) := \lim_{t \to 0^+} \frac{g_t(s) - g_0(s)}{t}$, et où $\Gamma$ se décompose, à un ensemble de mesure nulle près, en $\Gamma_{\mathrm{N}}^{u_0,g_0} \cup \Gamma_{\mathrm{D}}^{u_0,g_0} \cup \Gamma_{\mathrm{S-}}^{u_0,g_0} \cup \Gamma_{\mathrm{S+}}^{u_0,g_0}$, avec

$$\begin{aligned}
\Gamma_{\mathrm{N}}^{u_0,g_0} &:= \{s \in \Gamma \mid u_0(s) \neq 0\}\,,\\
\Gamma_{\mathrm{D}}^{u_0,g_0} &:= \{s \in \Gamma \mid u_0(s) = 0 \text{ et } \partial_{\mathrm{n}} u_0(s) \in (-g_0(s), g_0(s))\}\,,\\
\Gamma_{\mathrm{S-}}^{u_0,g_0} &:= \{s \in \Gamma \mid u_0(s) = 0 \text{ et } \partial_{\mathrm{n}} u_0(s) = g_0(s)\}\,,\\
\Gamma_{\mathrm{S+}}^{u_0,g_0} &:= \{s \in \Gamma \mid u_0(s) = 0 \text{ et } \partial_{\mathrm{n}} u_0(s) = -g_0(s)\}\,.
\end{aligned}$$

Dans le Chapitre 6, nous nous plaçons dans le modèle de l'élasticité linéaire (voir Chapitre 1 pour des détails sur ce modèle), avec $d \in \{2,3\}$. Nous supposons que $\Omega$ est un solide élastique, et que la frontière $\Gamma$ se décompose en deux parties, notées $\Gamma_{\mathrm{D}}$ et $\Gamma_{\mathrm{S}}$, telles que $\Gamma = \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{S}}$ et $\Gamma_{\mathrm{D}} \cap \Gamma_{\mathrm{S}} = \emptyset$, avec $\Gamma_{\mathrm{D}}$ de mesure strictement positive, et nous notons $\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d) := \left\{v \in \mathrm{H}^1(\Omega,\mathbb{R}^d) \mid v = 0 \text{ } p.p. \text{ sur } \Gamma_{\mathrm{D}}\right\}$ qui est un sous-espace vectoriel de $\mathrm{H}^1(\Omega,\mathbb{R}^d)$. Nous étudions dans ce modèle, entre autres :

(iii) L'analyse de sensibilité, par rapport au terme source et aux conditions sur le bord, du problème de Signorini

$$\begin{cases}
-\mathrm{div}\,(\mathrm{Ae}\,(u_t)) = f_t & \text{dans } \Omega,\\
u_t = 0 & \text{sur } \Gamma_{\mathrm{D}},\\
\sigma_\tau(u_t) = 0 & \text{sur } \Gamma_{\mathrm{S}},\\
u_{t\mathrm{n}} \leq 0,\ \sigma_{\mathrm{n}}(u_t) \leq g_t \text{ et } u_{t\mathrm{n}}\,(\sigma_{\mathrm{n}}(u_t) - g_t) = 0 & \text{sur } \Gamma_{\mathrm{S}},
\end{cases}$$

où $f_t \in \mathrm{L}^2(\Omega,\mathbb{R}^d)$, $g_t \in \mathrm{L}^2(\Gamma_{\mathrm{S}})$, $u_{t\mathrm{n}} := u_t \cdot \mathrm{n}$, A est le tenseur des rigidités, e est le tenseur des déformations, $\sigma_{\mathrm{n}}(u_t)$ est la contrainte normale et $\sigma_\tau(u_t)$ la contrainte tangentielle (voir Chapitre 1 pour plus de détails), pour tout $t \geq 0$. L'unique solution faible $u_t \in \mathrm{H}^1(\Omega,\mathbb{R}^d)$ de ce problème, qui appartient au cône convexe fermé non vide $\mathcal{K}^1_0(\Omega,\mathbb{R}^d) := \{v \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d) \mid v_{\mathrm{n}} \leq 0 \text{ } p.p. \text{ sur } \Gamma_{\mathrm{S}}\}$, satisfait l'inéquation variationnelle

$$\int_\Omega \mathrm{Ae}(u_t) : \mathrm{e}\,(v - u_t) \geq \int_\Omega f_t \cdot (v - u_t) + \int_{\Gamma_{\mathrm{S}}} g_t\,(v_{\mathrm{n}} - u_{t\mathrm{n}})\,, \qquad \forall v \in \mathcal{K}^1_0(\Omega,\mathbb{R}^d),$$

et est donnée par $u_t = \mathrm{proj}_{\mathcal{K}^1_0(\Omega,\mathbb{R}^d)}(F_t)$, où $F_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ est l'unique solution d'un certain problème de Dirichlet-Neumann (voir Section 6.1 pour les détails), et où $\mathrm{proj}_{\mathcal{K}^1_0(\Omega,\mathbb{R}^d)}$ est l'opérateur de projection sur $\mathcal{K}^1_0(\Omega,\mathbb{R}^d)$. Pour utiliser notre méthodologie, nous écrivons

$$u_t = \mathrm{prox}_{\iota_{\mathcal{K}^1_0(\Omega,\mathbb{R}^d)}}(F_t),$$

où $\mathrm{prox}_{\iota_{\mathcal{K}^1_0(\Omega,\mathbb{R}^d)}}$ est l'opérateur proximal associé à la fonction indicatrice de Signorini $\iota_{\mathcal{K}^1_0(\Omega,\mathbb{R}^d)}$. Nous prouvons alors, en utilisant notamment l'épi-différentiabilité du second ordre de la fonction indicatrice de Signorini $\iota_{\mathcal{K}^1_0(\Omega,\mathbb{R}^d)}$ que, sous certaines hypothèses (voir Corollaire 6.1.5), l'application $t \in \mathbb{R}_+ \mapsto u_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ est dérivable en $t = 0$, et que sa dérivée $u_0' \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$

est l'unique solution faible du problème de Signorini

$$\begin{cases} -\mathrm{div}\,(\mathrm{Ae}(u_0')) = f_0' & \text{dans } \Omega, \\ u_0' = 0 & \text{sur } \Gamma_{\mathrm{D}}, \\ \sigma_\tau(u_0') = 0 & \text{sur } \Gamma_{\mathrm{S}}, \\ \sigma_{\mathrm{n}}(u_0') = g_0' & \text{sur } \Gamma_{\mathrm{S_N}}^{u_0,g_0}, \\ u_{0\mathrm{n}}' = 0 & \text{sur } \Gamma_{\mathrm{S_D}}^{u_0,g_0}, \\ u_{0\mathrm{n}}' \leq 0,\ \sigma_{\mathrm{n}}(u_0') \leq g_0' \text{ et } u_{0\mathrm{n}}'\,(\sigma_{\mathrm{n}}(u_0') - g_0') = 0 & \text{sur } \Gamma_{\mathrm{S-}}^{u_0,g_0}, \end{cases}$$

où $f_0' \in \mathrm{L}^2(\Omega, \mathbb{R}^d)$ (resp. $g_0' \in \mathrm{L}^2(\Gamma_{\mathrm{S}})$) est la dérivée en $t = 0$ de l'application $t \in \mathbb{R}_+ \mapsto f_t \in \mathrm{L}^2(\Omega, \mathbb{R}^d)$ (resp. $t \in \mathbb{R}_+ \mapsto g_t \in \mathrm{L}^2(\Gamma_{\mathrm{S}})$), et où $\Gamma_{\mathrm{S}}$ se décompose, à un ensemble de mesure nulle près, en $\Gamma_{\mathrm{S_N}}^{u_0,g_0} \cup \Gamma_{\mathrm{S_D}}^{u_0,g_0} \cup \Gamma_{\mathrm{S-}}^{u_0,g_0}$, avec

$$\begin{aligned} \Gamma_{\mathrm{S_N}}^{u_0,g_0} &:= \{s \in \Gamma_{\mathrm{S}} \mid u_{0\mathrm{n}}(s) \neq 0\}\,, \\ \Gamma_{\mathrm{S_D}}^{u_0,g_0} &:= \{s \in \Gamma_{\mathrm{S}} \mid u_{0\mathrm{n}}(s) = 0 \text{ et } \sigma_{\mathrm{n}}(u_0)(s) < g_0(s)\}\,, \\ \Gamma_{\mathrm{S-}}^{u_0,g_0} &:= \{s \in \Gamma_{\mathrm{S}} \mid u_{0\mathrm{n}}(s) = 0 \text{ et } \sigma_{\mathrm{n}}(u_0)(s) = g_0(s)\}\,. \end{aligned}$$

(iv) L'analyse de sensibilité, par rapport au terme source et aux conditions sur le bord, du problème de Tresca

$$\begin{cases} -\mathrm{div}\,(\mathrm{Ae}(u_t)) = f_t & \text{dans } \Omega, \\ u_t = 0 & \text{sur } \Gamma_{\mathrm{D}}, \\ \sigma_{\mathrm{n}}(u_t) = h_t & \text{sur } \Gamma_{\mathrm{S}}, \\ \|\sigma_\tau(u_t)\| \leq g_t \text{ et } u_{t\tau} \cdot \sigma_\tau(u_t) + g_t\,\|u_{t\tau}\| = 0 & \text{sur } \Gamma_{\mathrm{S}}, \end{cases}$$

où $f_t \in \mathrm{L}^2(\Omega, \mathbb{R}^d)$, $g_t \in \mathrm{L}^2(\Gamma_{\mathrm{S}})$, avec $g_t > 0$ *p.p.* sur $\Gamma_{\mathrm{S}}$, pour tout $t \geq 0$. L'unique solution faible $u_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ de ce problème satisfait l'inéquation variationnelle

$$\int_\Omega \mathrm{Ae}(u_t) : \mathrm{e}\,(v - u_t) + \int_{\Gamma_{\mathrm{S}}} g_t\,\|v_\tau\| - \int_{\Gamma_{\mathrm{S}}} g\,\|u_{t\tau}\| \geq \int_\Omega f_t \cdot (v - u_t) + \int_{\Gamma_{\mathrm{S}}} h_t\,(v_{\mathrm{n}} - u_{t\mathrm{n}})\,,$$

pour tout $v \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$, et est donnée par

$$u_t = \mathrm{prox}_{\Phi(t,\cdot)}(F_t),$$

où $F_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ est l'unique solution d'un certain problème de Dirichlet-Neumann (voir Sous-section 6.2.1 pour les détails) et où $\mathrm{prox}_{\Phi(t,\cdot)}$ est l'opérateur proximal associé à la fonctionnelle de Tresca dépendant d'un paramètre $\Phi(t,\cdot)$ définie par

$$\begin{array}{rccl} \Phi(t,\cdot): & \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d) & \longrightarrow & \mathbb{R} \\ & v & \longmapsto & \Phi(t,v) := \displaystyle\int_{\Gamma_{\mathrm{S}}} g_t\,\|v_\tau\|\,. \end{array}$$

Nous prouvons alors, sous certaines hypothèses (voir Corollaire 6.2.8) telles que l'épi-différenti abilité du second ordre (dépendant d'un paramètre) de $\Phi$, que l'application $t \in \mathbb{R}_+ \mapsto u_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ est dérivable en $t = 0$, et que sa dérivée $u_0' \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ est l'unique solution faible du problème de Signorini tangentiel

$$\begin{cases} -\operatorname{div}(\mathrm{A}\mathrm{e}(u_0')) = f_0' & \text{dans } \Omega, \\ u_0' = 0 & \text{sur } \Gamma_{\mathrm{D}}, \\ \sigma_{\mathrm{n}}(u_0') = h_0' & \text{sur } \Gamma_{\mathrm{S}}, \\ \sigma_\tau(u_0') + \frac{g_0}{\|u_{0_\tau}\|}\left(u_{0_\tau}' - \left(u_{0_\tau}' \cdot \frac{u_{0_\tau}}{\|u_{0_\tau}\|}\right)\frac{u_{0_\tau}}{\|u_{0_\tau}\|}\right) = -g_0'\frac{u_{0_\tau}}{\|u_{0_\tau}\|} & \text{sur } \Gamma_{\mathrm{S}_{\mathrm{R}}^{u_0,g_0}}, \\ u_{0_\tau}' = 0 & \text{sur } \Gamma_{\mathrm{S}_{\mathrm{D}}^{u_0,g_0}}, \\ u_{0_\tau}' \in \mathbb{R}_-\frac{\sigma_\tau(u_0)}{g_0}, \left(\sigma_\tau(u_0') - g_0'\frac{\sigma_\tau(u_0)}{g_0}\right)\cdot\frac{\sigma_\tau(u_0)}{g_0} \leq 0 \text{ et} & \\ u_{0_\tau}' \cdot \left(\sigma_\tau(u_0') - g_0'\frac{\sigma_\tau(u_0)}{g_0}\right) = 0 & \text{sur } \Gamma_{\mathrm{S}_{\mathrm{S}}^{u_0,g_0}}. \end{cases}$$

où $f_0' \in \mathrm{L}^2(\Omega)$ est la dérivée en $t = 0$ de l'application $t \in \mathbb{R}_+ \mapsto f_t \in \mathrm{L}^2(\Omega)$, $g_0' \in \mathrm{L}^2(\Gamma_{\mathrm{S}})$ est la fonction définie, pour presque tout $s \in \Gamma_{\mathrm{S}}$, par $g_0'(s) := \lim_{t\to 0^+}\frac{g_t(s)-g_0(s)}{t}$, et où $\Gamma_{\mathrm{S}}$ se décompose, à un ensemble de mesure nulle près, en $\Gamma_{\mathrm{S}_{\mathrm{R}}^{u_0,g_0}} \cup \Gamma_{\mathrm{S}_{\mathrm{D}}^{u_0,g_0}} \cup \Gamma_{\mathrm{S}_{\mathrm{S}}^{u_0,g_0}}$, avec

$$\begin{aligned} \Gamma_{\mathrm{S}_{\mathrm{R}}^{u_0,g_0}} &:= \left\{s \in \Gamma_{\mathrm{S}} \mid u_{0_\tau}(s) \neq 0\right\}, \\ \Gamma_{\mathrm{S}_{\mathrm{D}}^{u_0,g_0}} &:= \left\{s \in \Gamma_{\mathrm{S}} \mid u_{0_\tau}(s) = 0 \text{ et } \frac{\sigma_\tau(u_0)(s)}{g_0(s)} \in \mathrm{B}(0,1) \cap (\mathbb{R}\mathrm{n}(s))^\perp\right\}, \\ \Gamma_{\mathrm{S}_{\mathrm{S}}^{u_0,g_0}} &:= \left\{s \in \Gamma_{\mathrm{S}} \mid u_{0_\tau}(s) = 0 \text{ et } \frac{\sigma_\tau(u_0)(s)}{g_0(s)} \in \partial\mathrm{B}(0,1) \cap (\mathbb{R}\mathrm{n}(s))^\perp\right\}. \end{aligned}$$

Notons que les conditions sur $\Gamma_{\mathrm{S}_{\mathrm{S}}^{u_0,g_0}}$ sont appelées les conditions tangentielles de Signorini (voir Remarque 4.2.12).

Finalement, cette partie est conclue avec le Chapitre 7 où des simulations numériques sont effectuées afin d'illustrer le résultat obtenu dans (ii), lequel affirme que, l'approximation du premier ordre de la solution $u_t$ du problème de Tresca scalaire (1) est donnée par $u_0 + tu_0'$ pour de petites valeurs de $t \geq 0$, où $u_0'$ est la solution du problème de Signorini scalaire (2) (voir les Figures 1 et 2).

Il est important de noter que dans la partie II, nous faisons l'analyse de sensibilité de problèmes de Signorini (resp. Tresca) bien plus généraux (c'est-à-dire avec davantage de données perturbées) que ceux présentés précédemment. En effet, le but est de prendre en compte l'analyse de sensibilité par rapport à la forme, des problèmes de Signorini (resp. Tresca) ci-dessus, puisque si la forme est perturbée, alors ces problèmes plus généraux apparaîtront naturellement après un changement de variables approprié dans les formulations variationnelles du problème de Signorini (resp. Tresca) (voir, par exemple, les résumés ci-dessous des Chapitres 8 et 9 pour plus de détails).

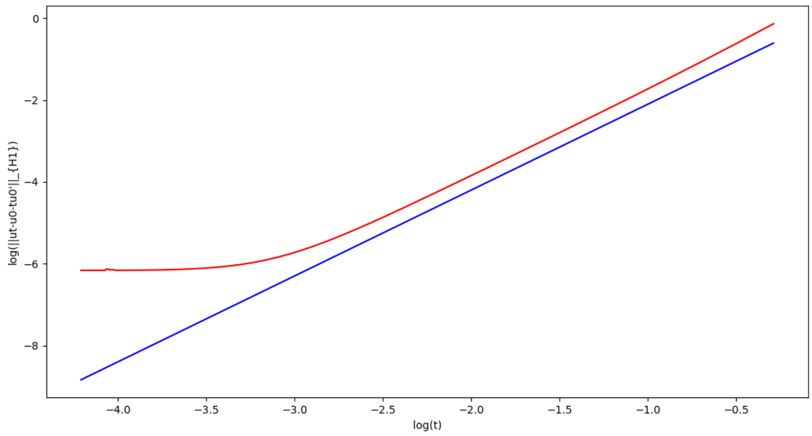


Figure 1 – La réprésentation en échelle logarithmique de l'application $t \in \mathbb{R}_+ \mapsto \|u_t - u_0 - tu_0'\|_{\mathrm{H}^1(\Omega)} \in \mathbb{R}_+$ (courbe rouge) et de l'application $t \in \mathbb{R}_+ \mapsto t^2 \in \mathbb{R}_+$ (courbe bleue).

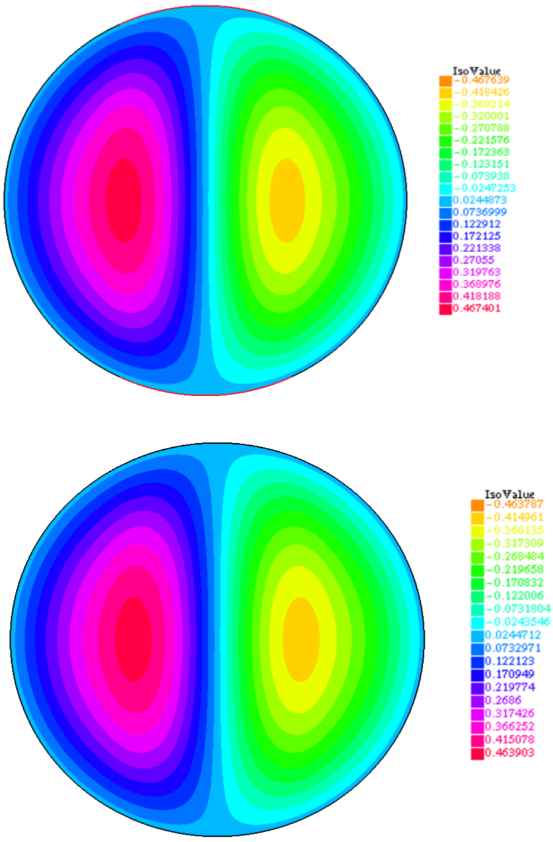


Figure 2 – La première figure affiche les valeurs de $u_t$, tandis que la seconde affiche les valeurs de $u_0 + \ tu_0'$ pour $t = 0.1$.

**Résumé Partie III.** Dans cette partie, nous présentons des applications des analyses de sensibilité effectuées dans la Partie II pour la résolution de problèmes d'optimisation de forme et de contrôle optimal.

(i) Nous commençons par le Chapitre 8 où nous traitons un problème d'optimisation de forme impliquant la loi de frottement de Tresca dans le modèle simplifié du cas scalaire. Plus précisément, avec $d \in \mathbb{N}^*$, $f \in \mathrm{H}^1(\mathbb{R}^d)$ et $g \in \mathrm{H}^2(\mathbb{R}^d)$ telle que $g > 0$ *p.p.* sur $\mathbb{R}^d$, nous considérons le problème d'optimisation de forme suivant

$$\underset{\substack{\Omega \in \mathcal{U} \\ |\Omega| = \lambda}}{\text{minimiser}} \ \mathcal{J}(\Omega), \tag{3}$$

où

$$\mathcal{U} := \{\Omega \subset \mathbb{R}^d \mid \Omega \text{ ouvert non vide borné connexe lipschitzien de } \mathbb{R}^d\},$$

avec une contrainte de volume fixé $|\Omega| = \lambda > 0$, où $\mathcal{J} : \mathcal{U} \to \mathbb{R}$ est la *fonctionnelle énergie de Tresca* définie par

$$\mathcal{J}(\Omega) := \frac{1}{2} \int_\Omega \left( \|\nabla u_\Omega\|^2 + |u_\Omega|^2 \right) + \int_\Gamma g|u_\Omega| - \int_\Omega f u_\Omega,$$

où $\Gamma := \partial\Omega$ est la frontière de $\Omega$ et où $u_\Omega \in \mathrm{H}^1(\Omega)$ est l'unique solution du problème de Tresca scalaire

$$\begin{cases} -\Delta u + u = f & \text{dans } \Omega, \\ |\partial_\mathrm{n} u| \leq g \text{ et } u\partial_\mathrm{n} u + g|u| = 0 & \text{sur } \Gamma, \end{cases} \tag{$\mathrm{TP}_\Omega$}$$

pour tout $\Omega \in \mathcal{U}$.

Pour le traitement numérique du problème d'optimisation de forme ci-dessus, une expression adéquate du gradient de forme de $\mathcal{J}$ est nécessaire. Pour ce faire, nous suivons la stratégie classique développée dans la littérature sur l'optimisation des formes (voir, par exemple, [8, 46]). Considérons $\Omega_0 \in \mathcal{U}$ et une direction $\theta \in \mathcal{C}^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d) := \mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d) \cap \mathrm{W}^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$. Alors, pour tout $t \geq 0$ suffisamment petit tel que $\mathrm{id} + t\theta$ est un difféomorphisme de classe $\mathcal{C}^1$ de $\mathbb{R}^d$, nous notons $\Omega_t := (\mathrm{id} + t\theta)(\Omega_0) \in \mathcal{U}$ et $u_t := u_{\Omega_t} \in \mathrm{H}^1(\Omega_t)$, où $\mathrm{id} : \mathbb{R}^d \to \mathbb{R}^d$ est l'opérateur identité. Afin d'obtenir une expression du gradient de forme de $\mathcal{J}$ en $\Omega_0$ dans la direction $\theta$, le premier pas consiste à obtenir une expression de la dérivée de l'application $t \in \mathbb{R}_+ \mapsto u_t \in \mathrm{H}^1(\Omega_t)$ en $t = 0$. Cependant, cette application est mal définie puisque l'espace d'arrivée $\mathrm{H}^1(\Omega_t)$ dépend de la variable $t$. Pour surmonter cette difficulté, nous considérons le changement de variables $\mathrm{id} + t\theta$ et nous prouvons que $\overline{u}_t := u_t \circ (\mathrm{id} + t\theta) \in \mathrm{H}^1(\Omega_0)$ est l'unique solution du problème de Tresca scalaire général

$$\begin{cases} -\mathrm{div}\,(\mathrm{A}_t \nabla \overline{u}_t) + \overline{u}_t \mathrm{J}_t = f_t \mathrm{J}_t & \text{dans } \Omega_0, \\ |\mathrm{A}_t \nabla \overline{u}_t \cdot \mathrm{n}| \leq g_t \mathrm{J}_{\mathrm{T}_t} \text{ et } \overline{u}_t \mathrm{A}_t \nabla \overline{u}_t \cdot \mathrm{n} + g_t \mathrm{J}_{\mathrm{T}_t} |\overline{u}_t| = 0 & \text{sur } \Gamma_0, \end{cases}$$

considéré sur le domaine fixe $\Omega_0$, où $\Gamma_0 := \partial\Omega_0$, n est le vecteur normal unitaire extérieur à la frontière $\Gamma_0$, $f_t := f \circ (\mathrm{id} + t\theta) \in \mathrm{H}^1(\mathbb{R}^d)$, $g_t := g \circ (\mathrm{id} + t\theta) \in \mathrm{H}^1(\mathbb{R}^d)$ et où $\mathrm{J}_t$, $\mathrm{A}_t$ et $\mathrm{J}_{\mathrm{T}_t}$ sont les termes Jacobien standards, résultant du changement de variable utilisé dans la formulation faible du Problème ($\mathrm{TP}_{\Omega_t}$) (voir Section 8.1 pour les détails). Maintenant, pour obtenir l'expression de la dérivée de l'application $t \in \mathbb{R}_+ \mapsto \overline{u}_t \in \mathrm{H}^1(\Omega_0)$ en $t = 0$, noté $\overline{u}'_0 \in \mathrm{H}^1(\Omega_0)$ et appelée la dérivée matérielle directionnelle (la terminologie *directionnelle* a été ajoutée par rapport à la littérature, puisque l'expression de $\overline{u}_0'$ n'est pas linéaire par rapport à la direction $\theta$, voir Remarque 8.1.2), et nous écrivons alors $\overline{u}_t = \mathrm{prox}_{\phi_t}(F_t)$, où $F_t \in \mathrm{H}^1(\Omega_0)$ est l'unique solution d'un certain problème de Neumann scalaire, et où $\mathrm{prox}_{\phi_t}$ est l'opérateur proximal associé à la fonctionnelle de Tresca perturbée $\phi_t$ définie par

$$\begin{array}{rccl} \phi_t : & \mathrm{H}^1(\Omega_0) & \longrightarrow & \mathbb{R} \\ & v & \longmapsto & \phi_t(v) := \displaystyle\int_{\Gamma_0} g_t \mathrm{J}_{\mathrm{T}_t} |v|, \end{array}$$

et considéré sur l'espace de Hilbert perturbé $(\mathrm{H}^1(\Omega_0), \langle\cdot,\cdot\rangle_{\mathrm{A}_t,\mathrm{J}_t})$ (voir les détails sur le produit scalaire dans la Section 8.1). En utilisant alors les résultats issus de la Partie II, nous obtenons les principaux résultats théoriques, énoncés dans les Théorèmes 8.1.1 et 8.2.1, et résumés ci-dessous. Cependant, pour rendre leurs expressions plus explicites et élégantes, nous les présentons sous certaines hypothèses de régularité supplémentaires, telles que $u_0 \in \mathrm{H}^3(\Omega_0)$, dans le cadre des Corollaires 8.1.3, 8.1.5 et 8.2.2, ce qui les rend plus adaptés à cette introduction.

1. Sous certaines hypothèses décrites dans le Corollaire 8.1.3, la dérivée matérielle directionnelle $\overline{u}_0' \in \mathrm{H}^1(\Omega_0)$ est l'unique solution faible du problème de Signorini scalaire

$$\begin{cases} -\Delta\overline{u}_0' + \overline{u}_0' = -\Delta\left(\theta\cdot\nabla u_0\right) + \theta\cdot\nabla u_0 & \text{dans } \Omega_0, \\ \overline{u}_0' = 0 & \text{sur } \Gamma_{\mathrm{D}}^{u_0,g}, \\ \partial_{\mathrm{n}}\overline{u}_0' = h^m(\theta) & \text{sur } \Gamma_{\mathrm{N}}^{u_0,g}, \\ \overline{u}_0' \leq 0,\ \partial_{\mathrm{n}}\overline{u}_0' \leq h^m(\theta) \text{ et } \overline{u}_0'\left(\partial_{\mathrm{n}}\overline{u}_0' - h^m(\theta)\right) = 0 & \text{sur } \Gamma_{\mathrm{S-}}^{u_0,g}, \\ \overline{u}_0' \geq 0,\ \partial_{\mathrm{n}}\overline{u}_0' \geq h^m(\theta) \text{ et } \overline{u}_0'\left(\partial_{\mathrm{n}}\overline{u}_0' - h^m(\theta)\right) = 0 & \text{sur } \Gamma_{\mathrm{S+}}^{u_0,g}, \end{cases}$$

où $h^m(\theta) := (\frac{\nabla g}{g}\cdot\theta - \nabla\theta\mathrm{n}\cdot\mathrm{n})\partial_{\mathrm{n}}u_0 + (\nabla\theta + \nabla\theta^\top)\nabla u_0\cdot\mathrm{n} \in \mathrm{L}^2(\Gamma_0)$, $\nabla\theta$ est la matrice Jacobienne de $\theta$, et où $\Gamma_0$ se décompose, à un ensemble de mesure nulle près, en $\Gamma_{\mathrm{N}}^{u_0,g} \cup \Gamma_{\mathrm{D}}^{u_0,g} \cup \Gamma_{\mathrm{S-}}^{u_0,g} \cup \Gamma_{\mathrm{S+}}^{u_0,g}$, avec

$$\begin{aligned} \Gamma_{\mathrm{N}}^{u_0,g} &:= \left\{s\in\Gamma_0 \mid u_0(s) \neq 0\right\}, \\ \Gamma_{\mathrm{D}}^{u_0,g} &:= \left\{s\in\Gamma_0 \mid u_0(s) = 0 \text{ et } \partial_{\mathrm{n}}u_0(s) \in (-g(s), g(s))\right\}, \\ \Gamma_{\mathrm{S-}}^{u_0,g} &:= \left\{s\in\Gamma_0 \mid u_0(s) = 0 \text{ et } \partial_{\mathrm{n}}u_0(s) = g(s)\right\}, \\ \Gamma_{\mathrm{S+}}^{u_0,g} &:= \left\{s\in\Gamma_0 \mid u_0(s) = 0 \text{ et } \partial_{\mathrm{n}}u_0(s) = -g(s)\right\}. \end{aligned}$$

2. Nous déduisons dans le Corollaire 8.1.5 que, sous certaines hypothèses, la dérivée de forme directionnelle, définie par $u_0' := \overline{u}_0' - \nabla u_0\cdot\theta \in \mathrm{H}^1(\Omega_0)$, est l'unique solution faible du problème de Signorini scalaire

$$\begin{cases} -\Delta u_0' + u_0' = 0 & \text{dans } \Omega_0, \\ u_0' = -\theta\cdot\nabla u_0 & \text{sur } \Gamma_{\mathrm{D}}^{u_0,g}, \\ \partial_{\mathrm{n}}u_0' = h^s(\theta) & \text{sur } \Gamma_{\mathrm{N}}^{u_0,g}, \\ u_0' \leq -\theta\cdot\nabla u_0,\ \partial_{\mathrm{n}}u_0' \leq h^s(\theta) \text{ et } \left(u_0' + \theta\cdot\nabla u_0\right)\left(\partial_{\mathrm{n}}u_0' - h^s(\theta)\right) = 0 & \text{sur } \Gamma_{\mathrm{S-}}^{u_0,g}, \\ u_0' \geq -\theta\cdot\nabla u_0,\ \partial_{\mathrm{n}}u_0' \geq h^s(\theta) \text{ et } \left(u_0' + \theta\cdot\nabla u_0\right)\left(\partial_{\mathrm{n}}u_0' - h^s(\theta)\right) = 0 & \text{sur } \Gamma_{\mathrm{S+}}^{u_0,g}, \end{cases}$$

où $h^s(\theta) := \theta\cdot\mathrm{n}(\partial_{\mathrm{n}}(\partial_{\mathrm{n}}u_0) - \frac{\partial^2 u_0}{\partial\mathrm{n}^2}) + \nabla_{\Gamma_0}u_0\cdot\nabla_{\Gamma_0}(\theta\cdot\mathrm{n}) - g\nabla(\frac{\partial_{\mathrm{n}}u_0}{g})\cdot\theta \in \mathrm{L}^2(\Gamma_0)$.

3. Finalement, les deux points précédents sont utilisés afin d'obtenir le Corollaire 8.2.2, affirmant que, sous certaines hypothèses, le gradient de forme de $\mathcal{J}$ en $\Omega_0$ dans la direction $\theta \in \mathcal{C}^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ est donné par

$$\mathcal{J}'(\Omega_0)(\theta) = \int_{\Gamma_0} \theta \cdot \mathrm{n} \left( \frac{\|\nabla u_0\|^2 + |u_0|^2}{2} - f u_0 + Hg\,|u_0| - \partial_{\mathrm{n}}\left(u_0 \partial_{\mathrm{n}} u_0\right) + g u_0 \nabla\left(\frac{\partial_{\mathrm{n}} u_0}{g}\right) \cdot \mathrm{n} \right),$$

où $H$ est la courbure moyenne de $\Gamma_0$. Soulignons que $\mathcal{J}'(\Omega_0)$ dépend uniquement de $u_0$ (et non de $u_0'$). Par conséquent, son expression est explicite (et aussi linéaire) par rapport à la direction $\theta$. En particulier, cela implique qu'il n'y a pas besoin d'introduire un problème adjoint afin d'effectuer des simulations numériques (voir Remarque 8.2.4 pour plus de détails).

L'expression du gradient de forme de $\mathcal{J}$ obtenue dans le point 3. permet d'obtenir une direction de descente de $\mathcal{J}$ (voir la Section 8.3 pour les détails). Par conséquent, en utilisant cette direction de descente, combinée avec l'algorithme d'Uzawa pour prendre en compte la contrainte de volume fixé, nous effectuons des simulations numériques afin de résoudre numériquement le problème d'optimisation de forme (3) sur un exemple en deux dimensions présenté dans la Sous-section 8.3.2 (voir Figure 3).

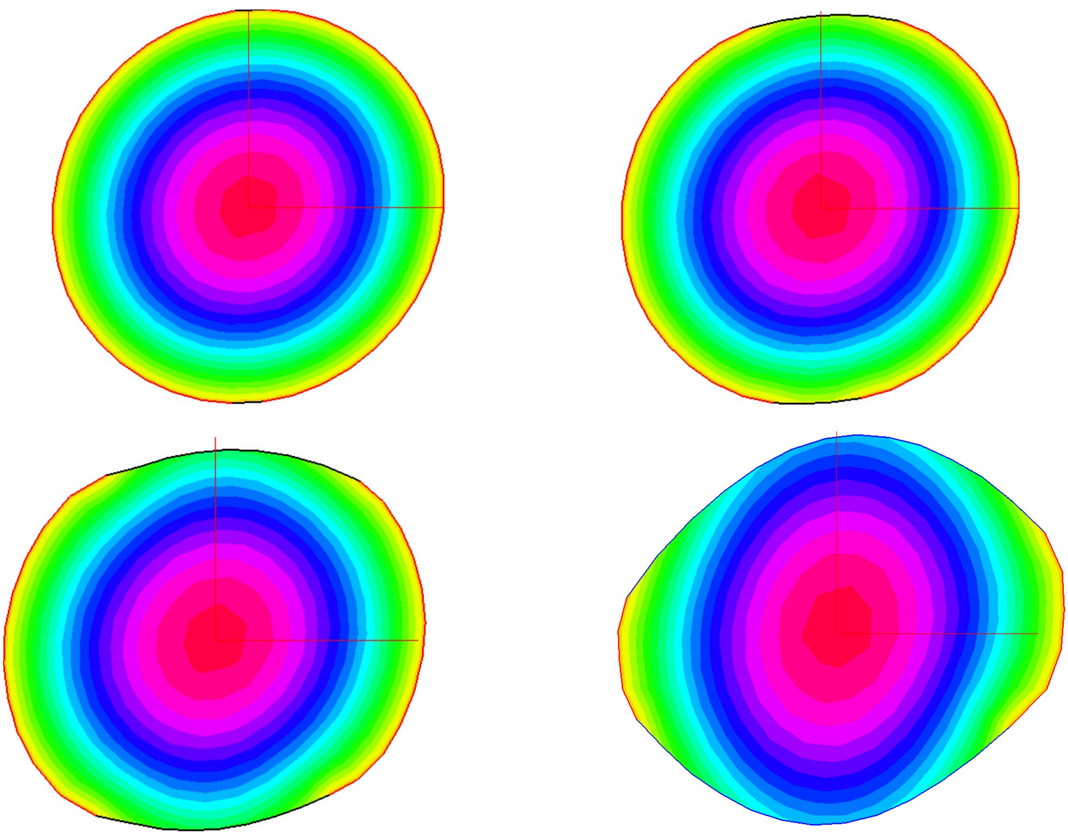

Figure 3 – Formes minimisant la fonctionnelle de Tresca $\mathcal{J}$ avec la contrainte de volume fixé $\lambda = \pi$ en fonction de différentes valeurs de $g$. La frontière rouge montre où $u = 0$ et la frontière noire montre où $|\partial_{\mathrm{n}} u| = g$.

(ii) Dans le chapitre 9, nous traitons un problème d'optimisation de forme impliquant des conditions unilatérales de Signorini dans le modèle de l'élasticité linéaire. Plus précisément, considérons $d \in \{2,3\}$, $f \in \mathrm{H}^1(\mathbb{R}^d,\mathbb{R}^d)$, et $\Omega_{\mathrm{ref}}$ un ensemble ouvert non vide borné connexe lipschitzien de $\mathbb{R}^d$, avec une frontière notée $\Gamma_{\mathrm{ref}} := \partial\Omega$, telle que $\Gamma_{\mathrm{ref}} = \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{S}_{\mathrm{ref}}}$, où $\Gamma_{\mathrm{D}}$ et $\Gamma_{\mathrm{S}_{\mathrm{ref}}}$ sont deux ensembles mesurables disjoints et inclus dans $\Gamma_{\mathrm{ref}}$, avec $\Gamma_{\mathrm{D}}$ de mesure strictement positive. Nous considérons alors le problème d'optimisation de forme suivant

$$\underset{\substack{\Omega\in\mathcal{U}_{\mathrm{ref}}\\ |\Omega|=|\Omega_{\mathrm{ref}}|}}{\text{minimiser}}\ \mathcal{J}(\Omega), \tag{4}$$

où

$$\mathcal{U}_{\mathrm{ref}} := \Big\{\Omega \subset \mathbb{R}^d \mid \Omega \text{ ouvert non vide borné connexe lipschitzien de } \mathbb{R}^d$$
$$\text{avec une frontière } \Gamma := \partial\Omega \text{ telle que } \Gamma_{\mathrm{D}} \subset \Gamma\Big\},$$

avec une contrainte de volume fixé $|\Omega| = |\Omega_{\mathrm{ref}}| > 0$, $\Omega$ est un solide élastique qui satisfait le modèle de l'élasticité linéaire, pour tout $\Omega \in \mathcal{U}_{\mathrm{ref}}$, et où $\mathcal{J} : \mathcal{U}_{\mathrm{ref}} \to \mathbb{R}$ est la *fonctionnelle énergie de Signorini* definie par

$$\mathcal{J}(\Omega) := \frac{1}{2}\int_\Omega \mathrm{Ae}\,(u_\Omega) : \mathrm{e}\,(u_\Omega) - \int_\Omega f \cdot u_\Omega,$$

où $u_\Omega \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ est l'unique solution faible du problème de Signorini

$$\begin{cases} -\mathrm{div}(\mathrm{Ae}(u)) = f & \text{dans } \Omega,\\ u = 0 & \text{sur } \Gamma_{\mathrm{D}},\\ \sigma_\tau(u) = 0 & \text{sur } \Gamma_{\mathrm{S}},\\ u_{\mathrm{n}} \leq 0, \sigma_{\mathrm{n}}(u) \leq 0 \text{ et } u_{\mathrm{n}}\sigma_{\mathrm{n}}(u) = 0 & \text{sur } \Gamma_{\mathrm{S}}, \end{cases} \tag{SP$_\Omega$}$$

où, pour tout $\Omega \in \mathcal{U}_{\mathrm{ref}}$, $\Gamma := \partial\Omega$, $\Gamma_{\mathrm{S}} := \Gamma\backslash\Gamma_{\mathrm{D}}$, n est le vecteur normal unitaire extérieur à la frontière $\Gamma$. Similairement au Chapitre 8, nous considérons $\Omega_0 \in \mathcal{U}_{\mathrm{ref}}$ et une direction $\theta \in \mathcal{C}^{2,\infty}_{\mathrm{D}}(\mathbb{R}^d,\mathbb{R}^d)$, où

$$\mathcal{C}^{2,\infty}_{\mathrm{D}}(\mathbb{R}^d,\mathbb{R}^d) := \left\{\theta \in \mathcal{C}^2(\mathbb{R}^d,\mathbb{R}^d) \cap \mathrm{W}^{2,\infty}(\mathbb{R}^d,\mathbb{R}^d) \mid \theta = 0 \text{ sur } \Gamma_{\mathrm{D}}\right\},$$

et, pour tout $t \geq 0$ suffisamment petit tel que $\mathrm{id} + t\theta$ est un difféomorphisme de classe $\mathcal{C}^2$ de $\mathbb{R}^d$, nous notons $\Omega_t := (\mathrm{id} + t\theta)(\Omega_0) \in \mathcal{U}_{\mathrm{ref}}$ et $u_t := u_{\Omega_t} \in \mathrm{H}^1_{\mathrm{D}}(\Omega_t,\mathbb{R}^d)$. Nous considèrons ensuite le changement de variables $\mathrm{id} + t\theta$ et nous prouvons que $\overline{u}_t := u_t \circ (\mathrm{id} + t\theta) \in \mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)$ est l'unique solution faible du problème de Signorini général

$$\begin{cases} -\mathrm{div}\left(\mathrm{J}_t\mathrm{A}\left[\nabla\overline{u}_t\mathrm{B}_t\right]\mathrm{B}_t^\top\right) = f_t\mathrm{J}_t & \text{dans } \Omega_0,\\ \overline{u}_t = 0 & \text{sur } \Gamma_{\mathrm{D}},\\ \left(\left(\mathrm{J}_t\mathrm{A}\left[\nabla\overline{u}_t\mathrm{B}_t\right]\mathrm{B}_t^\top\right)\mathrm{n}\right)_{\left(\mathrm{M}_t^{-1^\top}\mathrm{n}\right)^\perp} = 0 & \text{sur } \Gamma_{\mathrm{S}_0},\\ \overline{u}_t \cdot \mathrm{M}_t^{-1^\top}\mathrm{n} \leq 0,\ \left(\mathrm{J}_t\mathrm{A}\left[\nabla\overline{u}_t\mathrm{B}_t\right]\mathrm{B}_t^\top\right)\mathrm{n}\cdot\mathrm{M}_t^{-1^\top}\mathrm{n} \leq 0 \text{ et} & \\ \qquad \overline{u}_t \cdot \mathrm{M}_t^{-1^\top}\mathrm{n}\left(\left(\mathrm{J}_t\mathrm{A}\left[\nabla\overline{u}_t\mathrm{B}_t\right]\mathrm{B}_t^\top\right)\mathrm{n}\cdot\mathrm{M}_t^{-1^\top}\mathrm{n}\right) = 0 & \text{sur } \Gamma_{\mathrm{S}_0}, \end{cases}$$

où $f_t := f\circ(\mathrm{id}+t\theta) \in \mathrm{H}^1(\mathbb{R}^d,\mathbb{R}^d)$, $\mathrm{J}_t := \det(\mathrm{I}+t\nabla\theta) \in \mathrm{L}^\infty(\mathbb{R}^d,\mathbb{R})$ est le Jacobien, $\mathrm{M}_t := (\mathrm{I}+t\nabla\theta) \in \mathrm{C}^{1,\infty}(\mathbb{R}^d,\mathbb{R}^{d\times d})$, $\mathrm{B}_t := (\mathrm{I}+t\nabla\theta)^{-1} \in \mathrm{L}^\infty(\mathbb{R}^d,\mathbb{R}^{d\times d})$ et où

$$\left(\left(\mathrm{J}_t\mathrm{A}\left[\nabla\overline{u}_t\mathrm{B}_t\right]\mathrm{B}_t^\top\right)\mathrm{n}\right)_{\left(\mathrm{M}_t^{-1\top}\mathrm{n}\right)^\perp} := \left(\mathrm{J}_t\mathrm{A}\left[\nabla\overline{u}_t\mathrm{B}_t\right]\mathrm{B}_t^\top\right)\mathrm{n} - \left(\left(\mathrm{J}_t\mathrm{A}\left[\nabla\overline{u}_t\mathrm{B}_t\right]\mathrm{B}_t^\top\right)\mathrm{n}\cdot\frac{\mathrm{M}_t^{-1\top}\mathrm{n}}{||\mathrm{M}_t^{-1\top}\mathrm{n}||^2}\right)\mathrm{M}_t^{-1\top}\mathrm{n}.$$

Par conséquent, $\overline{u}_t$ est donnée par $\overline{u}_t = \mathrm{prox}_{\iota_{\mathcal{K}_t^1(\Omega_0,\mathbb{R}^d)}}(F_t)$, où $F_t \in \mathrm{H}^1_\mathrm{D}(\Omega_0,\mathbb{R}^d)$ est l'unique solution d'un certain problème de Dirichlet-Neumann (voir les détails dans la Section 9.1), et où $\mathrm{prox}_{\iota_{\mathcal{K}_t^1(\Omega_0,\mathbb{R}^d)}}$ est l'opérateur proximal associé à la fonction indicatrice de Signorini $\iota_{\mathcal{K}_t^1(\Omega_0,\mathbb{R}^d)}$, où $\mathcal{K}_t^1(\Omega_0,\mathbb{R}^d) := \{v \in \mathrm{H}^1_\mathrm{D}(\Omega_0,\mathbb{R}^d) \mid v\cdot(\mathrm{I}+t\nabla\theta^\top)^{-1}\mathrm{n} \leq 0 \text{ } p.p. \text{ sur } \Gamma_{\mathrm{S}_0}\}$, et considéré sur l'espace de Hilbert perturbé $(\mathrm{H}^1_\mathrm{D}(\Omega_0,\mathbb{R}^d), \langle\cdot,\cdot\rangle_{\mathrm{J}_t,\mathrm{A},\mathrm{B}_t})$ (voir les détails sur le produit scalaire dans la Section 9.1). En utilisant alors les résultats issus de la Partie II, nous obtenons les principaux résultats théoriques, énoncés dans les Théorèmes 9.1.2 et 9.2.1, et résumés ci-dessous. Cependant, nous les présentons sous certaines hypothèses de régularité supplémentaires, dans le cadre des Corollaires 9.1.3, 9.1.5 et 9.2.2.

1. Sous certaines hypothèses décrites dans le Corollaire 9.1.3, l'application $t \in \mathbb{R}_+ \mapsto \overline{u}_t \in \mathrm{H}^1_\mathrm{D}(\Omega_0,\mathbb{R}^d)$ est dérivable en $t = 0$, et la dérivée matérielle directionnelle $\overline{u}'_0 \in \mathrm{H}^1_\mathrm{D}(\Omega_0,\mathbb{R}^d)$ est l'unique solution faible du problème de Signorini

$$\begin{cases} -\mathrm{div}(\mathrm{Ae}(\overline{u}'_0)) = \ell(\theta) & \text{dans } \Omega_0,\\ \overline{u}'_0 = 0 & \text{sur } \Gamma_\mathrm{D},\\ \sigma_\tau(\overline{u}'_0) = h^m(\theta)_\tau & \text{sur } \Gamma_{\mathrm{S}_0},\\ \sigma_\mathrm{n}(\overline{u}'_0) = h^m(\theta)_\mathrm{n} & \text{sur } \Gamma_{\mathrm{S}^{u_{0\mathrm{n}}}_{0,\mathrm{N}}},\\ \overline{u}'_{0\mathrm{n}} = (\nabla\theta u_0)_\mathrm{n} & \text{sur } \Gamma_{\mathrm{S}^{u_{0\mathrm{n}}}_{0,\mathrm{D}}},\\ \overline{u}'_{0\mathrm{n}} \leq (\nabla\theta u_0)_\mathrm{n}, \sigma_\mathrm{n}(\overline{u}'_0) \leq h^m(\theta)_\mathrm{n} \text{ et } \left(\overline{u}'_{0\mathrm{n}} - (\nabla\theta u_0)_\mathrm{n}\right)(\sigma_\mathrm{n}(\overline{u}'_0) - h^m(\theta)_\mathrm{n}) = 0 & \text{sur } \Gamma_{\mathrm{S}^{u_{0\mathrm{n}}}_{0,\mathrm{S}}}, \end{cases}$$

où $h^m(\theta) := ((\mathrm{Ae}(u_0))\nabla\theta^\top + \mathrm{A}(\nabla u_0\nabla\theta) - \sigma_\mathrm{n}(u_0)(\mathrm{div}(\theta)\mathrm{I} + \nabla\theta^\top))\mathrm{n} \in \mathrm{L}^2(\Gamma_{\mathrm{S}_0},\mathbb{R}^d)$, $\ell(\theta) = -\mathrm{div}(\mathrm{Ae}(\nabla u_0\theta)) \in \mathrm{L}^2(\Omega_0,\mathbb{R}^d)$, et où $\Gamma_{\mathrm{S}_0}$ se décompose, à un ensemble de mesure nulle près, en $\Gamma^{u_{0\mathrm{n}}}_{\mathrm{S}_{0\mathrm{N}}} \cup \Gamma^{u_{0\mathrm{n}}}_{\mathrm{S}_{0\mathrm{D}}} \cup \Gamma^{u_{0\mathrm{n}}}_{\mathrm{S}_0-}$, avec

$$\begin{aligned} \Gamma^{u_{0\mathrm{n}}}_{\mathrm{S}_{0\mathrm{N}}} &:= \{s \in \Gamma_{\mathrm{S}_0} \mid u_{0\mathrm{n}}(s) \neq 0\},\\ \Gamma^{u_{0\mathrm{n}}}_{\mathrm{S}_{0\mathrm{D}}} &:= \{s \in \Gamma_{\mathrm{S}_0} \mid u_{0\mathrm{n}}(s) = 0 \text{ et } \sigma_\mathrm{n}(u_0)(s) < 0\},\\ \Gamma^{u_{0\mathrm{n}}}_{\mathrm{S}_0-} &:= \{s \in \Gamma_{\mathrm{S}_0} \mid u_{0\mathrm{n}}(s) = 0 \text{ et } \sigma_\mathrm{n}(u_0)(s) = 0\}. \end{aligned}$$

2. Nous déduisons dans le Corollaire 9.1.5 que, sous certaines hypothèses, la dérivée de forme directionnelle, définie par $u'_0 := \overline{u}'_0 - \nabla u_0\theta \in \mathrm{H}^1_\mathrm{D}(\Omega_0,\mathbb{R}^d)$, est l'unique solution faible du problème de Signorini

$$\begin{cases} -\mathrm{div}(\mathrm{Ae}(u'_0)) = 0 & \text{dans } \Omega_0,\\ u'_0 = 0 & \text{sur } \Gamma_\mathrm{D},\\ \sigma_\tau(u'_0) = h^s(\theta)_\tau & \text{sur } \Gamma_{\mathrm{S}_0},\\ \sigma_\mathrm{n}(u'_0) = h^s(\theta)_\mathrm{n} & \text{sur } \Gamma_{\mathrm{S}^{u_{0\mathrm{n}}}_{0,\mathrm{N}}},\\ u'_{0\mathrm{n}} = W(\theta)_\mathrm{n} & \text{sur } \Gamma_{\mathrm{S}^{u_{0\mathrm{n}}}_{0,\mathrm{D}}},\\ u'_{0\mathrm{n}} \leq W(\theta)_\mathrm{n}, \sigma_\mathrm{n}(u'_0) \leq h^s(\theta)_\mathrm{n} \text{ et } (u'_{0\mathrm{n}} - W(\theta)_\mathrm{n})(\sigma_\mathrm{n}(u'_0) - h^s(\theta)_\mathrm{n}) = 0 & \text{sur } \Gamma_{\mathrm{S}^{u_{0\mathrm{n}}}_{0,\mathrm{S}}}, \end{cases}$$

où $W(\theta) := (\nabla\theta u_0) - (\nabla u_0 \theta) \in \mathrm{H}^{1/2}(\Gamma_0, \mathbb{R}^d)$,

$$h^s(\theta) := \theta \cdot \mathrm{n} \left(\partial_{\mathrm{n}} \left(\mathrm{Ae}(u_0)\mathrm{n}\right) - \partial_{\mathrm{n}} \left(\mathrm{Ae}(u_0)\right) \mathrm{n}\right) + \mathrm{Ae}(u_0) \nabla_\tau \left(\theta \cdot \mathrm{n}\right) \\ - \nabla(\mathrm{Ae}(u_0)\mathrm{n})\theta - \sigma_{\mathrm{n}}(u_0) \left(\mathrm{div}_\tau(\theta)\mathrm{I} + \nabla\theta^\top\right) \mathrm{n} \in \mathrm{L}^2(\Gamma_{\mathrm{S}_0}, \mathbb{R}^d),$$

et où $\mathrm{div}_\tau(\theta) := \mathrm{div}(\theta) - (\nabla\theta\mathrm{n} \cdot \mathrm{n}) \in \mathrm{L}^\infty(\Gamma_0)$ est la divergence tangentielle de $\theta$, $\partial_{\mathrm{n}} \left(\mathrm{Ae}(u_0)\mathrm{n}\right) := \nabla(\mathrm{Ae}(u_0)\mathrm{n})\mathrm{n}$ est la dérivée normale de $\mathrm{Ae}(u_0)\mathrm{n}$, et $\partial_{\mathrm{n}} \left(\mathrm{Ae}(u_0)\right)$ est la matrice dont la $i$-ième ligne est le vecteur $\partial_{\mathrm{n}} \left(\mathrm{Ae}(u_0)_i\right) := \nabla(\mathrm{Ae}(u_0)_i)\mathrm{n}$, où $\mathrm{Ae}(u_0)_i$ est la $i$-ième ligne de la matrice $\mathrm{Ae}(u_0)$ pour tout $i \in [[1, d]]$.

3. Finalement les deux précédents points sont utilisés afin d'obtenir le Corollaire 9.2.2, lequel affirme que, sous certaines hypothèses, le gradient de forme de $\mathcal{J}$ en $\Omega_0$ dans la direction $\theta \in \mathcal{C}^{2,\infty}_{\mathrm{D}}(\mathbb{R}^d, \mathbb{R}^d)$ est donné par

$$\mathcal{J}'(\Omega_0)(\theta) = \int_{\Gamma_{\mathrm{S}_0}} \left(\theta \cdot \mathrm{n} \left(\frac{\mathrm{Ae}(u_0) : \mathrm{e}(u_0)}{2} - f \cdot u_0\right) + \mathrm{Ae}(u_0)\mathrm{n} \cdot (\nabla\theta u_0 - \nabla u_0 \theta)\right).$$

Notons que, similairement au problème d'optimisation de forme impliquant la loi de frottement de Tresca dans le modèle scalaire traité dans le Chapitre 8, $\mathcal{J}'(\Omega_0)$ est explicite et linéaire par rapport à la direction $\theta$, et par conséquent nous permet d'exhiber une direction de descente. Ainsi, en utilisant cette direction de descente ainsi que l'algorithme d'Uzawa, nous effectuons des simulations numériques afin de résoudre le problème d'optimisation de forme (4) sur un exemple en deux dimensions présenté dans la Sous-section 9.3.2 (voir Figure 4).

(iii) Nous concluons avec le Chapitre 10, dans lequel nous traitons un problème de contrôle optimal impliquant la loi de frottement de Tresca dans le modèle de l'élasticité linéaire. Plus précisément, nous considérons $d \in \{2, 3\}$, $\Omega$ un ensemble de $\mathbb{R}^d$ qui est ouvert non vide borné connexe de classe $\mathcal{C}^1$, avec une frontière notée $\Gamma := \partial\Omega$, et nous notons n le vecteur normal unitaire extérieur à la frontière $\Gamma$. Nous supposons que $\Gamma$ se décompose en $\Gamma = \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{S}}$, où $\Gamma_{\mathrm{D}}$ et $\Gamma_{\mathrm{S}}$ sont deux ensembles mesurables (de mesure strictement positive) disjoints, inclus dans $\Gamma$, et que presque tout les points de $\Gamma_{\mathrm{S}}$ sont dans $\mathrm{int}_\Gamma(\Gamma_{\mathrm{S}})$. Nous supposons que $\Omega$ est un solide élastique qui satisfait le modèle de l'élasticité linéaire, et nous considérons $f \in \mathrm{L}^2(\Omega, \mathbb{R}^d)$, $h \in \mathrm{L}^2(\Gamma_{\mathrm{S}})$, $g_1 \in \mathrm{L}^\infty(\Gamma_{\mathrm{S}})$ qui satisfait $g_1 \geq m$ *p.p.* sur $\Gamma_{\mathrm{S}}$, où $m > 0$ est une constante positive, et $g_2 \in \mathrm{L}^\infty(\Gamma_{\mathrm{S}})$ qui satisfait $||g_2||_{\mathrm{L}^\infty(\Gamma_{\mathrm{S}})} > 0$. Nous considérons alors le problème de contrôle optimal suivant

$$\underset{z \in \mathcal{U}}{\text{minimiser}} \ \mathcal{J}(z), \tag{5}$$

où $\mathcal{J}$ est la fonctionnelle coût définie par

$$\begin{array}{rccl} \mathcal{J}: & \mathrm{V} & \longrightarrow & \mathbb{R} \\ & z & \longmapsto & \mathcal{J}(z) := \frac{1}{2} \left\|u(\ell(z))\right\|^2_{\mathrm{A}} + \frac{\beta}{2} \left\|\ell(z)\right\|^2_{\mathrm{L}^2(\Gamma_{\mathrm{S}})}, \end{array}$$

où $|| \cdot ||_{\mathrm{A}}$ est la norme associée au produit scalaire $\langle \cdot, \cdot \rangle_{\mathrm{A}}$ (voir (4.1)), V est un sous-ensemble ouvert

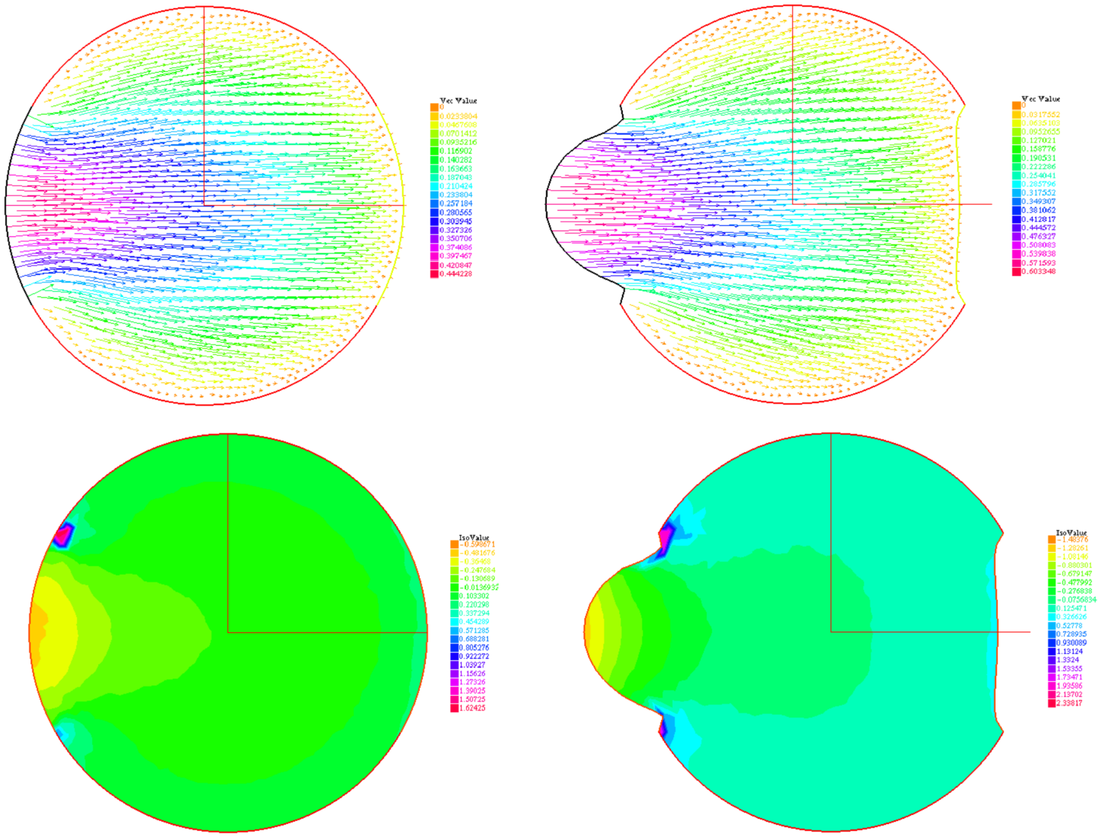

Figure 4 – Forme initiale (gauche) et la forme qui minimise $\mathcal{J}$ (droite), sous la contrainte de volume fixé $|\Omega_{\mathrm{ref}}| = \pi$. En haut sont affichées les valeurs de la solution du problème de Signorini, tandis qu'en bas sont affichées les valeurs de l'intégrande de $\mathcal{J}$.

de $\mathrm{L}^\infty(\Gamma_\mathrm{S})$ défini par

$$\mathrm{V} := \left\{ z \in \mathrm{L}^\infty(\Gamma_\mathrm{S}) \mid \exists C(z) > 0,\, \ell(z) > C(z) \text{ } p.p. \text{ sur } \Gamma_\mathrm{S} \right\},$$

où $\ell$ est l'application définie par $z \in \mathrm{L}^\infty(\Gamma_\mathrm{S}) \mapsto \ell(z) := g_1 + zg_2 \in \mathrm{L}^\infty(\Gamma_\mathrm{S})$, et où $u(\ell(z)) \in \mathrm{H}^1_\mathrm{D}(\Omega, \mathbb{R}^d)$ est l'unique solution du problème de Tresca

$$\begin{cases} -\mathrm{div}(\mathrm{A}\mathrm{e}(u)) = f & \text{dans } \Omega, \\ u = 0 & \text{sur } \Gamma_\mathrm{D}, \\ \sigma_\mathrm{n}(u) = h & \text{sur } \Gamma_\mathrm{S}, \\ \|\sigma_\tau(u)\| \leq \ell(z) \text{ et } u_\tau \cdot \sigma_\tau(u) + \ell(z)\, \|u_\tau\| = 0 & \text{sur } \Gamma_\mathrm{S}, \end{cases} \tag{6}$$

où $\beta > 0$ est une constante positive et où $\mathcal{U}$ est un sous-ensemble non vide convexe de V tel que $\mathcal{U}$ est un sous-ensemble fermé borné de $\mathrm{L}^2(\Gamma_\mathrm{S})$. Nous prouvons dans la Section 10.1 l'existence d'une solution au problème (5), et dans la Section 10.2 nous prouvons, en utilisant les résultats issus de la Partie II, que, sous certaines hypothèses (voir Théorème 10.2.1), $\mathcal{J}$ est Gateaux différentiable sur V et, pour tout $z_0 \in \mathrm{V}$, sa Gateaux différentielle est donnée par

$$\mathrm{d}_G\mathcal{J}(z_0)(z) = \int_{\Gamma_{\mathrm{S}_\mathrm{R}^{u_0,\ell(z_0)}}} zg_2\left(\beta\left(g_1 + z_0 g_2\right) - \|u_{0_\tau}\|\right) + \int_{\Gamma_{\mathrm{S}_\mathrm{D}^{u_0,\ell(z_0)}} \cup \Gamma_{\mathrm{S}_\mathrm{S}^{u_0,\ell(z_0)}}} \beta z g_2\left(g_1 + z_0 g_2\right),$$

pour tout $z \in \mathrm{L}^\infty(\Gamma_\mathrm{S})$, où $u_0 := u(\ell(z_0))$ est l'unique solution du problème de Tresca (6) avec $\ell(z_0) = g_1 + z_0 g_2$, et où $\Gamma_\mathrm{S}$ se décompose, à un ensemble de mesure nulle près, en $\Gamma_{\mathrm{S}_\mathrm{R}^{u_0,\ell(z_0)}} \cup \Gamma_{\mathrm{S}_\mathrm{D}^{u_0,\ell(z_0)}} \cup \Gamma_{\mathrm{S}_\mathrm{S}^{u_0,\ell(z_0)}}$, avec

$$\begin{aligned}
\Gamma_{\mathrm{S}_\mathrm{R}^{u_0,\ell(z_0)}} &:= \left\{ s \in \Gamma_\mathrm{S} \mid u_{0_\tau}(s) \neq 0 \right\}, \\
\Gamma_{\mathrm{S}_\mathrm{D}^{u_0,\ell(z_0)}} &:= \left\{ s \in \Gamma_\mathrm{S} \mid u_{0_\tau}(s) = 0 \text{ et } \tfrac{\sigma_\tau(u_0)(s)}{\ell(z_0)(s)} \in \mathrm{B}(0,1) \cap (\mathbb{R}\mathrm{n}(s))^\perp \right\}, \\
\Gamma_{\mathrm{S}_\mathrm{S}^{u_0,\ell(z_0)}} &:= \left\{ s \in \Gamma_\mathrm{S} \mid u_{0_\tau}(s) = 0 \text{ et } \tfrac{\sigma_\tau(u_0)(s)}{\ell(z_0)(s)} \in \partial\mathrm{B}(0,1) \cap (\mathbb{R}\mathrm{n}(s))^\perp \right\}.
\end{aligned}$$

Enfin, dans la Section 10.3, nous supposons $||g_2||_{\mathrm{L}^\infty(\Gamma_\mathrm{S})} < m$ où $m > 0$ est la constante déjà introduite précédemment, et nous considérons $\mathcal{U} := \{z \in \mathrm{L}^2(\Gamma_\mathrm{S}) \mid -1 \leq z \leq 1 \text{ } p.p. \text{ sur } \Gamma_\mathrm{S}\}$, qui est alors bien un ensemble convexe inclus dans V et également un fermé borné de $\mathrm{L}^2(\Gamma_\mathrm{S})$. Nous utilisons alors l'expression de la Gateaux différentielle de $\mathcal{J}$, laquelle nous permet d'obtenir une direction de descente, que l'on combine avec un algorithme de gradient projeté pour prendre en compte les contraintes présentes dans l'ensemble $\mathcal{U}$, afin de résoudre numériquement le problème de contrôle optimal (5) sur un exemple en deux dimensions présenté dans la Sous-section 10.3.2 (voir Figure 5).

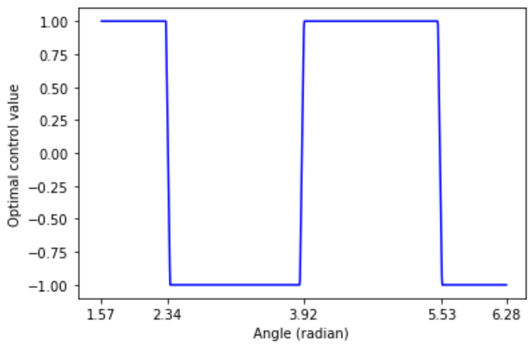


Figure 5 – Valeurs du contrôle optimal sur $\Gamma_\mathrm{S} := \left\{(\cos\theta, \sin\theta) \in \Gamma \mid \frac{\pi}{2} < \theta < 2\pi\right\}$.

# General notations

| | |
|---|---|
| $\mathbb{N}$ | The set of nonnegative integers |
| $\mathbb{N}^*$ | The set of positive integers |
| $\mathbb{R}$ | The set of real numbers |
| $\mathbb{R}_+$ | The set of nonnegative real numbers |
| $\mathbb{R}_-$ | The set of nonpositive real numbers |
| $\mathbb{R}_+^*$ | The set of positive real numbers |
| $d$ | The dimension, $d \in \mathbb{N}^*$ or $d \in \{2,3\}$ depending on the section |
| $\lvert \cdot \rvert$ | The absolute value map |
| $\cdot$ | The Euclidean scalar product of $\mathbb{R}^d$ |
| $\lvert\lvert \cdot \rvert\rvert$ | The Euclidean norm of $\mathbb{R}^d$ |
| $\mathbb{R}_- x$ | The linear subspace of $\mathbb{R}^d$ defined by $\mathbb{R}_- x := \{y \in \mathbb{R}^d \mid \exists \lambda \leq 0 \text{ such that } y = \lambda x\}$ for any $x \in \mathbb{R}^d$ |
| $\Omega$ | Nonempty bounded connected open subset of $\mathbb{R}^d$ with a Lipschitz boundary |
| $\Gamma$ | The boundary of $\Omega$ |
| $\mathrm{n}$ | The outward-pointing unit normal vector to $\Gamma$ |
| $\mathrm{int}_\Gamma$ | The interior relative to $\Gamma$ |
| $x_\mathrm{n}$ | The normal component of $x \in \mathbb{R}^d$ defined by $x_\mathrm{n} := x \cdot \mathrm{n}$ |
| $x_\tau$ | The tangential component of $x \in \mathbb{R}^d$ defined by $x_\tau := x - x_\mathrm{n}\mathrm{n}$ |
| $x_{(\mathrm{Mn})^\perp}$ | The component of $x \in \mathbb{R}^d$ defined by $x_{(\mathrm{Mn})^\perp} := x - (x \cdot \frac{\mathrm{Mn}}{\lvert\lvert \mathrm{Mn} \rvert\rvert^2})\mathrm{Mn} \in \mathbb{R}^d$, where $\mathrm{M} \in \mathbb{R}^{d \times d}$ |
| *a.e.* | Abbreviation for *almost everywhere* |
| *q.e.* | Abbreviation for *quasi everywhere* |
| $\mathcal{D}(\Omega, \mathbb{R}^d)$ | The set of functions that are infinitely differentiable with compact support in $\Omega$ |
| $\mathcal{D}(\Omega)$ | The set $\mathcal{D}(\Omega, \mathbb{R})$ |
| $\mathcal{D}'(\Omega, \mathbb{R}^d)$ | The set of distributions on $\Omega$ |
| $\mathcal{D}'(\Omega)$ | The set $\mathcal{D}'(\Omega, \mathbb{R})$ |
| $\mathrm{L}^p(\mathrm{X}, \mathbb{R}^d)$ | The usual Lebesgue spaces endowed with their standard norms, where X is measurable set and $p \in \mathbb{N}^* \cup \{+\infty\}$ |
| $\mathrm{L}^p(\mathrm{X})$ | The Lebesgue space $\mathrm{L}^p(\mathrm{X}, \mathbb{R})$ |
| $\mathrm{W}^{m,p}(\Omega, \mathbb{R}^d)$ | The usual Sobolev spaces endowed with their standard norms, where $(m,p) \in \mathbb{N} \times \mathbb{N}^* \cup \{+\infty\}$ |
| $\mathrm{W}^{m,p}(\Omega)$ | The Sobolev space $\mathrm{W}^{m,p}(\Omega, \mathbb{R})$ |
| $\mathrm{H}^m(\Omega, \mathbb{R}^d)$ | The Sobolev space $\mathrm{W}^{m,2}(\Omega, \mathbb{R}^d)$ |

| | |
|---|---|
| $\mathrm{H}^m(\Omega)$ | The Sobolev space $\mathrm{H}^m(\Omega,\mathbb{R})$ |
| $\mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d)$ | The linear subspace of $\mathrm{H}^1(\Omega,\mathbb{R}^d)$ given by $\{v \in \mathrm{H}^1(\Omega,\mathbb{R}^d) \mid v = 0$ *a.e.* on $\Gamma_\mathrm{D}\}$ where $\Gamma_\mathrm{D} \subset \Gamma$ |
| $\mathrm{H}^{1/2}(\Gamma,\mathbb{R}^d)$ | The usual trace space |
| $\mathrm{H}^{1/2}(\Gamma)$ | The space $\mathrm{H}^{1/2}(\Gamma,\mathbb{R})$ |
| $\mathrm{H}^{-1/2}(\Gamma,\mathbb{R}^d)$ | The dual space of $\mathrm{H}^{1/2}(\Gamma,\mathbb{R}^d)$ |
| $\mathrm{H}^{-1/2}(\Gamma)$ | The dual space of $\mathrm{H}^{1/2}(\Gamma)$ |
| $\partial_\mathrm{n}(u)$ | The normal derivative of u |
| A | The stiffness tensor (see Chapter 1) |
| e | The infinitesimal strain tensor defined by $\mathrm{e}(u) := \frac{1}{2}(\nabla u + \nabla u^\top)$ for all $u \in \mathrm{H}^1(\Omega,\mathbb{R}^d)$ |
| $\sigma_\mathrm{n}(u)$ | The normal stress defined by $\sigma_\mathrm{n}(u) := \mathrm{Ae}(u)\mathrm{n}\cdot\mathrm{n}$ for all $u \in \mathrm{H}^1(\Omega,\mathbb{R}^d)$ |
| $\sigma_\tau(u)$ | The shear stress defined by $\sigma_\tau(u) := \mathrm{Ae}(u)\mathrm{n} - \sigma_\mathrm{n}(u)\mathrm{n}$ for all $u \in \mathrm{H}^1(\Omega,\mathbb{R}^d)$ |
| $\langle\cdot,\cdot\rangle_{\mathrm{H}^1(\Omega)}$ | The usual scalar product on $\mathrm{H}^1(\Omega)$ |
| $\|\|\cdot\|\|_{\mathrm{H}^1(\Omega)}$ | The norm associated with the scalar product $\langle\cdot,\cdot\rangle_{\mathrm{H}^1(\Omega)}$ |
| $\langle\cdot,\cdot\rangle_\mathrm{A}$ | The scalar product on $\mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d)$ defined by (4.1) |
| $\|\|\cdot\|\|_\mathrm{A}$ | The norm associated with the scalar product $\langle\cdot,\cdot\rangle_\mathrm{A}$ |

Part I

# Reminders and preliminaries

In this first part, we introduce the results and tools involved all along the manuscript. In Chapter 1, the linear elastic model and the definitions of the Signorini unilateral conditions and of the Tresca friction law are presented. In Chapter 2, some notions from differential geometry, functional analysis and capacity theory are reminded. In Chapter 3, concepts and results from convex and variational analyses are described, such as the proximal operator, the Mosco epi-convergence, the twice epi-differentiability, etc. Finally, in Chapter 4, some boundary value problems are detailed: a Neumann problem, a Signorini problem and a Tresca friction problem.

We mention that some results presented here are only reminders, while some are new. In this last case their proofs are given.

Chapter 1

# Linear elastic model

For details on this chapter, we refer, for instance, to [17, 30, 34, 73]). Let $d \in \{2,3\}$ and $\Omega$ be an elastic solid, which means that it returns to its initial shape when stresses are no longer exerted. The elastic solid is characterized by the Cauchy stress tensor $\sigma : u \in \mathrm{H}^1(\Omega, \mathbb{R}^d) \mapsto \sigma(u) \in \mathbb{R}^d$, a symmetric tensor of rank 2 which describes the stress state of the elastic solid, and where $u \in \mathrm{H}^1(\Omega, \mathbb{R}^d)$ is called a displacement field. Moreover, the stress vector $T$ is defined by

$$\begin{array}{rccl} T: & \mathrm{H}^1(\Omega,\mathbb{R}^d) & \longrightarrow & \mathbb{R}^d \\ & u & \longmapsto & T(u) := \sigma(u)\mathrm{n} = \sigma(u)\mathrm{n}\cdot\mathrm{n} + \sigma_\tau(u), \end{array}$$

where $\sigma_\mathrm{n}(u)$ is the normal stress and $\sigma_\tau(u)$ the shear stress. Note that, if $\sigma_\mathrm{n} < 0$, then the exterior exerts compressive forces on the boundary of the elastic solid, while $\sigma_\mathrm{n} > 0$ means that the exterior exerts tractions forces. In the case where the elastic solid is subject to small deformations, then it satisfies the linear elastic model given by,

$$\sigma(u) = \mathrm{Ae}(u),$$

for all $u \in \mathrm{H}^1(\Omega, \mathbb{R}^d)$, where A is a symmetric tensor of rank 4 called the stiffness tensor, and e is the infinitesimal strain tensor defined by

$$\begin{array}{rccl} \mathrm{e}: & \mathrm{H}^1(\Omega,\mathbb{R}^d) & \longrightarrow & \mathbb{R}^{d\times d} \\ & u & \longmapsto & \mathrm{e}(u) := \dfrac{1}{2}\left(\nabla u + \nabla u^\top\right). \end{array}$$

In this work, we assume that all coefficients of A are constant (denoted $a_{ijkl}$ for all $(i,j,k,l) \in \{1,...,d\}^4$), and there exists one constant $\alpha > 0$ such that, all coefficients of A and $e$ (denoted $\epsilon_{ij}$ for all $(i,j) \in \{1,...,d\}^2$), satisfy

$$a_{ijkl} = a_{jikl} = a_{lkij}, \ \sum_{i=1}^d\sum_{j=1}^d\sum_{k=1}^d\sum_{l=1}^d a_{ijkl}\epsilon_{ij}(v)(x)\epsilon_{kl}(u)(x) \geq \alpha \sum_{i=1}^d\sum_{j=1}^d \epsilon_{ij}(v)(x)\epsilon_{ij}(u)(x), \tag{1.1}$$

for all displacement field $v, u \in \mathrm{H}^1(\Omega, \mathbb{R}^d)$ and for almost all $x \in \Omega$. Note that, from the assumptions on A, one has $\mathrm{Ae}(v) = \mathrm{A}\nabla v$, for all $v \in \mathrm{H}^1_\mathrm{D}(\Omega, \mathbb{R}^d)$.

**Remark 1.0.1.** In the particular case where the elastic solid is isotropic, which means that its mechanical properties are identical in all directions, then the stiffness tensor A is given, for all symmetric matrix S, by

$$\mathrm{AS} = 2\mu\mathrm{S} + \lambda\mathrm{Tr}(\mathrm{S})\mathrm{I},$$

where $\mu > 0$ and $\lambda > 0$ are Lamé parameters (see, e.g., [30, 34]), Tr(S) is the trace of the matrix S, and I stands for the identity operator. Thus, one deduces the Hooke's law given by

$$\sigma(u) = 2\mu\mathrm{e}(u) + \lambda\mathrm{Tr}(\mathrm{e}(u))\mathrm{I}, \qquad \forall u \in \mathrm{H}^1(\Omega, \mathbb{R}^d).$$

In this work, we assume that the elastic solid is static and subjected to volume forces $f \in \mathrm{L}^2(\Omega, \mathbb{R}^d)$. Therefore, the equilibrium equation is given by (see, e.g., [30, Chapter 5 p.45]),

$$-\mathrm{div}\,(\mathrm{Ae}(u)) = f \text{ in } \Omega. \tag{1.2}$$

Now let us introduce the Signorini unilateral conditions and the Tresca friction law, on which we will focus in this manuscript.

## 1.1 Signorini unilateral conditions

A. Signorini proposed in 1933 (see [75]) some boundary conditions which describe a non-permeable contact between an elastic solid and a rigid foundation. More precisely, assume that $\Omega$ is in contact with a rigid foundation on a part $\Gamma_{\mathrm{S}}$ of its boundary. Then the Signorini unilateral conditions are given by

$$u_{\mathrm{n}} \leq 0, \quad \sigma_{\mathrm{n}}(u) \leq 0 \quad \text{and} \quad u_{\mathrm{n}}\sigma_{\mathrm{n}}(u) = 0 \quad \text{on } \Gamma_{\mathrm{S}},$$

where $u_{\mathrm{n}} \in \mathrm{L}^2(\Gamma)$ is the normal component of the displacement field $u \in \mathrm{H}^1(\Omega, \mathbb{R}^d)$. The physical sense of those conditions is:

(i) $u_{\mathrm{n}} \leq 0$ describes the non-permeability of $\Gamma_{\mathrm{S}}$ with the rigid foundation;

(ii) $\sigma_{\mathrm{n}}(u) \leq 0$ means that there are only compressive stresses exerted on $\Gamma_{\mathrm{S}}$ by the rigid foundation;

(iii) $u_{\mathrm{n}}\sigma_{\mathrm{n}}(u) = 0$ is the complementarity relation. If $u_{\mathrm{n}} < 0$, then the rigid foundation is not in contact with $\Gamma_{\mathrm{S}}$, thus it does not exert compressive stresses i.e. $\sigma_{\mathrm{n}}(u) = 0$. If $\sigma_{\mathrm{n}}(u) < 0$, then the elastic solid and the rigid foundation are in contact, thus $u_{\mathrm{n}} = 0$ since they is no permeability.

Moreover, in some sections of the manuscript, we will consider the generalized Signorini unilateral conditions, that are

$$u_{\mathrm{n}} \leq w_{\mathrm{n}} \text{ (resp. } \geq w_{\mathrm{n}}), \quad \sigma_{\mathrm{n}}(u) \leq g \text{ (resp. } \geq g) \quad \text{and} \quad (u_{\mathrm{n}} - w)(\sigma_{\mathrm{n}}(u) - g) = 0 \quad \text{on } \Gamma_{\mathrm{S}},$$

where $w \in \mathrm{H}^1(\Omega, \mathbb{R}^d)$ and $g \in \mathrm{L}^2(\Gamma_{\mathrm{S}})$ are given data.

**Remark 1.1.1.** In this work, we will also consider the scalar model, i.e. displacement fields are in $\mathrm{H}^1(\Omega)$. In that case the scalar Signorini unilateral conditions (see, e.g., [52, Section 1]) are given by

$$u \leq w \text{ (resp. } \geq w), \quad \partial_{\mathrm{n}} u \leq g \text{ (resp. } \geq g) \quad \text{and} \quad (u - w)(\partial_{\mathrm{n}} u - g) = 0 \quad \text{on } \Gamma_{\mathrm{S}},$$

where $w \in \mathrm{H}^1(\Omega)$, $g \in \mathrm{L}^2(\Gamma_{\mathrm{S}})$ and $\partial_{\mathrm{n}} u$ is the normal derivative (when it exists) of $u \in \mathrm{H}^1(\Omega)$ which solves a Laplace type equation instead of the linear elasticity equation (1.2).

## 1.2 Coulomb friction law and Tresca friction law

The Coulomb friction law was introduced by C.A. Coulomb in 1785 (see [24]) and allows to modelize the friction between an elastic solid and a rigid foundation. More precisely, assume that $\Omega$ is in contact with a rigid foundation on a part $\Gamma_{\mathrm{S}}$ of its boundary. Then the Coulomb friction law is given by

$$\left\{\begin{array}{rcll} \|\sigma_\tau(u)\| & \leq & -\mu_s\sigma_{\mathrm{n}}(u), & \\ \text{if } \|\sigma_\tau(u)\| & < & -\mu_s\sigma_{\mathrm{n}}(u), & \text{then } u_\tau = 0, \\ \text{if } \|\sigma_\tau(u)\| & = & -\mu_s\sigma_{\mathrm{n}}(u), & \text{then there exists } \lambda \geq 0 \text{ such that } u_\tau = -\lambda\sigma_\tau(u), \end{array}\right.$$

on $\Gamma_{\mathrm{S}}$, where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^d$, $u_\tau$ is the tangential component of the displacement field $u \in \mathrm{H}^1(\Omega, \mathbb{R}^d)$ and $\mu_s > 0$ is the static coefficient of friction, which is an empirical property of the elastic solid and of the rigid foundation. Roughly speaking, this law physically means that, as long as the tangential stress is inferior to the friction threshold given by $-\mu_s\sigma_{\mathrm{n}}(u)$, then the elastic solid does not slide on the rigid foundation. When the friction threshold is reached, then the elastic solid slides in the same direction but in the opposite sense of the shear stress.

The Tresca friction law is a simplification of the Coulomb friction law, where the friction threshold does not depend on the normal stress $\sigma_{\mathrm{n}}(u)$. More precisely, this law is given by

$$\left\{\begin{array}{rcll} \|\sigma_\tau(u)\| & \leq & g, & \\ \text{if } \|\sigma_\tau(u)\| & < & g, & \text{then } u_\tau = 0, \\ \text{if } \|\sigma_\tau(u)\| & = & g, & \text{then there exists } \lambda \geq 0 \text{ such that } u_\tau = -\lambda\sigma_\tau(u), \end{array}\right.$$

where $g \in \mathrm{L}^2(\Gamma_{\mathrm{S}})$ is the friction term and is a positive function almost everywhere on $\Gamma_{\mathrm{S}}$. Moreover, using Cauchy–Schwarz inequality, the Tresca friction law is equivalent to

$$\|\sigma_\tau(u)\| \leq g \quad \text{and} \quad u_\tau \cdot \sigma_\tau(u) = -g\,\|u_\tau\|,$$

and it is with this formulation that we will use it, all along the manuscript.

**Remark 1.2.1.** In the scalar model i.e. the displacement fields are in $\mathrm{H}^1(\Omega)$, the scalar Tresca friction law is given by

$$|\partial_{\mathrm{n}} u| \leq g \quad \text{and} \quad u\partial_{\mathrm{n}} u = -g\,|u|,$$

where $\partial_{\mathrm{n}} u$ is the normal derivative (when it exists) of $u \in \mathrm{H}^1(\Omega)$ (see, e.g., [37, Section 1.3 Chapter 1]).

Chapter 2

# Differential geometry, functional analysis and capacity theory

This chapter concerns classical reminders that we recall here for the sake of completeness. The first section claims some results on differential geometry and functional analysis, such as the divergence formula or some Sobolev embeddings. The second section recalls notions on capacity theory used in this manuscript.

## 2.1 Results on differential geometry and functional analysis

Let us introduce

$$\mathrm{H}_{\mathrm{div}}(\Omega,\mathbb{R}^{d\times d}) := \{w \in \mathrm{L}^2(\Omega,\mathbb{R}^{d\times d}) \mid \mathrm{div}(w) \in \mathrm{L}^2(\Omega,\mathbb{R}^d)\}.$$

The next proposition, known as divergence formula, can be found in [7, Theorem 4.4.7 p.104].

**Proposition 2.1.1** (Divergence formula)**.** *If $w \in \mathrm{H}_{\mathrm{div}}(\Omega,\mathbb{R}^{d\times d})$, then $w$ admits a normal trace, denoted by $w\mathrm{n} \in \mathrm{H}^{-1/2}(\Gamma,\mathbb{R}^d)$, satisfying*

$$\int_\Omega \mathrm{div}(w)\cdot v + \int_\Omega w : \nabla v = \langle w\mathrm{n}, v\rangle_{\mathrm{H}^{-1/2}(\Gamma,\mathbb{R}^d)\times\mathrm{H}^{1/2}(\Gamma,\mathbb{R}^d)}, \qquad \forall v \in \mathrm{H}^1(\Omega,\mathbb{R}^d).$$

From the divergence formula, one deduces the Green formula.

**Proposition 2.1.2** (Green formula)**.** *Let $w \in \mathrm{H}^1(\Omega)$. If $\Delta w \in \mathrm{L}^2(\Omega)$, then $\nabla w$ admits a normal trace $\partial_\mathrm{n} w \in \mathrm{H}^{-1/2}(\Gamma)$ such that*

$$\int_\Omega v\Delta w + \int_\Omega \nabla w\cdot\nabla v = \langle \partial_\mathrm{n} w, v\rangle_{\mathrm{H}^{-1/2}(\Gamma)\times\mathrm{H}^{1/2}(\Gamma)}, \qquad \forall v \in \mathrm{H}^1(\Omega).$$

The following propositions will be useful and their proofs can be found in [46].

**Proposition 2.1.3.** *Let $\theta \in \mathcal{C}^1(\mathbb{R}^d,\mathbb{R}^d)\cap\mathrm{W}^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d)$ and $w \in \mathrm{H}^1(\Omega)$ such that $\Delta w \in \mathrm{L}^2(\Omega)$. Then the equality*

$$\Delta(\theta\cdot\nabla w) = \mathrm{div}\left((\Delta w)\,\theta - \mathrm{div}(\theta)\nabla w + (\nabla\theta + \nabla\theta^\top)\nabla w\right),$$

*holds true in* $\mathcal{D}'(\Omega)$.

**Proposition 2.1.4.** *Assume that* $\Gamma$ *is of class* $\mathcal{C}^2$ *and let* $\theta \in \mathcal{C}^1(\mathbb{R}^d,\mathbb{R}^d)$. *It holds that*

$$\int_\Gamma (\theta \cdot \nabla v + v\mathrm{div}_\tau(\theta)) = \int_\Gamma \theta \cdot \mathrm{n}(\partial_\mathrm{n} v + Hv), \qquad \forall v \in \mathrm{W}^{2,1}(\Omega,\mathbb{R}),$$

*where* $\mathrm{div}_\tau(\theta) := \mathrm{div}(\theta) - (\nabla\theta\mathrm{n}\cdot\mathrm{n}) \in \mathrm{L}^\infty(\Gamma)$ *is the tangential divergence of* $\theta$, $\partial_\mathrm{n} v := \nabla v \cdot \mathrm{n} \in \mathrm{L}^1(\Gamma,\mathbb{R})$ *stands for the normal derivative of* $v$, *and* $H$ *stands for the mean curvature of* $\Gamma$.

**Proposition 2.1.5.** *Assume that* $\Gamma$ *is of class* $\mathcal{C}^2$ *and let* $w \in \mathrm{H}^2(\Omega,\mathbb{R}^{d\times d})$. *It holds that*

$$\mathrm{div}(w) = \mathrm{div}_\tau(w_\tau) + Hw\mathrm{n} + (\partial_\mathrm{n} w)\,\mathrm{n} \qquad \text{a.e. on } \Gamma,$$

*where* $\mathrm{div}_\tau(w_\tau) \in \mathrm{L}^2(\Gamma,\mathbb{R}^d)$ *is the vector whose the i-th component is defined by* $(\mathrm{div}_\tau(w_\tau))_i := \mathrm{div}_\tau((w_i)_\tau) \in \mathrm{L}^2(\Gamma,\mathbb{R})$, *where* $w_i \in \mathbb{R}^d$ *is the i-th line of* $w$, *and where* $\partial_\mathrm{n} w \in \mathrm{L}^2(\Gamma,\mathbb{R}^{d\times d})$ *is the matrix whose the i-th line is the vector* $\partial_\mathrm{n} w_i := (\nabla w_i)\mathrm{n} \in \mathrm{L}^2(\Gamma,\mathbb{R}^d)$, *for all* $i \in [[1,d]]$. *Moreover it holds that*

$$\int_\Gamma v \cdot \mathrm{div}_\tau(w_\tau) = -\int_\Gamma w : \nabla_\tau v, \qquad \forall v \in \mathrm{H}^2(\Omega,\mathbb{R}^d),$$

*where* $\nabla_\tau v$ *is the matrix whose the i-th line is the tangential gradient* $\nabla_\tau v_i := \nabla v_i - (\partial_\mathrm{n} v_i)\mathrm{n} \in \mathrm{H}^{1/2}(\Gamma,\mathbb{R}^d)$, *for all* $i \in [[1,d]]$.

From the previous proposition, one can deduce the following result.

**Proposition 2.1.6.** *Assume that* $\Gamma$ *is of class* $\mathcal{C}^2$ *and let* $w \in \mathrm{H}^3(\Omega)$. *It holds that*

$$\Delta w = \Delta_\tau w + H\partial_\mathrm{n} w + \frac{\partial^2 w}{\partial \mathrm{n}^2} \qquad \text{a.e. on } \Gamma,$$

*where* $\Delta_\tau w \in \mathrm{L}^2(\Gamma)$ *stands for the Laplace-Beltrami operator of* $w$ *(see, e.g., [46, Definition 5.4.11 p.196]), and* $\frac{\partial^2 w}{\partial \mathrm{n}^2} := \mathrm{D}^2(w)\mathrm{n}\cdot\mathrm{n} \in \mathrm{L}^2(\Gamma)$, *where* $\mathrm{D}^2(w)$ *stands for the Hessian matrix of* $w$. *Moreover it holds that*

$$\int_\Gamma v\Delta_\tau w = -\int_\Gamma \nabla_\tau v \cdot \nabla_\tau w, \qquad \forall v \in \mathrm{H}^2(\Omega).$$

Now, let us consider a decomposition $\Gamma =: \Gamma_1 \cup \Gamma_2$ where $\Gamma_1$ and $\Gamma_2$ are two measurable disjoint subsets of $\Gamma$. Let us recall some embeddings useful in this work, that can be found for instance in [1, Chapter 4, p.79], [13], [16], [25, Chapter 7, Section 2 p.395], [33] and [38, Chapter 1, p.27].

**Proposition 2.1.7.** *The continuous and dense embeddings:*

- $\mathrm{H}^1(\Omega,\mathbb{R}^d) \hookrightarrow \mathrm{H}^{1/2}(\Gamma,\mathbb{R}^d) \hookrightarrow \mathrm{L}^2(\Gamma,\mathbb{R}^d) \hookrightarrow \mathrm{H}^{-1/2}(\Gamma,\mathbb{R}^d)$;
- $\mathrm{L}^2(\Gamma,\mathbb{R}^d) \hookrightarrow \mathrm{L}^1(\Gamma,\mathbb{R}^d)$;
- $\mathrm{H}^1(\Omega,\mathbb{R}^d) \hookrightarrow \mathrm{L}^2(\Omega,\mathbb{R}^d)$;
- $\mathrm{H}^{1/2}_{00}(\Gamma_1,\mathbb{R}^d) \hookrightarrow \mathrm{L}^2(\Gamma_1,\mathbb{R}^d) \hookrightarrow \mathrm{H}^{-1/2}_{00}(\Gamma_1,\mathbb{R}^d)$;

*are satisfied, where* $\mathrm{H}^{1/2}_{00}(\Gamma_1,\mathbb{R}^d)$ *can be identified to a linear subspace of* $\mathrm{H}^{1/2}(\Gamma,\mathbb{R}^d)$ *defined by*

$$\mathrm{H}^{1/2}_{00}(\Gamma_1,\mathbb{R}^d) := \left\{ v \in \mathrm{L}^2(\Gamma_1,\mathbb{R}^d) \mid \exists w \in \mathrm{H}^1(\Omega,\mathbb{R}^d),\ w = v \text{ a.e. on } \Gamma_1 \text{ and } w = 0 \text{ a.e. on } \Gamma_2 \right\},$$

*and* $\mathrm{H}_{00}^{-1/2}(\Gamma_1,\mathbb{R}^d)$ *stands for its dual space. Furthermore the dense and compact embedding*

$$\mathrm{H}^1(\Omega,\mathbb{R}^d) \hookrightarrow\!\!\!\!\rightarrow \mathrm{L}^2(\Gamma,\mathbb{R}^d),$$

*holds true, and, if* $d \in \{2,3\}$*, then we have the continuous embedding* $\mathrm{H}^1(\Omega,\mathbb{R}^d)\hookrightarrow\mathrm{L}^4(\Gamma,\mathbb{R}^d)$*.*

The next proposition can be found in [80, Section 2.9 p.56].

**Proposition 2.1.8.** *Let* $(\mathrm{V},\langle\cdot,\cdot\rangle_{\mathrm{V}})$ *and* $(\mathcal{H},\langle\cdot,\cdot\rangle_{\mathcal{H}})$ *be two real Hilbert spaces such that the continuous and dense embedding* $\mathrm{V}\hookrightarrow\mathcal{H}$ *holds, and where* $\|\cdot\|_{\mathcal{H}}$ *is the norm corresponding to the scalar product* $\langle\cdot,\cdot\rangle_{\mathcal{H}}$*. Let* $w \in \mathrm{V}'$*, where* $\mathrm{V}'$ *is the dual space of* $\mathrm{V}$*. If there exists* $C \geq 0$ *such that*

$$\langle w,v\rangle_{\mathrm{V}'\times\mathrm{V}} \leq C\,\|v\|_{\mathcal{H}}\,,$$

*for all* $v\in\mathrm{V}$*, then* $w$ *can be identified to an element* $h\in\mathcal{H}$ *with*

$$\langle w,v\rangle_{\mathrm{V}'\times\mathrm{V}} = \langle h,v\rangle_{\mathcal{H}}\,,$$

*for all* $v\in\mathrm{V}$*.*

## 2.2 Notions from capacity theory

Let us recall some notions from capacity theory (we refer to standard references such as [27, 40, 46, 58]). Let us consider $(\mathrm{X},\mathcal{B}(\mathrm{X}),\xi)$ be a positively measured topological space with its borelian $\sigma$-algebra, $\xi$ a Radon measure, and where $\mathrm{X}\subset\mathbb{R}^d$ is a locally compact set, admitting a countable compact covering. Let $\mathcal{H}\subset\mathrm{L}^2(\mathrm{X},\xi)$ be a vector space endowed with a scalar product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}}$ the corresponding norm.

**Definition 2.2.1.** *Consider* $\mathrm{B}\in\mathcal{B}(\mathrm{X})$ *and let us introduce the closed convex subset*

$$\mathrm{C}_{\mathrm{B}} := \{v\in\mathcal{H} \mid v\geq 1 \ \xi\text{-a.e. on a neighborhood of } \mathrm{B}\}\,.$$

*The capacity of* B *is defined by*

$$\mathrm{cap}(\mathrm{B}) := \left\|\mathrm{proj}_{\mathrm{C}_{\mathrm{B}}}(0)\right\|_{\mathcal{H}}^2\,,$$

*where* $\mathrm{proj}_{\mathrm{C}_{\mathrm{B}}}$ *is the projection operator onto the nonempty closed convex set* $\mathrm{C}_{\mathrm{B}}$*.*

**Definition 2.2.2.** *A property holds quasi everywhere (denoted q.e.) if it holds for all elements in a set except a subset of null capacity.*

**Definition 2.2.3.** *A function* $v:\mathrm{X}\to\mathbb{R}$ *is said quasi-continuous if it exists a decreasing sequence of open sets* $(w_n)_{n\in\mathbb{N}}$ *such that* $\mathrm{cap}(w_n)\to 0$ *when* $n\to+\infty$ *and* $f_{|\mathrm{X}\backslash w_n}$ *is continuous for all* $n\in\mathbb{N}$*.*

Now, let us assume that $(\mathcal{H},\langle\cdot,\cdot\rangle_{\mathcal{H}})$ is a Dirichlet space (see, e.g., [58] for the definition). Thus, one can prove the following proposition (see, e.g., [40, 46, 58]).

**Proposition 2.2.4.** *For all $v \in \mathcal{H}$, there exists a unique quasi-continuous representative in the class of $v$, for the q.e. equivalence relation.*

To conclude, let us give three examples of Dirichlet space (see [58] for the first example, [40] for the second one and [78, Chapter 4] for the third one).

**Example 2.2.5.** *The space $\mathcal{H} := \mathrm{H}^1(\Omega)$ endowed with its standard scalar product $\langle \cdot, \cdot \rangle_{\mathrm{H}^1(\Omega)}$ is a Dirichlet space.*

**Example 2.2.6.** *The space $\mathcal{H} := \mathrm{H}^{1/2}(\Gamma)$ endowed with the scalar product defined in [40, Example 6 p.629] is a Dirichlet space.*

**Example 2.2.7.** *Assume that $\Gamma$ is given by the following decomposition $\Gamma =: \Gamma_1 \cup \Gamma_2$, where $\Gamma_1$ and $\Gamma_2$ are two measurable disjoint subsets of $\Gamma$. Then,*

$$\mathcal{H} := \left\{ v \cdot \mathrm{n} \in \mathrm{H}^{1/2}(\Gamma_2, \mathbb{R}) \mid v \in \mathrm{H}^1(\Omega, \mathbb{R}^d) \text{ and } v = 0 \text{ a.e. on } \Gamma_1 \right\},$$

*is a Dirichlet space endowed with the scalar product defined in [78, Chapter 4, Eq. (4.192) p.208].*

CHAPTER 3

# CONVEX AND VARIATIONAL ANALYSES

In this chapter, we first recall some notions of convex analysis, in particular the so-called proximal operator which will have an importance in the analysis of the contact problems presented in the introduction. Then we give the definition of the notion of twice epi-differentiability and several propositions based on this, since the methodology proposed in this manuscript in order to perform sensitivity analysis of variational inequalities is highly based on this notion.

For notions and results presented in this chapter, we refer to [15, 26, 48, 59, 67] and [70, Chapter 12]. In what follows $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ stands for a general real Hilbert space.

## 3.1 Notions from convex analysis

Let us start with some classical definitions.

**Definition 3.1.1** (Orthogonal complement)**.** *Let* $\mathrm{P}$ *be a nonempty subset of* $\mathcal{H}$*. The orthogonal complement to* $\mathrm{P}$ *is the linear subspace of* $\mathcal{H}$ *defined by*

$$\mathrm{P}^{\perp} := \left\{ z \in \mathcal{H} \mid \langle z, p \rangle_{\mathcal{H}} = 0, \forall p \in \mathrm{P} \right\}.$$

**Definition 3.1.2** (Polar cone)**.** *Let* $\mathrm{P}$ *be a nonempty subset of* $\mathcal{H}$*. The polar cone to* $\mathrm{P}$ *is the nonempty closed convex cone of* $\mathcal{H}$ *defined by*

$$\mathrm{P}^{\circ} := \left\{ z \in \mathcal{H} \mid \langle z, p \rangle_{\mathcal{H}} \leq 0, \forall p \in \mathrm{P} \right\}.$$

**Definition 3.1.3** (Normal cone)**.** *Let* $\mathrm{C}$ *be a nonempty closed convex subset of* $\mathcal{H}$ *and* $x \in \mathrm{C}$*. The normal cone to* $\mathrm{C}$ *at* $x$ *is the nonempty closed convex cone of* $\mathcal{H}$ *defined by*

$$\mathrm{N}_{\mathrm{C}}(x) := \left\{ z \in \mathcal{H} \mid \langle z, c - x \rangle_{\mathcal{H}} \leq 0, \forall c \in \mathrm{C} \right\}.$$

**Definition 3.1.4** (Tangent cone)**.** *Let* $\mathrm{C}$ *be a nonempty closed convex subset of* $\mathcal{H}$ *and* $x \in \mathrm{C}$*. The tangent cone to* $\mathrm{C}$ *at* $x$ *is the nonempty closed convex cone of* $\mathcal{H}$ *defined by*

$$\mathrm{T}_{\mathrm{C}}(x) := \overline{\{ z \in \mathcal{H} \mid \exists \lambda > 0, x + \lambda z \in \mathrm{C} \}}.$$

The following definition of polyhedric sets will be important in the sensitivity analysis of contact problems involving Signorini unilateral conditions.

**Definition 3.1.5** (Polyhedric set). *Let* C *be a nonempty closed convex subset of* $\mathcal{H}$*. We say that* C *is polyhedric at* $x \in \mathrm{C}$ *for* $y \in \mathrm{N_C}(x)$ *if*

$$\mathrm{T_C}(x) \cap (\mathbb{R}y)^{\perp} = \overline{\{z \in \mathcal{H} \mid \exists \lambda > 0, x + \lambda z \in \mathrm{C}\} \cap (\mathbb{R}y)^{\perp}}.$$

**Remark 3.1.6.** Recall that, in finite dimension, polyhedric sets reduce to polyhedral sets, which is the intersection of a finite set of closed half-spaces (see, e.g., [54]).

The following classical result is important to prove that there exists a unique weak solution to the tangential Signorini problem (see Subsection 4.2.3).

**Proposition 3.1.7.** *Let us consider* $\phi : \mathcal{H} \to \mathbb{R}$ *a convex and Fréchet differentiable function on* $\mathcal{H}$*,* C *a nonempty convex subset of* $\mathcal{H}$*,* $y \in \mathrm{C}$ *and* $x \in \mathcal{H}$*. Then the following variational inequalities are equivalent:*

*(i)* $\varphi(z) - \varphi(y) \geq \langle x - y, z - y \rangle_{\mathcal{H}}, \qquad \forall z \in \mathrm{C};$

*(ii)* $\langle \nabla\varphi(y), z - y \rangle_{\mathcal{H}} \geq \langle x - y, z - y \rangle_{\mathcal{H}}, \qquad \forall z \in \mathrm{C}.$

**Definition 3.1.8** (Domain and epigraph). *Let* $\phi : \mathcal{H} \to \mathbb{R} \cup \{\pm\infty\}$*. The domain and the epigraph of* $\phi$ *are respectively defined by*

$$\mathrm{dom}\,(\phi) := \{x \in \mathcal{H} \mid \phi(x) < +\infty\} \quad \textit{and} \quad \mathrm{epi}\,(\phi) := \{(x,t) \in \mathcal{H} \times \mathbb{R} \mid \phi(x) \leq t\}.$$

Recall that $\phi : \mathcal{H} \to \mathbb{R} \cup \{\pm\infty\}$ is said to be *proper* if $\mathrm{dom}(\phi) \neq \emptyset$ and $\phi(x) > -\infty$ for all $x \in \mathcal{H}$. Moreover, $\phi$ is a convex (resp. lower semi-continuous) function on $\mathcal{H}$ if and only if $\mathrm{epi}(\phi)$ is a convex (resp. closed) subset of $\mathcal{H} \times \mathbb{R}$.

The following example is used in this work, in order to investigate the sensitivity analysis of a Signorini problem.

**Example 3.1.9.** *Let* C *be a nonempty closed convex subset of* $\mathcal{H}$*, and* $\iota_{\mathrm{C}}$ *be the indicator function of* C*, defined by* $\iota_{\mathrm{C}}(x) := 0$ *if* $x \in \mathrm{C}$*, and* $\iota_{\mathrm{C}}(x) := +\infty$ *otherwise. Then, for all* $x \in \mathrm{C}$*,*

$$\partial\iota_{\mathrm{C}}(x) = \mathrm{N_C}(x).$$

**Definition 3.1.10** (Proximal operator). *Let* $\phi : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ *be a proper lower semi-continuous convex function. The proximal operator associated with* $\phi$ *is the map* $\mathrm{prox}_{\phi} : \mathcal{H} \to \mathcal{H}$ *defined by*

$$\mathrm{prox}_{\phi}(x) := \underset{y \in \mathcal{H}}{\mathrm{argmin}} \left[\phi(y) + \frac{1}{2}\|y - x\|^2_{\mathcal{H}}\right] = (\mathrm{I} + \partial\phi)^{-1}(x),$$

*for all* $x \in \mathcal{H}$*, where* $\mathrm{I} : \mathcal{H} \to \mathcal{H}$ *stands for the identity operator.*

**Remark 3.1.11.** Note that, if $\phi := \iota_{\mathrm{C}}$, where $\iota_{\mathrm{C}}$ is the indicator function of the nonempty closed convex subset $\mathrm{C} \subset \mathcal{H}$ (see Example 3.1.9 for the definition), then $\iota_{\mathrm{C}}$ is a proper lower semi-continuous convex function and

$$\mathrm{prox}_{\iota_{\mathrm{C}}} = \mathrm{proj}_{\mathrm{C}},$$

where $\text{proj}_C$ is the projection operator on C. This equality is important for the sensitivity analysis of a Signorini problem (see Sections 5.1 and 6.1).

We underline that the proximal operator has been introduced by J.J. Moreau in 1965 (see [60, 61]) and can be seen as a generalization of the classical projection operator onto a nonempty closed convex subset. Recall that, if $\psi : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ is a proper, lower semi-continuous and convex function, then its subdifferential $\partial\psi$ is a maximal monotone operator (see, e.g., [67]), and thus its proximal operator $\text{prox}_\psi : \mathcal{H} \to \mathcal{H}$ is well-defined, single-valued and nonexpansive, i.e. Lipschitz continuous with modulus 1 (see, e.g., [15, Chapter II]).

## 3.2 Twice epi-differentiability

As mentioned in Introduction, the solutions to the contact problems considered in this work can be expressed via the proximal operator. Therefore, the sensitivity analysis of those problems is related to the differentiability (in a generalized sense) of the involved proximal operator. To investigate this issue, we will use the notion of twice epi-differentiability introduced by R.T. Rockafellar in 1985 (see [68]) defined as the Mosco epi-convergence of second-order difference quotient functions. Our aim in what follows is to provide reminders and backgrounds on these notions for the reader's convenience. For more details, we refer to [70, Chapter 7, Section B p.240] for the finite-dimensional case and to [28] for the infinite-dimensional case. The strong (resp. weak) convergence of a sequence in $\mathcal{H}$ will be denoted by $\to$ (resp. $\rightharpoonup$) and note that all limits with respect to $t$ will be considered for $t \to 0^+$.

**Definition 3.2.1** (Mosco convergence)**.** *The outer, weak-outer, inner and weak-inner limits of a parameterized family $(S_t)_{t>0}$ of subsets of $\mathcal{H}$ are respectively defined by*

$$\begin{aligned}
\limsup S_t &:= \left\{x \in \mathcal{H} \mid \exists (t_n)_{n\in\mathbb{N}} \to 0^+,\ \exists\,(x_n)_{n\in\mathbb{N}} \to x,\ \forall n \in \mathbb{N},\ x_n \in S_{t_n}\right\},\\
\text{w-}\limsup S_t &:= \left\{x \in \mathcal{H} \mid \exists (t_n)_{n\in\mathbb{N}} \to 0^+,\ \exists\,(x_n)_{n\in\mathbb{N}} \rightharpoonup x,\ \forall n \in \mathbb{N},\ x_n \in S_{t_n}\right\},\\
\liminf S_t &:= \left\{x \in \mathcal{H} \mid \forall (t_n)_{n\in\mathbb{N}} \to 0^+,\ \exists\,(x_n)_{n\in\mathbb{N}} \to x,\ \exists N \in \mathbb{N},\ \forall n \geq N,\ x_n \in S_{t_n}\right\},\\
\text{w-}\liminf S_t &:= \left\{x \in \mathcal{H} \mid \forall (t_n)_{n\in\mathbb{N}} \to 0^+,\ \exists\,(x_n)_{n\in\mathbb{N}} \rightharpoonup x,\ \exists N \in \mathbb{N},\ \forall n \geq N,\ x_n \in S_{t_n}\right\}.
\end{aligned}$$

*The family $(S_t)_{t>0}$ is said to be Mosco convergent if* $\text{w-}\limsup S_t \subset \liminf S_t$*. In that case all the previous limits are equal and we write*

$$\text{M-lim } S_t := \liminf S_t = \limsup S_t = \text{w-}\liminf S_t = \text{w-}\limsup S_t.$$

**Definition 3.2.2** (Mosco epi-convergence)**.** *Let $(\psi_t)_{t>0}$ be a parameterized family of functions $\psi_t : \mathcal{H} \to \mathbb{R} \cup \{\pm\infty\}$ for all $t > 0$. We say that $(\psi_t)_{t>0}$ is Mosco epi-convergent if* $(\text{epi}(\psi_t))_{t>0}$ *is Mosco convergent in $\mathcal{H} \times \mathbb{R}$. Then we denote by* $\text{ME-lim }\psi_t : \mathcal{H} \to \mathbb{R} \cup \{\pm\infty\}$ *the function characterized by its epigraph* $\text{epi}\,(\text{ME-lim }\psi_t) := \text{M-lim epi}\,(\psi_t)$ *and we say that $(\psi_t)_{t>0}$ Mosco epi-converges to* $\text{ME-lim }\psi_t$.

**Remark 3.2.3.** In Definition 3.2.2, the abbreviation ME stands for the *Mosco Epi-convergence* (which is related to functions), while the abbreviation M stands for the *Mosco convergence* (related to subsets).

The proof of the next proposition can be found in [11, Proposition 3.19 p.297].

**Proposition 3.2.4** (Characterization of Mosco epi-convergence)**.** *Let* $(\psi_t)_{t>0}$ *be a parameterized family of functions* $\psi_t : \mathcal{H} \to \mathbb{R} \cup \{\pm\infty\}$ *for all* $t > 0$ *and let* $\psi : \mathcal{H} \to \mathbb{R} \cup \{\pm\infty\}$*. Then* $(\psi_t)_{t>0}$ *Mosco epi-converges to* $\psi$ *if and only if, for all* $x \in \mathcal{H}$*, the two conditions*

(i) *there exists* $(x_t)_{t>0} \to x$ *such that* $\limsup \psi_t(x_t) \leq \psi(x)$*;*

(ii) *for all* $(x_t)_{t>0} \rightharpoonup x$*,* $\liminf \psi_t(x_t) \geq \psi(x)$*;*

*are satisfied.*

Now let us recall the notion of twice epi-differentiability introduced by R.T. Rockafellar in [68] that generalizes the classical notion of second-order derivative to nonsmooth convex functions.

**Definition 3.2.5** (Twice epi-differentiability)**.** *A proper lower semi-continuous convex function* $\Psi : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ *is said to be twice epi-differentiable at* $x \in \mathrm{dom}(\Psi)$ *for* $y \in \partial\Psi(x)$ *if the family of second-order difference quotient functions* $(\delta_t^2\Psi(x|y))_{t>0}$ *defined by*

$$\begin{array}{rccl} \delta_t^2\Psi(x|y): & \mathcal{H} & \longrightarrow & \mathbb{R} \cup \{+\infty\} \\ & z & \longmapsto & \dfrac{\Psi(x+tz) - \Psi(x) - t\,\langle y, z\rangle_{\mathcal{H}}}{t^2}, \end{array}$$

*for all* $t > 0$*, is Mosco epi-convergent. In that case we denote by*

$$\mathrm{d}_e^2\Psi(x|y) := \text{ME-lim } \delta_t^2\Psi(x|y),$$

*which is called the second-order epi-derivative of* $\Psi$ *at* $x$ *for* $y$*.*

**Remark 3.2.6.** In the case where $\Psi$ is twice Fréchet differentiable at $x \in \mathcal{H}$, then $\Psi$ is twice epi-differentiable at $x$ for $\nabla\Psi(x)$ and

$$\mathrm{d}_e^2\Psi(x|\nabla\Psi(x))(z) = \frac{1}{2}\mathrm{D}^2\Psi(x)(z,z), \qquad \forall z \in \mathcal{H},$$

where $\mathrm{D}^2\Psi(x)$ is the twice Fréchet differential of $\Psi$ at $x$.

It is well-known that the convexity and the lower-semicontinuity are preserved by the Mosco epi-convergence. Moreover the properness of the Mosco epi-limit is also preserved (we refer the reader to [69, 70] for the finite-dimensional case and to [28] for the infinite dimensional one).

**Proposition 3.2.7.** *Let* $\Psi : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ *be a proper lower semi-continuous convex function. If* $\Psi$ *is twice epi-differentiable at* $x \in \mathrm{dom}(\Psi)$ *for* $y \in \partial\Psi(x)$*, then* $\mathrm{d}_e^2\Psi(x|y)$ *is a proper lower semi-continuous convex function.*

The following proposition is an extension of a result proved in [28, Example 2.7 p.286], that will be useful in Part II.

**Proposition 3.2.8.** *Let* $\mathrm{C}$ *be a nonempty closed convex subset of* $\mathcal{H}$ *and* $h_{\mathrm{C}}$ *the support function of* $\mathrm{C}$ *defined by*

$$\begin{array}{rccl} h_{\mathrm{C}}: & \mathcal{H} & \longrightarrow & \mathbb{R} \cup \{+\infty\} \\ & x & \longmapsto & h_{\mathrm{C}}(x) := \displaystyle\sup_{y \in \mathrm{C}} \langle x, y\rangle_{\mathcal{H}}. \end{array}$$

*Then, for all $x \in \mathrm{C}^{\perp}$, one gets*

$$\partial h_{\mathrm{C}}(x) = \mathrm{C}.$$

*Moreover $h_{\mathrm{C}}$ is twice epi-differentiable at $x$ for any $y \in \mathrm{C}$ and*

$$\mathrm{d}_e^2 h_{\mathrm{C}}(x|y) = \iota_{\mathrm{N}_{\mathrm{C}}(y)},$$

*where $\iota_{\mathrm{N}_{\mathrm{C}}(y)}$ stands for the indicator function of the normal cone to $\mathrm{C}$ at $y$.*

*Proof.* Let $x \in \mathrm{C}^{\perp}$. Since C is a nonempty closed convex subset of $\mathcal{H}$ one easily gets using the convex conjugate of $h_{\mathrm{C}}$ (see, e.g., [48, Lesson E]) that

(i) $y \in \partial h_{\mathrm{C}}(x) \Leftrightarrow x \in \partial \iota_{\mathrm{C}}(y) \Leftrightarrow y \in \mathrm{C}$;

(ii) if $y \in \mathrm{C}$ then: $z \in \mathrm{N}_{\mathrm{C}}(y) \Leftrightarrow h_{\mathrm{C}}(z) = \langle z, y \rangle_{\mathcal{H}}$.

From the first item one deduces that $\partial h_{\mathrm{C}}(x) = \mathrm{C}$. Let us prove that $h_{\mathrm{C}}$ is twice epi-differentiable at $x$ for $y \in \mathrm{C}$. To this aim we use Proposition 3.2.4. Consider $z \in \mathrm{N}_{\mathrm{C}}(y)$, then $h_{\mathrm{C}}(z) = \langle y, z \rangle$. Thus, by considering the sequence $z_t := z$ for all $t > 0$, one gets

$$\delta_t^2 h_{\mathrm{C}}(x|y)(z_t) = \frac{h_{\mathrm{C}}(x+tz) - h_{\mathrm{C}}(x) - t\langle y, z \rangle_{\mathcal{H}}}{t^2} = \frac{\sup_{c \in \mathrm{C}} \langle x+tz, c \rangle_{\mathcal{H}} - t\langle y, z \rangle_{\mathcal{H}}}{t^2} = \frac{h_{\mathrm{C}}(z) - h_{\mathrm{C}}(z)}{t} = 0.$$

Moreover, since $\delta_t^2 h_{\mathrm{C}}(x|y)(v) \geq 0$, for all $v \in \mathcal{H}$, one deduces that $\mathrm{d}_e^2 h_{\mathrm{C}}(x|y)(z) = 0$. Now consider $z \notin \mathrm{N}_{\mathrm{C}}(y)$. There exists $c_0 \in \mathrm{C}$ such that $\langle z, c_0 \rangle_{\mathcal{H}} > \langle y, z \rangle_{\mathcal{H}}$, thus $h_{\mathrm{C}}(z) > \langle y, z \rangle_{\mathcal{H}}$. Consider any sequence $(z_t)_{t>0} \rightharpoonup z$. Since $h_{\mathrm{C}}$ is convex and lower semi-continuous, then $h_{\mathrm{C}}$ is also weakly lower semi-continuous (see, e.g., [16, Corollary 3.9 p.61]), thus one has

$$\liminf h_{\mathrm{C}}(z_t) \geq h_{\mathrm{C}}(z) \text{ and } \langle y, z_t \rangle_{\mathcal{H}} \longrightarrow \langle y, z \rangle_{\mathcal{H}},$$

when $t \to 0^+$. Therefore, there exists $\varepsilon > 0$ such that, for all $t \leq \varepsilon$,

$$h_{\mathrm{C}}(z_t) \geq h_{\mathrm{C}}(z) - \frac{h_{\mathrm{C}}(z) - \langle y, z \rangle_{\mathcal{H}}}{4} \text{ and } -\langle y, z_t \rangle_{\mathcal{H}} \geq -\langle y, z \rangle_{\mathcal{H}} - \frac{h_{\mathrm{C}}(z) - \langle y, z \rangle_{\mathcal{H}}}{4},$$

then one has for all $t \leq \varepsilon$,

$$\delta_t^2 h_{\mathrm{C}}(x|y)(z_t) = \frac{h_{\mathrm{C}}(z_t) - \langle y, z_t \rangle_{\mathcal{H}}}{t} \geq \frac{h_{\mathrm{C}}(z) - \langle y, z \rangle_{\mathcal{H}}}{2t} \longrightarrow +\infty,$$

when $t \to 0^+$. Thus,

$$\mathrm{d}_e^2 h_{\mathrm{C}}(x|y)(z) = +\infty.$$

This concludes the proof. □

The following result, due to C.N. Do. (see [28, Chapter 2, Example 2.10 p.287]), shows that the indicator function of a nonempty closed convex set is twice epi-differentiable under an additional assumption.

**Proposition 3.2.9.** *Let* $\mathrm{C}$ *be a nonempty closed convex subset of* $\mathcal{H}$. *If* $\mathrm{C}$ *is polyhedric at* $x \in \mathrm{C}$ *for* $y \in \mathrm{N_C}(x)$, *then* $\iota_{\mathrm{C}}$ *is twice epi-differentiable at* $x$ *for* $y$ *and*

$$\mathrm{d}_e^2 \iota_{\mathrm{C}}(x|y) = \iota_{\mathrm{T_C}(x) \cap (\mathbb{R}y)^{\perp}},$$

*where* $\mathrm{N_C}(x)$ *(resp.* $\mathrm{T_C}(x)$*) is the normal cone (resp. tangent cone) to* $\mathrm{C}$ *at* $x$.

The notion of twice epi-differentiability leads to the ***protodifferentiability*** of the proximal operator, thus to the next proposition (see, e.g., [2, Theorem 4.15 p.1714]), which is one of the key points to derive our main results in this manuscript.

**Proposition 3.2.10.** *Let* $\Psi : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ *be a proper lower semi-continuous convex function on* $\mathcal{H}$. *Let* $F : \mathbb{R}^+ \to \mathcal{H}$ *and let* $u : \mathbb{R}^+ \to \mathcal{H}$ *be defined by*

$$u(t) := \mathrm{prox}_{\Psi}(F(t)),$$

*for all* $t \geq 0$. *If the conditions*

1. $F$ *is differentiable at* $t = 0$;
2. $\Psi$ *is twice epi-differentiable at* $u(0)$ *for* $F(0) - u(0) \in \partial\Psi(u(0))$;

*are both satisfied, then* $u$ *is differentiable at* $t = 0$ *with*

$$u'(0) = \mathrm{prox}_{\mathrm{d}_e^2 \Psi(u(0)|F(0)-u(0))}(F'(0)).$$

As mentioned in Introduction, in order to perturb Tresca friction law and to investigate the sensitivity analysis of a boundary value problem involving this law, an extended version of the twice epi-differentiability for parameterized convex functions, recently developed in [2], is required. To provide recalls on this extended notion, when considering a function $\Psi : \mathbb{R}_+ \times \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ such that, for all $t \geq 0$, $\Psi(t, \cdot) : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ is a proper function, we will make use of the following two notations: $\partial\Psi(0, \cdot)(x)$ stands for the convex subdifferential operator at $x \in \mathcal{H}$ of the function $\Psi(0, \cdot)$, and for each $t \geq 0$, $\Psi^{-1}(t, \mathbb{R}) := \{x \in \mathcal{H} \mid \Psi(t, x) \in \mathbb{R}\}$ and $\Psi^{-1}(\cdot, \mathbb{R}) := \cap_{t \geq 0} \Psi^{-1}(t, \mathbb{R})$.

**Definition 3.2.11** (Twice epi-differentiability depending on a parameter)**.** *Let* $\Psi : \mathbb{R}_+ \times \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ *be a function such that, for all* $t \geq 0$, $\Psi(t, \cdot) : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ *is a proper lower semi-continuous convex function. Then* $\Psi$ *is said to be twice epi-differentiable at* $x \in \Psi^{-1}(\cdot, \mathbb{R})$ *for* $y \in \partial\Psi(0, \cdot)(x)$ *if the family of second-order difference quotient functions* $(\Delta_t^2 \Psi(x|y))_{t>0}$ *defined by*

$$\begin{array}{rccl} \Delta_t^2 \Psi(x|y) : & \mathcal{H} & \longrightarrow & \mathbb{R} \cup \{+\infty\} \\ & z & \longmapsto & \Delta_t^2 \Psi(x|y)(z) := \dfrac{\Psi(t, x + tz) - \Psi(t, x) - t \langle y, z \rangle_{\mathcal{H}}}{t^2}, \end{array}$$

*for all* $t > 0$, *is Mosco epi-convergent. In that case we denote by*

$$\mathrm{D}_e^2 \Psi(x|y) := \text{ME-lim } \Delta_t^2 \Psi(x|y),$$

*which is called the second-order epi-derivative of* $\Psi$ *at* $x$ *for* $y$.

**Remark 3.2.12.** If the real-valued function $\Psi$ is $t$-independent, Definition 3.2.11 recovers the classical notion of twice epi-differentiability.

**Remark 3.2.13.** Compared to the $t$-independent case (see Proposition 3.2.7), the properness of the Mosco epi-limit may fail, even if the sequence is proper. If, for each $t \geq 0$, $\Psi(t,\cdot) : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ is a proper, lower semi-continuous and convex function, then the Mosco epi-limi $\mathrm{D}_e^2\Psi(x|y)$ (when it exists) is also lower semi-continuous and convex function. However, it may be possible that there exists some $z \in \mathcal{H}$ such that $\mathrm{D}_e^2\Psi(x|y)(z) = -\infty$ (see, e.g., [2, Example 4.4 p.1711]).

To illustrate the notion of twice epi-differentiability depending on a parameter, one example extracted from [2, Lemma 5.2 p.1717] is given below.

**Example 3.2.14.** Consider the function $\Psi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ defined by $\Psi(t,x) := |x - t^2|$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$. For each $t \geq 0$, $\Psi(t,\cdot)$ is a proper, lower semi-continuous and convex function. For all $x \in \mathbb{R}$ and all $y \in \partial\Psi(0,\cdot)(x)$, $\Psi$ is twice epi-differentiable at $x$ for $y$ and its second-order epi-derivative is given by

$$\mathrm{D}_e^2\Psi(x|y) = \begin{cases} \iota_{\mathbb{R}} & \text{if } x \neq 0, \\ \iota_{\mathbb{R}_-} & \text{if } x = 0 \text{ and } y = -1, \\ \iota_{\mathbb{R}_+} - 2 & \text{if } x = 0 \text{ and } y = 1, \\ \iota_{\{0\}} - y - 1 & \text{if } x = 0 \text{ and } y \in (-1,1). \end{cases}$$

Finally, the next proposition (see [2, Theorem 4.15 p.1714]) generalizes Proposition 3.2.10 for $t$-dependent function.

**Proposition 3.2.15.** *Let $\Psi : \mathbb{R}_+ \times \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ be a function such that, for all $t \geq 0$, $\Psi(t,\cdot) : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ is a proper, lower semi-continuous and convex function. Let $F : \mathbb{R}_+ \to \mathcal{H}$ and $u : \mathbb{R}_+ \to \mathcal{H}$ be defined by*

$$u(t) := \mathrm{prox}_{\Psi(t,\cdot)}(F(t)),$$

*for all $t \geq 0$. If the conditions*

(i) *$F$ is differentiable at $t = 0$;*

(ii) *$\Psi$ is twice epi-differentiable at $u(0)$ for $F(0) - u(0) \in \partial\Psi(0,\cdot)(u(0))$;*

(iii) *$\mathrm{D}_e^2\Psi(u(0)|F(0) - u(0))$ is a proper function on $\mathcal{H}$;*

*are satisfied, then $u$ is differentiable at $t = 0$ with*

$$u'(0) = \mathrm{prox}_{\mathrm{D}_e^2\Psi(u(0)|F(0)-u(0))}(F'(0)).$$

Chapter 4

# Some required boundary value problems

As mentioned in Introduction, the major part of the present work consists in performing the sensitivity analysis of a Signorini problem and a Tresca friction problem with respect to the data. To this aim three boundary value problems will be involved: a Neumann problem, a Signorini problem and a Tresca friction problem. In Section 4.1 we introduce those problems in the scalar model, i.e. the displacement fields are in $\mathrm{H}^1(\Omega)$ and the solution must satisfy a Laplace type equation. In Section 4.2, we present those problems in the linear elastic model. Only the proofs of Subsections 4.2.3 and 4.2.4 are detailed, the other ones being similar. We recall that the notation $\mathrm{int}_\Gamma$ stands for the interior relative to $\Gamma$.

## 4.1 Boundary value problems in the scalar model

In this section, let $d \in \mathbb{N}^*$ be a positive integer, $f \in \mathrm{L}^2(\Omega)$, $g \in \mathrm{L}^2(\Gamma)$, $w \in \mathrm{H}^{1/2}(\Gamma)$, $k \in \mathrm{L}^\infty(\Omega)$ and $\mathrm{M} \in \mathrm{L}^\infty(\Omega, \mathbb{R}^{d\times d})$ satisfying

$$k \geq \alpha \textit{ a.e.} \text{ on } \Omega \qquad \text{and} \qquad \mathrm{M}(x)y \cdot y \geq \gamma \|y\|^2, \quad \forall y \in \mathbb{R}^d,$$

for some $\alpha > 0$, $\gamma > 0$, where $M(x)$ is a symmetric matrix for almost every $x \in \Omega$, and where $\|\cdot\|$ stands for the usual Euclidean norm of $\mathbb{R}^d$. From those assumptions, note that the map

$$\begin{array}{rrcl} \langle \cdot,\cdot \rangle_{\mathrm{M},k} : & \mathrm{H}^1(\Omega) \times \mathrm{H}^1(\Omega) & \longrightarrow & \mathbb{R} \\ & (v_1, v_2) & \longmapsto & \langle v_1, v_2 \rangle_{\mathrm{M},k} := \displaystyle\int_\Omega \mathrm{M}\nabla v_1 \cdot \nabla v_2 + \int_\Omega k v_1 v_2, \end{array}$$

is a scalar product on $\mathrm{H}^1(\Omega)$.

**Remark 4.1.1.** The classical scalar Signorini (resp. Tresca friction problem) is for $\mathrm{M} = \mathrm{I}$ and $k = 1$ *a.e.* on $\Omega$ (see, e.g. [52, Section 1]) and [37, Section 1.3 Chapter 1]). Nevertheless, as we saw in the Introduction, a more general version of this problem appears when the sensitivity analysis with respect to the shape is performed. Thus, in the sequel, we introduce a general version of the scalar Signorini (resp.Tresca friction) problem.

### 4.1.1 A general scalar Neumann problem

Consider the general scalar Neumann problem given by

$$\begin{cases} -\mathrm{div}(\mathrm{M}\nabla F) + kF = f & \text{in } \Omega, \\ \mathrm{M}\nabla F \cdot \mathrm{n} = g & \text{on } \Gamma. \end{cases} \tag{GSNP}$$

**Definition 4.1.2** (Solution to the general scalar Neumann problem)**.** *A (strong) solution to the general scalar Neumann problem* (GSNP) *is a function* $F \in \mathrm{H}^1(\Omega)$ *such that* $-\mathrm{div}(\mathrm{M}\nabla F) + kF = f$ *in* $\mathcal{D}'(\Omega)$ *and* $\mathrm{M}\nabla F \cdot \mathrm{n} \in \mathrm{L}^2(\Gamma)$ *with* $\mathrm{M}\nabla F \cdot \mathrm{n} = g$ *a.e. on* $\Gamma$*.*

**Definition 4.1.3** (Weak solution to the general scalar Neumann problem)**.** *A weak solution to the general scalar Neumann problem* (GSNP) *is a function* $F \in \mathrm{H}^1(\Omega)$ *such that*

$$\int_\Omega \mathrm{M}\nabla F \cdot \nabla v + \int_\Omega kFv = \int_\Omega fv + \int_\Gamma gv, \qquad \forall v \in \mathrm{H}^1(\Omega).$$

**Proposition 4.1.4.** *A function* $F \in \mathrm{H}^1(\Omega)$ *is a (strong) solution to the general scalar Neumann problem* (GSNP) *if and only if* $F$ *is a weak solution to the general scalar Neumann problem* (GSNP)*.*

From the assumptions on M and $k$ and using the Riesz representation theorem, one can easily get the following existence/uniqueness result.

**Proposition 4.1.5.** *The general scalar Neumann problem* (GSNP) *possesses a unique (strong) solution* $F \in \mathrm{H}^1(\Omega)$*. Moreover there exists a constant* $C \geq 0$ *(depending only on* $\Omega$*,* M *and* $k$*) such that*

$$\|F\|_{\mathrm{M},k} \leq C \left( \|f\|_{\mathrm{L}^2(\Omega)} + \|g\|_{\mathrm{L}^2(\Gamma)} \right).$$

### 4.1.2 A general scalar Signorini problem

In this part we assume that $\Gamma$ is decomposed (up to a null set) as

$$\Gamma_{\mathrm{N}} \cup \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{S}-} \cup \Gamma_{\mathrm{S}+},$$

where $\Gamma_{\mathrm{N}}$, $\Gamma_{\mathrm{D}}$, $\Gamma_{\mathrm{S}-}$ and $\Gamma_{\mathrm{S}+}$ are four measurable pairwise disjoint subsets of $\Gamma$. Consider the general scalar Signorini problem given by

$$\begin{cases} -\mathrm{div}(\mathrm{M}\nabla u) + ku = f & \text{in } \Omega, \\ \mathrm{M}\nabla u \cdot \mathrm{n} = g & \text{on } \Gamma_{\mathrm{N}}, \\ u = w & \text{on } \Gamma_{\mathrm{D}}, \\ u \leq w,\ \mathrm{M}\nabla u \cdot \mathrm{n} \leq g \text{ and } (u-w)(\mathrm{M}\nabla u \cdot \mathrm{n} - g) = 0 & \text{on } \Gamma_{\mathrm{S}-}, \\ u \geq w,\ \mathrm{M}\nabla u \cdot \mathrm{n} \geq g \text{ and } (u-w)(\mathrm{M}\nabla u \cdot \mathrm{n} - g) = 0 & \text{on } \Gamma_{\mathrm{S}+}. \end{cases} \tag{GSSP}$$

**Definition 4.1.6** (Solution to the general scalar Signorini problem)**.** *A (strong) solution to the general scalar Signorini problem* (GSSP) *is a function* $u \in \mathrm{H}^1(\Omega)$ *such that* $-\mathrm{div}(\mathrm{M}\nabla u) + ku = f$ *in* $\mathcal{D}'(\Omega)$*,* $u = w$ *a.e. on* $\Gamma_{\mathrm{D}}$*, and also* $\mathrm{M}\nabla u \cdot \mathrm{n} \in \mathrm{L}^2(\Gamma)$ *with* $\mathrm{M}\nabla u \cdot \mathrm{n} = g$ *a.e. on* $\Gamma_{\mathrm{N}}$*,* $u \leq w$*,* $\mathrm{M}\nabla u \cdot \mathrm{n} \leq g$

*and* $(u-w)(\mathrm{M}\nabla u\cdot \mathrm{n}-g)=0$ *a.e. on* $\Gamma_{\mathrm{S}-}$, $u\geq w$, $\mathrm{M}\nabla u\cdot\mathrm{n}\geq g$ *and* $(u-w)(\mathrm{M}\nabla u\cdot\mathrm{n}-g)=0$ *a.e. on* $\Gamma_{\mathrm{S}+}$.

**Definition 4.1.7** (Weak solution to the general scalar Signorini problem)**.** *A weak solution to the general scalar Signorini problem* (GSSP) *is a function* $u\in\mathcal{K}^1_w(\Omega)$ *such that*

$$\int_\Omega \mathrm{M}\nabla u\cdot\nabla(v-u)+\int_\Omega ku\,(v-u)\geq\int_\Omega f\,(v-u)+\int_{\Gamma_\mathrm{N}\cup\Gamma_{\mathrm{S}-}\cup\Gamma_{\mathrm{S}+}} g\,(v-u)\,,\qquad \forall v\in\mathcal{K}^1_w(\Omega),$$

*where* $\mathcal{K}^1_w(\Omega)$ *is the nonempty closed convex subset of* $\mathrm{H}^1(\Omega)$ *defined by*

$$\mathcal{K}^1_w(\Omega):=\left\{v\in\mathrm{H}^1(\Omega)\mid v\leq w\ \text{a.e. on }\Gamma_{\mathrm{S}-},\ v=w\ \text{a.e. on }\Gamma_\mathrm{D}\text{ and }v\geq w\ \text{a.e. on }\Gamma_{\mathrm{S}+}\right\}.$$

One can easily prove that a (strong) solution to the general scalar Signorini problem (GSSP) is also a weak solution. However, to the best of our knowledge, one cannot prove the converse without additional assumptions. To get the equivalence, one can assume, in particular, that the decomposition $\Gamma_\mathrm{N}\cup\Gamma_\mathrm{D}\cup\Gamma_{\mathrm{S}-}\cup\Gamma_{\mathrm{S}+}$ of $\Gamma$ is *consistent* in the following sense.

**Definition 4.1.8** (Consistent decomposition)**.** *The decomposition* $\Gamma_\mathrm{N}\cup\Gamma_\mathrm{D}\cup\Gamma_{\mathrm{S}-}\cup\Gamma_{\mathrm{S}+}$ *of* $\Gamma$ *is said to be consistent if*

(i) *for almost all* $s\in\Gamma_{\mathrm{S}-}$ *(resp.* $\Gamma_{\mathrm{S}+}$*),* $s\in\mathrm{int}_\Gamma(\Gamma_{\mathrm{S}-})$ *(resp.* $s\in\mathrm{int}_\Gamma(\Gamma_{\mathrm{S}+})$*);*

(ii) *the nonempty closed convex subset* $\mathcal{K}^{1/2}_w(\Gamma)$ *of* $\mathrm{H}^{1/2}(\Gamma)$ *defined by*

$$\mathcal{K}^{1/2}_w(\Gamma):=\left\{v\in\mathrm{H}^{1/2}(\Gamma)\mid v\leq w\ \text{a.e. on }\Gamma_{\mathrm{S}-},\ v=w\ \text{a.e. on }\Gamma_\mathrm{D}\text{ and }v\geq w\ \text{a.e.on }\Gamma_{\mathrm{S}+}\right\},$$

*is dense in the nonempty closed convex subset* $\mathcal{K}^0_w(\Gamma)$ *of* $\mathrm{L}^2(\Gamma)$ *defined by*

$$\mathcal{K}^0_w(\Gamma):=\left\{v\in\mathrm{L}^2(\Gamma)\mid v\leq w\ \text{a.e. on }\Gamma_{\mathrm{S}-},\ v=w\ \text{a.e. on }\Gamma_\mathrm{D}\text{ and }v\geq w\ \text{a.e. on }\Gamma_{\mathrm{S}+}\right\}.$$

**Proposition 4.1.9.** *Let* $u\in\mathrm{H}^1(\Omega)$.

(i) *If* $u$ *is a (strong) solution to the general scalar Signorini problem* (GSSP)*, then* $u$ *is a weak solution to the general scalar Signorini problem* (GSSP).

(ii) *If* $u$ *is a weak solution to the general scalar Signorini problem* (GSSP) *such that* $\mathrm{M}\nabla u\cdot\mathrm{n}\in\mathrm{L}^2(\Gamma)$ *and the decomposition* $\Gamma_\mathrm{N}\cup\Gamma_\mathrm{D}\cup\Gamma_{\mathrm{S}-}\cup\Gamma_{\mathrm{S}+}$ *of* $\Gamma$ *is consistent, then* $u$ *is a (strong) solution to the general scalar Signorini problem* (GSSP).

Using the classical characterization of the projection operator, one can easily get the following existence/uniqueness result.

**Proposition 4.1.10.** *The general scalar Signorini problem* (GSSP) *admits a unique weak solution* $u\in\mathrm{H}^1(\Omega)$ *characterized by*

$$u=\mathrm{proj}_{\mathcal{K}^1_w(\Omega)}(F),$$

*where* $F\in\mathrm{H}^1(\Omega)$ *is the unique solution to the general scalar Neumann problem* (GSNP)*, and where* $\mathrm{proj}_{\mathcal{K}^1_w(\Omega)}:\mathrm{H}^1(\Omega)\to\mathrm{H}^1(\Omega)$ *stands for the classical projection operator onto the nonempty closed convex subset* $\mathcal{K}^1_w(\Omega)$ *of* $\mathrm{H}^1(\Omega)$ *for the scalar product* $\langle\cdot,\cdot\rangle_{\mathrm{M},k}$.

**Remark 4.1.11.** Note that, from Remark 3.1.11, the unique weak solution $u \in \mathrm{H}^1(\Omega)$ to the general scalar Signorini problem (GSSP) is also characterized by the proximal operator since $\mathrm{prox}_{\iota_{\mathcal{K}^1_w(\Omega)}} = \mathrm{proj}_{\mathcal{K}^1_w(\Omega)}$.

**Remark 4.1.12.** The notion of consistent decomposition is only used to prove that a weak solution to Problem (GSSP) is also a strong solution. Nevertheless, we can notice that the notion of consistent decomposition is quite unrestrictive.

### 4.1.3 A general scalar Tresca friction problem

In this part we assume that $g > 0$ *a.e.* on $\Gamma$. Consider the general scalar Tresca friction problem given by

$$\begin{cases} -\mathrm{div}(\mathrm{M}\nabla u) + ku = f & \text{in } \Omega, \\ |\mathrm{M}\nabla u \cdot \mathrm{n}| \leq g \text{ and } u\mathrm{M}\nabla u \cdot \mathrm{n} + g\,|u| = 0 & \text{on } \Gamma. \end{cases} \tag{GSTP}$$

**Definition 4.1.13** (Solution to the general scalar Tresca friction problem)**.** *A (strong) solution to the general scalar Tresca friction problem* (GSTP) *is a function* $u \in \mathrm{H}^1(\Omega)$ *such that* $-\mathrm{div}(\mathrm{M}\nabla u)+ku = f$ *in* $\mathcal{D}'(\Omega)$, $\mathrm{M}\nabla u \cdot \mathrm{n} \in \mathrm{L}^2(\Gamma)$ *with* $|\mathrm{M}(s)\nabla u(s) \cdot \mathrm{n}(s)| \leq g(s)$ *and* $u(s)\mathrm{M}(s)\nabla u(s) \cdot \mathrm{n}(s) + g(s)|u(s)| = 0$ *for almost all* $s \in \Gamma$.

**Definition 4.1.14** (Weak solution to the general scalar Tresca friction problem)**.** *A weak solution to the general scalar Tresca friction problem* (GSTP) *is a function* $u \in \mathrm{H}^1(\Omega)$ *such that*

$$\int_\Omega \mathrm{M}\nabla u \cdot \nabla(v-u) + \int_\Omega ku(v-u) + \int_\Gamma g|v| - \int_\Gamma g|u| \geq \int_\Omega f(v-u), \qquad \forall v \in \mathrm{H}^1(\Omega).$$

**Proposition 4.1.15.** *A function* $u \in \mathrm{H}^1(\Omega)$ *is a (strong) solution to the general scalar Tresca friction problem* (GSTP) *if and only if* $u$ *is a weak solution to the general scalar Tresca friction problem* (GSTP).

Using the classical characterization of the proximal operator, we obtain the following result.

**Proposition 4.1.16.** *The general scalar Tresca friction problem* (GSTP) *admits a unique (strong) solution* $u \in \mathrm{H}^1(\Omega)$ *characterized by*

$$u = \mathrm{prox}_\phi(F),$$

*where* $F \in \mathrm{H}^1(\Omega)$ *is the unique solution to the general scalar Neumann problem*

$$\begin{cases} -\mathrm{div}(\mathrm{M}\nabla F) + kF = f & \text{in } \Omega, \\ \mathrm{M}\nabla F \cdot \mathrm{n} = 0 & \text{on } \Gamma, \end{cases}$$

*and where* $\mathrm{prox}_\phi : \mathrm{H}^1(\Omega) \to \mathrm{H}^1(\Omega)$ *stands for the proximal operator associated with the Tresca friction functional given by*

$$\begin{array}{rccl} \phi : & \mathrm{H}^1(\Omega) & \longrightarrow & \mathbb{R} \\ & v & \longmapsto & \phi(v) := \displaystyle\int_\Gamma g|v|, \end{array}$$

*considered on the Hilbert space* $(\mathrm{H}^1(\Omega), \langle \cdot, \cdot \rangle_{\mathrm{M},k})$.

## 4.2 Boundary value problems in the linear elastic model

Let $d \in \{2,3\}$ and assume that $\Gamma$ is decomposed as $\Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{S}}$, where $\Gamma_{\mathrm{D}}$ and $\Gamma_{\mathrm{S}}$ are two measurable pairwise disjoint subsets of $\Gamma$, such that $\Gamma_{\mathrm{D}}$ and $\Gamma_{\mathrm{S}}$ have a positive measure. In that case,

$$\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d) := \left\{ v \in \mathrm{H}^1(\Omega,\mathbb{R}^d) \mid v = 0 \textit{ a.e. } \text{on } \Gamma_{\mathrm{D}} \right\},$$

is a linear subspace of $\mathrm{H}^1(\Omega,\mathbb{R}^d)$. We assume that $\Omega$ is an elastic solid satisfying the linear elastic model (see Chaper 1 for details and notations). Thus, from (1.1), it follows that

$$\begin{array}{rccl} \langle\cdot,\cdot\rangle_{\mathrm{A}}: & \left(\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)\right)^2 & \longrightarrow & \mathbb{R} \\ & (v_1,v_2) & \longmapsto & \displaystyle\int_\Omega \mathrm{Ae}(v_1):\mathrm{e}(v_2), \end{array} \tag{4.1}$$

is a scalar product on $\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ (see, e.g., [31, Chapter 3]), and one denotes $\|\cdot\|_{\mathrm{A}}$ the corresponding norm. If for some $v \in \mathrm{H}^1(\Omega,\mathbb{R}^d)$, the stress vector $T(v) = \mathrm{Ae}(v)\mathrm{n}$ is in $\mathrm{L}^2(\Gamma_{\mathrm{S}},\mathbb{R}^d)$, then one uses the notation

$$\mathrm{Ae}(v)\mathrm{n} = \sigma_{\mathrm{n}}(v)\mathrm{n} + \sigma_\tau(v),$$

where $\sigma_{\mathrm{n}}(v) \in \mathrm{L}^2(\Gamma_{\mathrm{S}},\mathbb{R})$ is the normal stress and $\sigma_\tau \in \mathrm{L}^2(\Gamma_{\mathrm{S}},\mathbb{R}^d)$ the shear stress.

Moreover, let $z \in \mathrm{L}^\infty(\Omega)$ and $\mathrm{B} \in \mathrm{L}^\infty(\Omega,\mathbb{R}^{d\times d})$ such that

$$\int_\Omega z\mathrm{A}\left[(\nabla v_1)\mathrm{B}\right]\mathrm{B}^\top : \nabla v_2 \geq C\int_\Omega \mathrm{Ae}(v_1):\mathrm{e}(v_2), \qquad \forall v_1,v_2 \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d),$$

where $C>0$ is a constant. In that case, the map

$$\begin{array}{rccl} \langle\cdot,\cdot\rangle_{z,\mathrm{A},\mathrm{B}}: & \left(\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)\right)^2 & \longrightarrow & \mathbb{R} \\ & (v_1,v_2) & \longmapsto & \langle v_1,v_2\rangle_{z,\mathrm{A},\mathrm{B}} := \displaystyle\int_\Omega z\mathrm{A}\left[(\nabla v_1)\mathrm{B}\right]\mathrm{B}^\top : \nabla v_2, \end{array}$$

is a scalar product on $\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$.

We consider $\mathrm{M} \in \mathrm{L}^\infty(\Gamma_{\mathrm{S}},\mathbb{R}^{d\times d})$ and we denote by $v_{(\mathrm{Mn})^\perp} := v - (v\cdot\frac{\mathrm{Mn}}{||\mathrm{Mn}||^2})\mathrm{Mn}$, for any $v \in \mathrm{L}^2(\Gamma_{\mathrm{S}},\mathbb{R}^d)$ and, to simplify, we assume in the sequel that $||\mathrm{Mn}|| = 1$ *a.e.* on $\Gamma_{\mathrm{S}}$. We also denote for all $\mathrm{S} \in \mathbb{R}^{d\times d}$, by $\mathrm{div}(\mathrm{S})$ the vector defined by $(\mathrm{div}(\mathrm{S}))_i := \mathrm{div}(\mathrm{S}_i)$, where $\mathrm{S}_i$ is the $i$-th line of S, for all $i \in [[1,d]]$. Moreover, all along the paper, we denote (when it is necessary to avoid any confusion) A[S] the stiffness tensor A applied to the matrix S.

Finally, in the sequel, consider $f \in \mathrm{L}^2(\Omega,\mathbb{R}^d)$ and $\ell \in \mathrm{L}^2(\Gamma_{\mathrm{S}},\mathbb{R}^d)$.

**Remark 4.2.1.** The classical Signorini (resp. Tresca friction problem) is for $z=1$, $\mathrm{B}=\mathrm{I}$ *a.e.* on $\Omega$, and $\mathrm{M}=\mathrm{I}$ *a.e.* on $\Gamma_{\mathrm{S}}$ (see, e.g. [25, Chapter 3 p.102]) or [29]). Nevertheless, in the same way as Section 4.1, a more general version of this problem appears when the sensitivity analysis with respect to the shape is performed. Thus, in the sequel, we introduce a general version of the Signorini (resp. Tresca friction) problem.

### 4.2.1 A general Dirichlet-Neumann problem

Consider the general Dirichlet-Neumann problem given by

$$\begin{cases} -\mathrm{div}\left(z\mathrm{A}\left[(\nabla F)\mathrm{B}\right]\mathrm{B}^\top\right) = f & \text{in } \Omega, \\ F = 0 & \text{on } \Gamma_\mathrm{D}, \\ \left(z\mathrm{A}\left[(\nabla F)\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n} = \ell & \text{on } \Gamma_\mathrm{S}. \end{cases} \tag{GDNP}$$

**Definition 4.2.2** (Strong solution to the general Dirichlet-Neumann problem)**.** *A (strong) solution to the general Dirichlet-Neumann problem* (GDNP) *is a function* $F \in \mathrm{H}^1(\Omega, \mathbb{R}^d)$ *such that* $-\mathrm{div}\left(z\mathrm{A}\left[(\nabla F)\mathrm{B}\right]\mathrm{B}^\top\right) = f$ *in* $\mathcal{D}'(\Omega, \mathbb{R}^d)$*,* $F = 0$ *a.e. on* $\Gamma_\mathrm{D}$*,* $\left(z\mathrm{A}\left[(\nabla F)\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n} \in \mathrm{L}^2(\Gamma_\mathrm{S}, \mathbb{R}^d)$ *with* $\left(z\mathrm{A}\left[(\nabla F)\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n} = \ell$ *a.e. on* $\Gamma_\mathrm{S}$*.*

**Definition 4.2.3** (Weak solution to the general Dirichlet-Neumann problem)**.** *A weak solution to the general Dirichlet-Neumann problem* (GDNP) *is a function* $F \in \mathrm{H}^1_\mathrm{D}(\Omega, \mathbb{R}^d)$ *such that*

$$\int_\Omega z\mathrm{A}\left[(\nabla F)\mathrm{B}\right]\mathrm{B}^\top : \nabla v = \int_\Omega f \cdot v + \int_{\Gamma_\mathrm{S}} \ell \cdot v, \qquad \forall v \in \mathrm{H}^1_\mathrm{D}(\Omega, \mathbb{R}^d).$$

**Proposition 4.2.4.** *A function* $F \in \mathrm{H}^1(\Omega, \mathbb{R}^d)$ *is a (strong) solution to the general Dirichlet-Neumann problem* (GDNP) *if and only if* $F$ *is a weak solution to the general Dirichlet-Neumann problem* (GDNP)*.*

**Proposition 4.2.5.** *The general Dirichlet-Neumann problem* (GDNP) *admits a unique solution* $F \in \mathrm{H}^1_\mathrm{D}(\Omega, \mathbb{R}^d)$*.*

### 4.2.2 A general Signorini problem

Assume that $\Gamma_\mathrm{S}$ is decomposed (up to a null set) as

$$\Gamma_{\mathrm{S_N}} \cup \Gamma_{\mathrm{S_D}} \cup \Gamma_{\mathrm{S}-} \cup \Gamma_{\mathrm{S}+},$$

where $\Gamma_{\mathrm{S_N}}$, $\Gamma_{\mathrm{S_D}}$, $\Gamma_{\mathrm{S}-}$ and $\Gamma_{\mathrm{S}+}$ are four measurable pairwise disjoint subsets of $\Gamma_\mathrm{S}$. Note that $\ell \in \mathrm{L}^2(\Gamma_\mathrm{S}, \mathbb{R}^d)$ can be decomposed as $\ell = (\ell \cdot \mathrm{Mn})\mathrm{Mn} + \ell_\tau$, and we denote by $g := \ell \cdot \mathrm{Mn} \in \mathrm{L}^2(\Gamma_\mathrm{S})$ and $h := \ell_\tau \in \mathrm{L}^2(\Gamma_\mathrm{S}, \mathbb{R}^d)$. Let $w \in \mathrm{H}^1_\mathrm{D}(\Omega, \mathbb{R}^d)$ and consider the general Signorini problem given by

$$\begin{cases} -\mathrm{div}\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_\mathrm{D}, \\ \left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\right)_{(\mathrm{Mn})^\perp} = h & \text{on } \Gamma_\mathrm{S}, \\ \left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n} \cdot \mathrm{Mn} = g & \text{on } \Gamma_{\mathrm{S_N}}, \\ (u-w) \cdot \mathrm{Mn} = 0 & \text{on } \Gamma_{\mathrm{S_D}}, \\ (u-w) \cdot \mathrm{Mn} \leq 0,\ \left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n} \cdot \mathrm{Mn} \leq g \text{ and} & \\ \qquad (u-w) \cdot \mathrm{Mn}\left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n} \cdot \mathrm{Mn} - g\right) = 0 & \text{on } \Gamma_{\mathrm{S}-}, \\ (u-w) \cdot \mathrm{Mn} \geq 0,\ \left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n} \cdot \mathrm{Mn} \geq g \text{ and} & \\ \qquad (u-w) \cdot \mathrm{Mn}\left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n} \cdot \mathrm{Mn} - g\right) = 0 & \text{on } \Gamma_{\mathrm{S}+}. \end{cases} \tag{GSP}$$

**Definition 4.2.6** (Strong solution to the general Signorini problem)**.** *A (strong) solution to the general Signorini problem* (GSP) *is a function* $u \in \mathrm{H}^1(\Omega, \mathbb{R}^d)$ *such that* $-\mathrm{div}\left(z\mathrm{A}\left[\nabla u \mathrm{B}\right]\mathrm{B}^\top\right) = f$ *in* $\mathcal{D}'(\Omega, \mathbb{R}^d)$*,* $u = 0$ *a.e. on* $\Gamma_{\mathrm{D}}$*,* $(u - w) \cdot \mathrm{Mn} = 0$ *a.e. on* $\Gamma_{\mathrm{S}_\mathrm{D}}$*,* $\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n} \in \mathrm{L}^2(\Gamma_{\mathrm{S}}, \mathbb{R}^d)$ *with* $\left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\right)_{(\mathrm{Mn})^\perp} = h$ *a.e. on* $\Gamma_{\mathrm{S}}$*,* $\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n} \cdot \mathrm{Mn} = g$ *a.e. on* $\Gamma_{\mathrm{S}_\mathrm{N}}$*,* $(u - w) \cdot \mathrm{Mn} \leq 0$*,* $\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\cdot\mathrm{Mn} \leq g$ *and* $(u-w)\cdot\mathrm{Mn}\left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\cdot\mathrm{Mn} - g\right) = 0$ *a.e. on* $\Gamma_{\mathrm{S}-}$*,* $(u-w)\cdot\mathrm{Mn} \geq 0$*,* $\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\cdot\mathrm{Mn} \geq g$ *and* $(u - w)\cdot\mathrm{Mn}\left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\cdot\mathrm{Mn} - g\right) = 0$ *a.e. on* $\Gamma_{\mathrm{S}+}$*.*

**Definition 4.2.7** (Weak solution to the general Signorini problem)**.** *A weak solution to the general Signorini problem* (GSP) *is a function* $u \in \mathcal{K}^1_{w,\mathrm{M}}(\Omega, \mathbb{R}^d)$ *such that*

$$\int_\Omega z\mathrm{A}\left[(\nabla u)\mathrm{B}\right]\mathrm{B}^\top : \nabla(v-u) \geq \int_\Omega f\cdot(v-u) + \int_{\Gamma_{\mathrm{S}_\mathrm{N}}\cup\Gamma_{\mathrm{S}-}\cup\Gamma_{\mathrm{S}+}} g\mathrm{Mn}\cdot(v-u) + \int_{\Gamma_\mathrm{S}} h\cdot(v-u)\,, \qquad \forall v\in\mathcal{K}^1_{w,\mathrm{M}}(\Omega,\mathbb{R}^d),$$

*where* $\mathcal{K}^1_{w,\mathrm{M}}(\Omega, \mathbb{R}^d)$ *is the nonempty closed convex subset of* $\mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ *given by*

$$\mathcal{K}^1_{w,\mathrm{M}}(\Omega,\mathbb{R}^d) := \Big\{ v\in\mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d) \mid (v-w)\cdot\mathrm{Mn} = 0 \text{ a.e. on } \Gamma_{\mathrm{S}_\mathrm{D}},\ (v-w)\cdot\mathrm{Mn}\leq 0 \text{ a.e. on } \Gamma_{\mathrm{S}-} \text{ and } (v-w)\cdot\mathrm{Mn}\geq 0 \text{ a.e. on } \Gamma_{\mathrm{S}+}\Big\}.$$

Similarly to Subsection 4.1.2, one can prove that a (strong) solution is a weak solution but, to the best of our knowledge, without additional assumptions, one cannot prove the converse. To get the equivalence, we need to assume, in particular, that the decomposition $\Gamma_\mathrm{D}\cup\Gamma_{\mathrm{S}_\mathrm{N}}\cup\Gamma_{\mathrm{S}_\mathrm{D}}\cup\Gamma_{\mathrm{S}-}\cup\Gamma_{\mathrm{S}+}$ of $\Gamma$ is consistent in the following sense.

**Definition 4.2.8** (Consistent decomposition)**.** *The decomposition* $\Gamma_\mathrm{D}\cup\Gamma_{\mathrm{S}_\mathrm{N}}\cup\Gamma_{\mathrm{S}_\mathrm{D}}\cup\Gamma_{\mathrm{S}-}\cup\Gamma_{\mathrm{S}+}$ *of* $\Gamma$ *is said to be consistent if*

(i) *for almost all* $s\in\Gamma_{\mathrm{S}-}$ *(resp.* $\Gamma_{\mathrm{S}+}$*),* $s\in\mathrm{int}_\Gamma(\Gamma_{\mathrm{S}-})$ *(resp.* $s\in\mathrm{int}_\Gamma(\Gamma_{\mathrm{S}+})$*);*

(ii) *the nonempty closed convex subset* $\mathcal{K}^{1/2}_{w,\mathrm{M}}(\Gamma,\mathbb{R}^d)$ *of* $\mathrm{H}^{1/2}(\Gamma,\mathbb{R}^d)$ *defined by*

$$\mathcal{K}^{1/2}_{w,\mathrm{M}}(\Gamma,\mathbb{R}^d) := \Big\{ v\in\mathrm{H}^{1/2}(\Gamma,\mathbb{R}^d) \mid v = 0 \text{ a.e. on } \Gamma_\mathrm{D},\ (v-w)\cdot\mathrm{Mn} = 0 \text{ a.e. on } \Gamma_{\mathrm{S}_\mathrm{D}},\ (v-w)\cdot\mathrm{Mn}\leq 0 \text{ a.e. on } \Gamma_{\mathrm{S}-} \text{ and } (v-w)\cdot\mathrm{Mn}\geq 0 \text{ a.e. on } \Gamma_{\mathrm{S}+}\Big\},$$

*is dense in the nonempty closed convex subset* $\mathcal{K}^0_{w,\mathrm{M}}(\Gamma,\mathbb{R}^d)$ *of* $\mathrm{L}^2(\Gamma,\mathbb{R}^d)$ *given by*

$$\mathcal{K}^0_{w,\mathrm{M}}(\Gamma,\mathbb{R}^d) := \Big\{ v\in\mathrm{L}^2(\Gamma,\mathbb{R}^d) \mid v = 0 \text{ a.e. on } \Gamma_\mathrm{D},\ (v-w)\cdot\mathrm{Mn} = 0 \text{ a.e. on } \Gamma_{\mathrm{S}_\mathrm{D}},\ (v-w)\cdot\mathrm{Mn}\leq 0 \text{ a.e. on } \Gamma_{\mathrm{S}-} \text{ and } (v-w)\cdot\mathrm{Mn}\geq 0 \text{ a.e. on } \Gamma_{\mathrm{S}+}\Big\}.$$

**Proposition 4.2.9.** *Let* $u\in\mathrm{H}^1(\Omega,\mathbb{R}^d)$*.*

(i) *If $u$ is a (strong) solution to the general Signorini problem* (GSP)*, then $u$ is a weak solution to the general Signorini problem* (GSP)*.*

(ii) *If $u$ is a weak solution to the general Signorini problem* (GSP) *such that $\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^{\top}\right)\mathrm{n} \in \mathrm{L}^2(\Gamma_{\mathrm{S}},\mathbb{R}^d)$ and the decomposition $\Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{S_N}} \cup \Gamma_{\mathrm{S_D}} \cup \Gamma_{\mathrm{S}-} \cup \Gamma_{\mathrm{S}+}$ of $\Gamma$ is consistent, then $u$ is a (strong) solution to the general Signorini problem* (GSP)*.*

**Proposition 4.2.10.** *The general Signorini problem* (GSP) *admits a unique weak solution $u \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ which is given by*

$$u = \mathrm{proj}_{\mathcal{K}^1_{w,\mathrm{M}}(\Omega,\mathbb{R}^d)}(F),$$

*where $F \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ is the unique solution to the general Dirichlet-Neumann problem* (GDNP) *and $\mathrm{proj}_{\mathcal{K}^1_w(\Omega)} : \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d) \rightarrow \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ stands for the classical projection operator onto the nonempty closed convex subset $\mathcal{K}^1_w(\Omega,\mathbb{R}^d)$ of $\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ for the scalar product $\langle\cdot,\cdot\rangle_{z,\mathrm{A},\mathrm{B}}$.*

**Remark 4.2.11.** Note that, from Remark 3.1.11, the unique weak solution $u \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ to the general Signorini problem (GSP) is also characterized by the proximal operator since $\mathrm{prox}_{\iota_{\mathcal{K}^1_{w,\mathrm{M}}(\Omega,\mathbb{R}^d)}} = \mathrm{proj}_{\mathcal{K}^1_{w,\mathrm{M}}(\Omega,\mathbb{R}^d)}$.

### 4.2.3 A general tangential Signorini problem

In this subection, we assume that $\mathrm{M} = \mathrm{I}$ *a.e.* on $\Gamma_{\mathrm{S}}$ and that $\Gamma_{\mathrm{S}}$ is decomposed (up to a null set) as

$$\Gamma_{\mathrm{S_R}} \cup \Gamma_{\mathrm{S_D}} \cup \Gamma_{\mathrm{S_S}},$$

where $\Gamma_{\mathrm{S_R}}$, $\Gamma_{\mathrm{S_D}}$ and $\Gamma_{\mathrm{S_S}}$ are three measurable pairwise disjoint subsets of $\Gamma_{\mathrm{S}}$. Moreover, let us consider $w \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$, $\xi \in \mathrm{L}^2(\Gamma_{\mathrm{S}},\mathbb{R}^d)$, $h \in \mathrm{L}^2(\Gamma_{\mathrm{S}})$, $\zeta \in \mathrm{L}^\infty(\Gamma_{\mathrm{S}},\mathbb{R}^d)$ such that $||\zeta||_{\mathrm{L}^\infty(\Gamma_{\mathrm{S_R}}\cup\Gamma_{\mathrm{S_S}},\mathbb{R}^d)} \leq 1$ and $k \in \mathrm{L}^4(\Gamma_{\mathrm{S_R}})$ such that $k \geq 0$ *a.e.* on $\Gamma_{\mathrm{S_R}}$. The general tangential Signorini problem is given by

$$\left\{\begin{aligned} -\mathrm{div}\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^{\top}\right) &= f && \text{in } \Omega,\\ u &= 0 && \text{on } \Gamma_{\mathrm{D}},\\ \left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^{\top}\right)\mathrm{n}\cdot\mathrm{n} &= \xi_{\mathrm{n}} && \text{on } \Gamma_{\mathrm{S}},\\ \left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^{\top}\right)\mathrm{n}\right)_\tau + k\left(u_\tau - (u_\tau\cdot\zeta_\tau)\,\zeta_\tau\right) &= h\zeta_\tau + \xi_\tau && \text{on } \Gamma_{\mathrm{S_R}},\\ u_\tau &= w_\tau && \text{on } \Gamma_{\mathrm{S_D}},\\ (u_\tau - w_\tau) \in \mathbb{R}_-\zeta_\tau, \left(\left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^{\top}\right)\mathrm{n}\right)_\tau - h\zeta_\tau - \xi_\tau\right)\cdot\zeta_\tau \leq 0 \text{ and}&&&\\ (u_\tau - w_\tau)\cdot\left(\left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^{\top}\right)\mathrm{n}\right)_\tau - h\zeta_\tau - \xi_\tau\right) &= 0 && \text{on } \Gamma_{\mathrm{S_S}}.\end{aligned}\right. \tag{GTSP}$$

**Remark 4.2.12.** *The boundary conditions on $\Gamma_{\mathrm{S_S}}$ are called the tangential Signorini conditions. Indeed they are close to the Signorini unilateral conditions, but they are on the tangential components instead of the normal components. We do not know if these conditions have a physical sense, however they appear naturally when we investigate the sensitivity analysis of a Tresca friction problem in the linear elastic model (see Section 6.2).*

**Definition 4.2.13** (Strong solution to the general tangential Signorini problem)**.** *A (strong) solution to the general tangential Signorini problem* (GTSP) *is a function $u \in \mathrm{H}^1(\Omega,\mathbb{R}^d)$ such that*

-div $\left(z\mathrm{A}\,[\nabla u\mathrm{B}]\,\mathrm{B}^\top\right) = f$ *in* $\mathcal{D}'(\Omega,\mathbb{R}^d)$, $u = 0$ *a.e. on* $\Gamma_{\mathrm{D}}$, $u_\tau = w_\tau$ *a.e. on* $\Gamma_{\mathrm{S_D}}$, $\left(z\mathrm{A}\,[\nabla u\mathrm{B}]\,\mathrm{B}^\top\right)\mathrm{n} \in \mathrm{L}^2(\Gamma_{\mathrm{S}},\mathbb{R}^d)$ *with* $\left(z\mathrm{A}\,[\nabla u\mathrm{B}]\,\mathrm{B}^\top\right)\mathrm{n}\cdot\mathrm{n} = \xi_{\mathrm{n}}$ *a.e. on* $\Gamma_{\mathrm{S}}$, $\left(\left(z\mathrm{A}\,[\nabla u\mathrm{B}]\,\mathrm{B}^\top\right)\mathrm{n}\right)_\tau + k\,(u_\tau - (u_\tau\cdot\zeta_\tau)\zeta_\tau) = h\zeta_\tau + \xi_\tau$ *a.e. on* $\Gamma_{\mathrm{S_R}}$, *and where* $(u_\tau - w_\tau) \in \mathbb{R}_-\zeta_\tau$, $\left(\left(\left(z\mathrm{A}\,[\nabla u\mathrm{B}]\,\mathrm{B}^\top\right)\mathrm{n}\right)_\tau - h\zeta_\tau - \xi_\tau\right)\cdot\zeta_\tau \leq 0$ *and* $(u_\tau - w_\tau)\cdot(((z\mathrm{A}\,[\nabla u\mathrm{B}]\,\mathrm{B}^\top)\mathrm{n})_\tau - h\zeta_\tau - \xi_\tau) = 0$ *a.e. on* $\Gamma_{\mathrm{S_S}}$.

**Definition 4.2.14** (Weak solution to the general tangential Signorini problem). *A weak solution to the general tangential Signorini problem* (GTSP) *is a function* $u \in \mathcal{K}^1_{\tau,w}(\Omega,\mathbb{R}^d)$ *such that*

$$\int_\Omega z\mathrm{A}\,[(\nabla u)\mathrm{B}]\,\mathrm{B}^\top : \nabla(v-u) \geq \int_\Omega f\cdot(v-u) + \int_{\Gamma_{\mathrm{S}}} \xi_{\mathrm{n}}(v_{\mathrm{n}} - u_{\mathrm{n}}) + \int_{\Gamma_{\mathrm{S_S}}} (h\zeta_\tau + \xi_\tau)\cdot(v_\tau - u_\tau)$$
$$+ \int_{\Gamma_{\mathrm{S_R}}} (h\zeta_\tau + \xi_\tau - k\,(u_\tau - (u_\tau\cdot\zeta_\tau)\zeta_\tau))\cdot(v_\tau - u_\tau), \qquad \forall v \in \mathcal{K}^1_{\tau,w}(\Omega,\mathbb{R}^d), \quad (4.2)$$

*where* $\mathcal{K}^1_{\tau,w}(\Omega,\mathbb{R}^d)$ *is the nonempty closed convex subset of* $\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ *given by*

$$\mathcal{K}^1_{\tau,w}(\Omega,\mathbb{R}^d) := \left\{ v \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d) \mid v_\tau = w_\tau \text{ a.e. on } \Gamma_{\mathrm{S_D}} \text{ and } (v_\tau - w_\tau) \in \mathbb{R}_-\zeta_\tau \text{ a.e. on } \Gamma_{\mathrm{S_S}} \right\}.$$

Similarly to Subsection 4.1.2, to prove that a weak solution is a (strong) solution, one can assume that the decomposition $\Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{S_R}} \cup \Gamma_{\mathrm{S_D}} \cup \Gamma_{\mathrm{S_S}}$ of $\Gamma$ is consistent in the following sense.

**Definition 4.2.15** (Consistent decomposition). *The decomposition* $\Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{S_R}} \cup \Gamma_{\mathrm{S_D}} \cup \Gamma_{\mathrm{S_S}}$ *of* $\Gamma$ *is said to be consistent if*

(i) *for almost all* $s \in \Gamma_{\mathrm{S_S}}$, $s \in \mathrm{int}_\Gamma(\Gamma_{\mathrm{S_S}})$;

(ii) *the nonempty closed convex subset* $\mathcal{K}^{1/2}_{\tau,w}(\Gamma,\mathbb{R}^d)$ *of* $\mathrm{H}^{1/2}(\Gamma,\mathbb{R}^d)$ *defined by*

$$\mathcal{K}^{1/2}_{\tau,w}(\Gamma,\mathbb{R}^d) := \Big\{ v \in \mathrm{H}^{1/2}(\Gamma,\mathbb{R}^d) \mid v = 0 \text{ a.e. on } \Gamma_{\mathrm{D}},\ v_\tau = w_\tau \text{ a.e. on } \Gamma_{\mathrm{S_D}} \text{ and } (v_\tau - w_\tau) \in \mathbb{R}_-\zeta_\tau \text{ a.e. on } \Gamma_{\mathrm{S_S}} \Big\},$$

*is dense in the nonempty closed convex subset* $\mathcal{K}^0_{\tau,w}(\Gamma,\mathbb{R}^d)$ *of* $\mathrm{L}^2(\Gamma,\mathbb{R}^d)$ *given by*

$$\mathcal{K}^0_{\tau,w}(\Gamma,\mathbb{R}^d) := \Big\{ v \in \mathrm{L}^2(\Gamma,\mathbb{R}^d) \mid v = 0 \text{ a.e. on } \Gamma_{\mathrm{D}},\ v_\tau = w_\tau \text{ a.e. on } \Gamma_{\mathrm{S_D}} \text{ and } (v_\tau - w_\tau) \in \mathbb{R}_-\zeta_\tau \text{ a.e. on } \Gamma_{\mathrm{S_S}} \Big\}.$$

**Proposition 4.2.16.** *Let* $u \in \mathrm{H}^1(\Omega,\mathbb{R}^d)$.

(i) *If* $u$ *is a (strong) solution to the general tangential Signorini problem* (GTSP), *then* $u$ *is a weak solution to the general tangential Signorini problem* (GTSP).

(ii) *If* $u$ *is a weak solution to the general tangential Signorini problem* (GTSP) *with the additional assumptions that* $(z\mathrm{A}\,[\nabla u\mathrm{B}]\,\mathrm{B}^\top)\mathrm{n} \in \mathrm{L}^2(\Gamma_{\mathrm{S}},\mathbb{R}^d)$ *and that the decomposition* $\Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{S_R}}\cup\Gamma_{\mathrm{S_D}}\cup\Gamma_{\mathrm{S_S}}$ *of* $\Gamma$ *is consistent, then* $u$ *is a (strong) solution to the general tangential Signorini problem* (GTSP).

*Proof.* (i) Assume that $u$ is a (strong) solution to the general tangential Signorini problem (GTSP). Then, from the boundary conditions, $u \in \mathcal{K}^1_{\tau,w}(\Omega,\mathbb{R}^d)$. Moreover, since $-\mathrm{div}\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right) = f$ in $\mathcal{D}'(\Omega,\mathbb{R}^d)$ and $f \in \mathrm{L}^2(\Omega,\mathbb{R}^d)$, thus $-\mathrm{div}\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right) = f$ in $\mathrm{L}^2(\Omega,\mathbb{R}^d)$. Hence, from divergence formula (see Proposition 2.1.1), one gets

$$\int_\Omega z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top : \nabla(v-u) - \left\langle \left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}, v-u\right\rangle_{\mathrm{H}^{-1/2}(\Gamma,\mathbb{R}^d)\times\mathrm{H}^{1/2}(\Gamma,\mathbb{R}^d)} = \int_\Omega f\cdot(v-u),$$

for all $v \in \mathcal{K}^1_{\tau,w}(\Omega,\mathbb{R}^d)$. Moreover, for all $v \in \mathcal{K}^1_{\tau,w}(\Omega,\mathbb{R}^d)$, one has $v \in \mathrm{H}^{1/2}_{00}(\Gamma_\mathrm{S},\mathbb{R}^d)$ which can be identified to a linear subspace of $\mathrm{H}^{1/2}(\Gamma,\mathbb{R}^d)$, hence

$$\int_\Omega z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top : \nabla(v-u) - \left\langle \left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}, v-u\right\rangle_{\mathrm{H}^{-1/2}_{00}(\Gamma_\mathrm{S},\mathbb{R}^d)\times\mathrm{H}^{1/2}_{00}(\Gamma_\mathrm{S},\mathbb{R}^d)} = \int_\Omega f\cdot(v-u),$$

for all $v \in \mathcal{K}^1_{\tau,w}(\Omega,\mathbb{R}^d)$. Since $\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n} \in \mathrm{L}^2(\Gamma_\mathrm{S},\mathbb{R}^d)$, one deduces

$$\left\langle \left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}, v-u\right\rangle_{\mathrm{H}^{-1/2}_{00}(\Gamma_\mathrm{S},\mathbb{R}^d))\times\mathrm{H}^{1/2}_{00}(\Gamma_\mathrm{S},\mathbb{R}^d)} = \int_{\Gamma_\mathrm{S}} \left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\cdot(v-u),$$

for all $v \in \mathcal{K}^1_{\tau,w}(\Omega,\mathbb{R}^d)$, which is also

$$\int_{\Gamma_\mathrm{S}} \left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\cdot(v-u) = \int_{\Gamma_\mathrm{S}} \left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\cdot\mathrm{n}(v_\mathrm{n}-u_\mathrm{n}) + \int_{\Gamma_{\mathrm{S_R}}\cup\Gamma_{\mathrm{S_S}}} \left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\right)_\tau\cdot(v_\tau-u_\tau),$$

for all $v \in \mathcal{K}^1_{\tau,w}(\Omega,\mathbb{R}^d)$. From the boundary conditions, one has $\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\cdot\mathrm{n} = \xi_\mathrm{n}$ *a.e.* on $\Gamma_\mathrm{S}$ and $\left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\right)_\tau = h\zeta_\tau + \xi_\tau - k\left(u_\tau - (u_\tau\cdot\zeta_\tau)\zeta_\tau\right)$ *a.e.* on $\Gamma_{\mathrm{S_R}}$. Moreover, for all $v \in \mathcal{K}^1_{\tau,w}(\Omega,\mathbb{R}^d)$, one has

$$\begin{aligned}\left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\right)_\tau\cdot(v_\tau-u_\tau) &= \left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\right)_\tau\cdot v_\tau - \left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\right)_\tau\cdot u_\tau \\ &\geq (h\zeta_\tau+\xi_\tau)\cdot v_\tau - (h\zeta_\tau+\xi_\tau)\cdot u_\tau = (h\zeta_\tau+\xi_\tau)\cdot(v_\tau-u_\tau),\end{aligned}$$

*a.e.* on $\Gamma_{\mathrm{S_S}}$. This concludes the proof of the first item.

(ii) Assume that $u$ is a weak solution to the general tangential Signorini problem (GTSP). Then $u \in \mathcal{K}^1_{\tau,w}(\Omega,\mathbb{R}^d)$. For all $\varphi \in \mathcal{D}(\Omega,\mathbb{R}^d)$, if we consider $v = u \pm \varphi \in \mathcal{K}^1_{\tau,w}(\Omega,\mathbb{R}^d)$ in Inequality (4.2), one gets $-\mathrm{div}\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right) = f$ in $\mathcal{D}'(\Omega,\mathbb{R}^d)$, thus in $\mathrm{L}^2(\Omega,\mathbb{R}^d)$ since $f \in \mathrm{L}^2(\Omega,\mathbb{R}^d)$. Hence, using divergence formula (see Proposition 2.1.1) in Inequality (4.2), one deduces

$$\begin{aligned}\left\langle \left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}, v-u\right\rangle_{\mathrm{H}^{-1/2}(\Gamma,\mathbb{R}^d)\times\mathrm{H}^{1/2}(\Gamma,\mathbb{R}^d)} \geq{}& \int_{\Gamma_\mathrm{S}} \xi_\mathrm{n}(v_\mathrm{n}-u_\mathrm{n}) + \int_{\Gamma_{\mathrm{S_S}}} (h\zeta_\tau+\xi_\tau)\cdot(v_\tau-u_\tau) \\ &+ \int_{\Gamma_{\mathrm{S_R}}} \left(h\zeta_\tau+\xi_\tau-k\left(u_\tau-(u_\tau\cdot\zeta_\tau)\zeta_\tau\right)\right)\cdot(v_\tau-u_\tau),\end{aligned}$$

for all $v \in \mathcal{K}^1_{\tau,w}(\Omega,\mathbb{R}^d)$. Moreover, similarly to (i) and from the assumption $\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n} \in$

$\mathrm{L}^2(\Gamma_{\mathrm{S}},\mathbb{R}^d)$, one gets

$$\int_{\Gamma_{\mathrm{S}}} \left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\cdot\mathrm{n}(v_\mathrm{n}-u_\mathrm{n}) + \int_{\Gamma_{\mathrm{S_R}}\cup\Gamma_{\mathrm{S_S}}} \left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\right)_\tau\cdot(v_\tau-u_\tau) \geq \int_{\Gamma_{\mathrm{S}}} \xi_\mathrm{n}(v_\mathrm{n}-u_\mathrm{n})$$
$$+\int_{\Gamma_{\mathrm{S_S}}}(h\zeta_\tau+\xi_\tau)\cdot(v_\tau-u_\tau) + \int_{\Gamma_{\mathrm{S_R}}}(h\zeta_\tau+\xi_\tau-k\left(u_\tau-(u_\tau\cdot\zeta_\tau)\zeta_\tau\right))\cdot(v_\tau-u_\tau), \quad (4.3)$$

for all $v\in\mathcal{K}^1_{\tau,w}(\Omega,\mathbb{R}^d)$, also for all $v\in\mathcal{K}^{1/2}_{\tau,w}(\Gamma,\mathbb{R}^d)$. Since $k\in\mathrm{L}^4(\Gamma_{\mathrm{S_R}})$, then from the continuous embedding $\mathrm{H}^1(\Omega,\mathbb{R}^d)\hookrightarrow\mathrm{L}^4(\Gamma,\mathbb{R}^d)$ (see Proposition 2.1.7), one deduces that $ku_\tau\in\mathrm{L}^2(\Gamma_{\mathrm{S_R}},\mathbb{R}^d)$. From the consistant decomposition of $\Gamma_{\mathrm{S_R}}\cup\Gamma_{\mathrm{S_D}}\cup\Gamma_{\mathrm{S_S}}$, one gets that $\mathcal{K}^{1/2}_{\tau,w}(\Gamma,\mathbb{R}^d)$ is dense in $\mathcal{K}^0_{\tau,w}(\Gamma,\mathbb{R}^d)$, and since $||\zeta||_{\mathrm{L}^\infty(\Gamma_{\mathrm{S_R}}\cup\Gamma_{\mathrm{S_S}},\mathbb{R}^d)}\leq 1$, one deduces that Inequality (4.3) holds true for all $v\in\mathcal{K}^0_{\tau,w}(\Gamma,\mathbb{R}^d)$. Let us consider the function $v:=u\pm\psi\mathrm{n}\in\mathcal{K}^0_{\tau,w}(\Gamma,\mathbb{R}^d)$ in Inequality (4.3), where $\psi\in\mathrm{L}^2(\Gamma)$ is defined by

$$\psi:=\begin{cases} 0 & \text{on } \Gamma_\mathrm{D},\\ \phi & \text{on } \Gamma_\mathrm{S},\end{cases}$$

for any function $\phi$ in $\mathrm{L}^2(\Gamma_\mathrm{S})$. Then it follows that $\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\cdot\mathrm{n}=\xi_\mathrm{n}$ *a.e.* on $\Gamma_\mathrm{S}$. Now let us consider $v:=u\pm v_\phi\in\mathcal{K}^0_{\tau,w}(\Gamma,\mathbb{R}^d)$ in Inequality (4.3), where $v_\phi\in\mathrm{L}^2(\Gamma,\mathbb{R}^d)$ is defined by

$$v_\phi:=\begin{cases} 0 & \text{on } \Gamma_\mathrm{D}\cup\Gamma_{\mathrm{S_D}}\cup\Gamma_{\mathrm{S_S}},\\ \phi & \text{on } \Gamma_{\mathrm{S_R}},\end{cases}$$

with any function $\phi$ in $\mathrm{L}^2(\Gamma_{\mathrm{S_R}},\mathbb{R}^d)$. Thus one deduces that $\left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\right)_\tau = h\zeta_\tau+\xi_\tau-k\left(u_\tau-(u_\tau\cdot\zeta_\tau)\zeta_\tau\right)$ *a.e.* on $\Gamma_{\mathrm{S_R}}$. Hence, Inequality (4.3) becomes

$$\int_{\Gamma_{\mathrm{S_S}}}\left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\right)_\tau\cdot(v_\tau-u_\tau)\geq\int_{\Gamma_{\mathrm{S_S}}}(h\zeta_\tau+\xi_\tau)\cdot(v_\tau-u_\tau), \quad (4.4)$$

for all $v\in\mathcal{K}^0_{\tau,w}(\Gamma,\mathbb{R}^d)$. Let $s\in\Gamma_{\mathrm{S_S}}$ be a Lebesgue point of $\left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\right)_\tau\cdot\zeta_\tau\in\mathrm{L}^2(\Gamma_{\mathrm{S_S}})$ and of $(h\zeta_\tau+\xi_\tau)\cdot\zeta_\tau\in\mathrm{L}^2(\Gamma_{\mathrm{S_S}})$, such that $s\in\mathrm{int}_\Gamma(\Gamma_{\mathrm{S_S}})$. Let us consider the function $v:=u-\psi\zeta_\tau\in\mathcal{K}^0_{\tau,w}(\Gamma,\mathbb{R}^d)$ in Inequality (4.4), where $\psi\in\mathrm{L}^2(\Gamma)$ is defined by

$$\psi:=\begin{cases} 1 & \text{on } B_\Gamma(s,\varepsilon),\\ 0 & \text{on } \Gamma\backslash B_\Gamma(s,\varepsilon),\end{cases}$$

for $\varepsilon>0$ such that $B_\Gamma(s,\varepsilon)\subset\Gamma_{\mathrm{S_S}}$. Thus, one gets

$$\frac{1}{|B_\Gamma(s,\varepsilon)|}\int_{B_\Gamma(s,\varepsilon)}\left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\right)_\tau\cdot\zeta_\tau\leq\frac{1}{|B_\Gamma(s,\varepsilon)|}\int_{B_\Gamma(s,\varepsilon)}(h\zeta_\tau+\xi_\tau)\cdot\zeta_\tau,$$

and thus $(\left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\right)_\tau(s)-h(s)\zeta_\tau(s)-\xi_\tau(s))\cdot\zeta_\tau(s)\leq 0$ by letting $\varepsilon\to 0^+$. Moreover, since almost every point of $\Gamma_{\mathrm{S_S}}$ is in $\mathrm{int}_\Gamma(\Gamma_{\mathrm{S_S}})$ and is a Lesbegue point of $\left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\right)_\tau\cdot\zeta_\tau\in\mathrm{L}^2(\Gamma_{\mathrm{S_S}})$ and of $(h\zeta_\tau+\xi_\tau)\cdot\zeta_\tau\in\mathrm{L}^2(\Gamma_{\mathrm{S_S}})$, one deduces $(\left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\right)_\tau-h\zeta_\tau-\xi_\tau)\cdot\zeta_\tau\leq 0$ *a.e.* on $\Gamma_{\mathrm{S_S}}$.

Finally, by considering $v = w$ and $v = 2u - w$ in Inequality (4.4), one deduces

$$\int_{\Gamma_{\mathrm{S_S}}} (u_\tau - w_\tau) \cdot \left( \left( \left( z\mathrm{A}\left[\nabla u \mathrm{B}\right] \mathrm{B}^\top \right) \mathrm{n} \right)_\tau - h\zeta_\tau - \xi_\tau \right) = 0,$$

thus, $(u_\tau - w_\tau) \cdot \left( \left( \left( z\mathrm{A}\left[\nabla u \mathrm{B}\right] \mathrm{B}^\top \right) \mathrm{n} \right)_\tau - h\zeta_\tau - \xi_\tau \right) = 0$ *a.e.* on $\Gamma_{\mathrm{S_S}}$ since $u_\tau \in \mathcal{K}^1_{\tau,w}(\Omega, \mathbb{R}^d)$. The proof is complete. □

Let us prove that there exists a unique solution to the general tangential Signorini problem (GTSP). To this aim, we introduce the functional $\Psi$ defined by

$$\begin{array}{rccl} \Psi : & \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d) & \longrightarrow & \mathbb{R} \\ & v & \longmapsto & \Psi(v) := \displaystyle\int_{\Gamma_{\mathrm{S_R}}} \frac{k}{2} \left( \|v_\tau\|^2 - |v_\tau \cdot \zeta_\tau|^2 \right). \end{array}$$

Note that $\Psi$ is well defined since $k \in \mathrm{L}^4(\Gamma_{\mathrm{S_R}})$, $||\zeta|| \leq 1$ *a.e.* on $\Gamma_{\mathrm{S_R}}$ and from the continuous embedding $\mathrm{H}^1(\Omega, \mathbb{R}^d) \hookrightarrow \mathrm{L}^4(\Gamma, \mathbb{R}^d)$.

**Lemma 4.2.17.** *The functional $\Psi$ is convex and Fréchet differentiable on $\mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$, and for all $v_0 \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$, its gradient denoted by $\nabla\Psi(v_0) \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$, is the unique solution to the Dirichlet-Neumann problem*

$$\begin{cases} -\mathrm{div}\left( z\mathrm{A}\left[\nabla(\nabla\Psi(v_0))\mathrm{B}\right] \mathrm{B}^\top \right) = 0 & \text{in } \Omega, \\ \nabla\Psi(v_0) = 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \left( z\mathrm{A}\left[\nabla(\nabla\Psi(v_0))\mathrm{B}\right] \mathrm{B}^\top \right) \mathrm{n} = 0 & \text{on } \Gamma_{\mathrm{S_D}} \cup \Gamma_{\mathrm{S_S}}, \\ \left( z\mathrm{A}\left[\nabla(\nabla\Psi(v_0))\mathrm{B}\right] \mathrm{B}^\top \right) \mathrm{n} \cdot \mathrm{n} = 0 & \text{on } \Gamma_{\mathrm{S_R}}, \\ \left( \left( z\mathrm{A}\left[\nabla(\nabla\Psi(v_0))\mathrm{B}\right] \mathrm{B}^\top \right) \mathrm{n} \right)_\tau = k\left( v_{0_\tau} - \left( v_{0_\tau} \cdot \zeta_\tau \right) \zeta_\tau \right) & \text{on } \Gamma_{\mathrm{S_R}}. \end{cases} \tag{4.5}$$

*Proof.* Let us start with the convexity of $\Psi$ and consider $v_1, v_2 \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$, $\lambda \in ]0,1[$. Then

$$\Psi(\lambda v_1 + (1-\lambda)v_2) - \lambda\Psi(v_1) - (1-\lambda)\Psi(v_2) =$$
$$\int_{\Gamma_{\mathrm{S_R}}} -\frac{k}{2}\lambda(1-\lambda)[\Big[ \|v_{1_\tau}\|^2 + \|v_{2_\tau}\|^2 + 2v_{1_\tau} \cdot v_{2_\tau} - |v_{1_\tau} \cdot \zeta_\tau|^2 - |v_{2_\tau} \cdot \zeta_\tau|^2 - 2(v_{1_\tau} \cdot \zeta_\tau)(v_{2_\tau} \cdot \zeta_\tau) \Big] =$$
$$\int_{\Gamma_{\mathrm{S_R}}} -\frac{k}{2}\lambda(1-\lambda) \|v_{1_\tau} + v_{2_\tau}\|^2 + \int_{\Gamma_{\mathrm{S_R}}} \frac{k}{2}\lambda(1-\lambda) \left|(v_{1_\tau} + v_{2_\tau}) \cdot \zeta_\tau\right|^2.$$

Since $k \geq 0$ and $\|\zeta\| \leq 1$ *a.e.* on $\Gamma_{\mathrm{S_R}}$, one deduces

$$\Psi(\lambda v_1 + (1-\lambda)v_2) - \lambda\Psi(v_1) - (1-\lambda)\Psi(v_2) \leq$$
$$\int_{\Gamma_{\mathrm{S_R}}} -\frac{k}{2}\lambda(1-\lambda) \|v_{1_\tau} + v_{2_\tau}\|^2 + \int_{\Gamma_{\mathrm{S_R}}} \frac{k}{2}\lambda(1-\lambda) \|v_{1_\tau} + v_{2_\tau}\|^2 \|\zeta_\tau\|^2 \leq 0,$$

thus $\Psi$ is convex on $\mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$.

Now let us prove that $\Psi$ is Fréchet differentiable. For $v_0 \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ and $v \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$,

$$\Psi(v_0 + v) = \Psi(v_0) + \int_{\Gamma_{\mathrm{S_R}}} k\left(v_{0_\tau} - \left(v_{0_\tau} \cdot \zeta_\tau\right)\zeta_\tau\right) \cdot v_\tau + \int_{\Gamma_{\mathrm{S_R}}} \frac{k}{2}\left(\|v_\tau\|^2 - |v_\tau \cdot \zeta_\tau|^2\right).$$

Moreover, from the continuous embedding $\mathrm{H}^1(\Omega, \mathbb{R}^d) \hookrightarrow \mathrm{L}^4(\Gamma, \mathbb{R}^d)$, $k \in \mathrm{L}^4(\Gamma_{\mathrm{S_R}})$ and $\|\zeta\| \leq 1$ a.e. on $\Gamma_{\mathrm{S_R}}$, one deduces

$$\int_{\Gamma_{\mathrm{S_R}}} \frac{k}{2}\left(\|v_\tau\|^2 - |v_\tau \cdot \zeta_\tau|^2\right) = o(v),$$

where $o$ stands for the standard Bachmann-Landau notation, and that the map

$$v \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d) \mapsto \int_{\Gamma_{\mathrm{S_R}}} k\left(v_{0_\tau} - \left(v_{0_\tau} \cdot \zeta_\tau\right)\zeta_\tau\right) \cdot v_\tau \in \mathbb{R},$$

is linear and continuous. Therefore $\Psi$ is Fréchet differentiable in $v_0 \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$, and

$$\langle \nabla\Psi(v_0), v\rangle_{z,\mathrm{A},\mathrm{B}} = \int_{\Gamma_{\mathrm{S_R}}} k\left(v_{0_\tau} - \left(v_{0_\tau} \cdot \zeta_\tau\right)\zeta_\tau\right) \cdot v_\tau, \qquad \forall v \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d),$$

i.e. $\nabla\Psi(v_0) \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ is the unique solution to the Dirichlet-Neumann problem (4.5). □

**Proposition 4.2.18.** *The general tangential Signorini problem* (GTSP) *admits a unique weak solution $u \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ which is given by*

$$u = \mathrm{prox}_{\Psi + \iota_{\mathcal{K}^1_{\tau,w}(\Omega,\mathbb{R}^d)}}(F)$$

*where $F \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ is the unique solution to the general Dirichlet-Neumann problem*

$$\begin{cases} -\mathrm{div}\left(z\mathrm{A}\left[(\nabla F)\mathrm{B}\right]\mathrm{B}^\top\right) = f & \text{in } \Omega, \\ F = 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \left(z\mathrm{A}\left[(\nabla F)\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n} = \xi + h\zeta_\tau & \text{on } \Gamma_{\mathrm{S}}, \end{cases} \tag{4.6}$$

*and* $\mathrm{prox}_{\Psi + \mathrm{I}_{\mathcal{K}^1_{\tau,w}(\Omega,\mathbb{R}^d)}}$ *stands for the proximal operator associated with the functional* $\Psi + \mathrm{I}_{\mathcal{K}^1_{\tau,w}(\Omega,\mathbb{R}^d)}$ *considered on the Hilbert space* $(\mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d), \langle\cdot,\cdot\rangle_{z,\mathrm{A},\mathrm{B}})$.

*Proof.* Let $F \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ be the solution to the general Dirichlet-Neumann problem (4.6), then

$$\langle F, v\rangle_{z,\mathrm{A},\mathrm{B}} = \int_\Omega f \cdot v + \int_{\Gamma_{\mathrm{S}}} \xi_{\mathrm{n}} v_{\mathrm{n}} + \int_{\Gamma_{\mathrm{S}}} (h\zeta_\tau + \xi_\tau) \cdot v_\tau,$$

for all $v \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$. Let us notice that $\Psi + \mathrm{I}_{\mathcal{K}^1(\Omega)}$ is a proper lower semi-continuous convex function on $\mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$. Then $u \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ is the weak solution to the general tangentiel Signorini problem (GTSP) if and only if $u \in \mathcal{K}^1_{\tau,w}(\Omega,\mathbb{R}^d)$ and

$$\langle u, v-u\rangle_{z,\mathrm{A},\mathrm{B}} \geq \int_\Omega f\cdot(v-u) + \int_{\Gamma_\mathrm{S}} \xi_\mathrm{n}(v_\mathrm{n}-u_\mathrm{n}) + \int_{\Gamma_{\mathrm{S}_\mathrm{S}}} (h\zeta_\tau+\xi_\tau)\cdot(v_\tau-u_\tau) + \int_{\Gamma_{\mathrm{S}_\mathrm{R}}} (h\zeta_\tau+\xi_\tau - k\,(u_\tau-(u_\tau\cdot\zeta_\tau)\zeta_\tau))\cdot(v_\tau-u_\tau),$$

for all $v\in\mathcal{K}^1_{\tau,w}(\Omega,\mathbb{R}^d)$, i.e. if and only if

$$\int_{\Gamma_{\mathrm{S}_\mathrm{R}}} k\,(u_\tau-(u_\tau\cdot\zeta_\tau)\,\zeta_\tau)\cdot(v_\tau-u_\tau) \geq \langle F-u, v-u\rangle_{z,\mathrm{A},\mathrm{B}}\,,$$

for all $v\in\mathcal{K}^1_{\tau,w}(\Omega,\mathbb{R}^d)$, i.e. if and only if (see Lemma 4.2.17 and Proposition 3.1.7)

$$\Psi(v)-\Psi(u)\geq \langle F-u, v-u\rangle_{z,\mathrm{A},\mathrm{B}}\,,$$

for all $v\in\mathcal{K}^1_{\tau,w}(\Omega,\mathbb{R}^d)$, i.e. if and only if

$$\langle F-u, v-u\rangle_{z,\mathrm{A},\mathrm{B}} \leq \Psi(v)-\Psi(u)+\iota_{\mathcal{K}^1_{\tau,w}(\Omega,\mathbb{R}^d)}(v)-\iota_{\mathcal{K}^1_{\tau,w}(\Omega,\mathbb{R}^d)}(u),$$

for all $v\in\mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d)$, i.e. if and only if

$$F-u\in\partial\Big(\Psi+\iota_{\mathcal{K}^1_{\tau,w}(\Omega,\mathbb{R}^d)}\Big)(u),$$

i.e. if and only if

$$u=\mathrm{prox}_{\Psi+\iota_{\mathcal{K}^1_{\tau,w}(\Omega,\mathbb{R}^d)}}(F).$$

This concludes the proof. □

### 4.2.4 A general Tresca friction problem

In this part, we assume that almost every point of $\Gamma_\mathrm{S}$ is in $\mathrm{int}_\Gamma(\Gamma_\mathrm{S})$, $h\in\mathrm{L}^2(\Gamma_\mathrm{S})$ and $g\in\mathrm{L}^2(\Gamma_\mathrm{S})$ such that $g>0$ *a.e.* on $\Gamma_\mathrm{S}$. Consider the general Tresca friction problem given by

$$\begin{cases} -\mathrm{div}\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right) = f & \text{in } \Omega,\\ u=0 & \text{on } \Gamma_\mathrm{D},\\ \left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\cdot\mathrm{Mn} = h & \text{on } \Gamma_\mathrm{S},\\ \left\|\left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\right)_{(\mathrm{Mn})^\perp}\right\|\leq g \text{ and} & \\ u_{(\mathrm{Mn})^\perp}\cdot\left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\right)_{(\mathrm{Mn})^\perp}+g\left\|u_{(\mathrm{Mn})^\perp}\right\| = 0 & \text{on } \Gamma_\mathrm{S}. \end{cases} \tag{GTP}$$

**Definition 4.2.19** (Strong solution to the general Tresca friction problem)**.** *A (strong) solution to the general Tresca friction problem* (GTP) *is a function* $u\in\mathrm{H}^1(\Omega,\mathbb{R}^d)$ *such that* $-\mathrm{div}\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)=f$ *in* $\mathcal{D}'(\Omega,\mathbb{R}^d)$, $u=0$ *a.e. on* $\Gamma_\mathrm{D}$, $\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\in\mathrm{L}^2(\Gamma_\mathrm{S},\mathbb{R}^d)$ *with* $\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\cdot\mathrm{Mn}=h$, $||\left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\right)_{(\mathrm{Mn})^\perp}||\leq g$ *and* $u_{(\mathrm{Mn})^\perp}\cdot\left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\right)_{(\mathrm{Mn})^\perp}+g||u_{(\mathrm{Mn})^\perp}||=0$ *a.e. on* $\Gamma_\mathrm{S}$.

**Definition 4.2.20** (Weak solution to the general Tresca friction problem)**.** *A weak solution to the general Tresca friction problem* (GTP) *is a function* $u \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ *such that*

$$\int_\Omega z\mathrm{A}\left[\nabla u \mathrm{B}\right]\mathrm{B}^\top : \nabla(v-u) + \int_{\Gamma_\mathrm{S}} g\left\|v_{(\mathrm{Mn})^\perp}\right\| - \int_{\Gamma_\mathrm{S}} g\left\|u_{(\mathrm{Mn})^\perp}\right\| \geq \int_\Omega f\cdot(v-u) + \int_{\Gamma_\mathrm{S}} h\mathrm{Mn}\cdot(v-u), \qquad \forall v \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d). \tag{4.7}$$

**Proposition 4.2.21.** *A function* $u \in \mathrm{H}^1(\Omega,\mathbb{R}^d)$ *is a (strong) solution to the general Tresca friction problem* (GTP) *if and only if* $u$ *is a weak solution to the general Tresca friction problem* (GTP)*.*

*Proof.* $\boxed{\Rightarrow}$ Let $u \in \mathrm{H}^1(\Omega,\mathbb{R}^d)$ be a (strong) solution to the general Tresca friction problem (GTP). Then $u \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ and $-\mathrm{div}\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right) = f$ in $\mathcal{D}'(\Omega,\mathbb{R}^d)$. Since $f \in \mathrm{L}^2(\Omega,\mathbb{R}^d)$, one deduces that $-\mathrm{div}\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right) = f$ in $\mathrm{L}^2(\Omega,\mathbb{R}^d)$. Thus it follows from the divergence formula that

$$\int_\Omega z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top : \nabla(v-u) - \left\langle\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}, v-u\right\rangle_{\mathrm{H}^{-1/2}_{00}(\Gamma_\mathrm{S},\mathbb{R}^d)\times\mathrm{H}^{1/2}_{00}(\Gamma_\mathrm{S},\mathbb{R}^d)} = \int_\Omega f\cdot(v-u),$$

for all $v \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$. Moreover, since $\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n} \in \mathrm{L}^2(\Gamma_\mathrm{S},\mathbb{R}^d)$ it follows that

$$\int_\Omega z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top : \nabla(v-u) - \int_{\Gamma_\mathrm{S}}\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\cdot(v-u) = \int_\Omega f\cdot(v-u),$$

for all $v \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$. Furthermore, one has

$$\int_{\Gamma_\mathrm{S}}\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\cdot(v-u) = \int_{\Gamma_\mathrm{S}}\left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\cdot\mathrm{Mn}\right)\mathrm{Mn}\cdot(v-u) + \int_{\Gamma_\mathrm{S}}\left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\right)_{(\mathrm{Mn})^\perp}\cdot(v-u),$$

for all $v \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$. From the Tresca friction law, one deduces that

$$-\left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\right)_{(\mathrm{Mn})^\perp}\cdot(v-u) \leq g\left(\left\|v_{(\mathrm{Mn})^\perp}\right\| - \left\|u_{(\mathrm{Mn})^\perp}\right\|\right),$$

*a.e.* on $\Gamma_\mathrm{S}$ and for all $v \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$, and since $\left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\cdot\mathrm{Mn}\right) = h$ *a.e.* on $\Gamma_\mathrm{S}$, we can conclude the first part of the proof.

$\boxed{\Leftarrow}$ Conversely, let $u \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ be a weak solution to the general Tresca friction problem (GTP). For all $\varphi \in \mathcal{D}(\Omega,\mathbb{R}^d)$, if we consider $v = u \pm \varphi \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ in Inequality (4.7), one gets the equality $-\mathrm{div}\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right) = f$ in $\mathcal{D}'(\Omega,\mathbb{R}^d)$, thus in $\mathrm{L}^2(\Omega,\mathbb{R}^d)$ since $f \in \mathrm{L}^2(\Omega,\mathbb{R}^d)$. Hence, using divergence formula in Inequality (4.7), one deduces

$$\left\langle\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}, v-u\right\rangle_{\mathrm{H}^{-1/2}_{00}(\Gamma_\mathrm{S},\mathbb{R}^d)\times\mathrm{H}^{1/2}_{00}(\Gamma_\mathrm{S},\mathbb{R}^d)} + \int_{\Gamma_\mathrm{S}} g\left\|v_{(\mathrm{Mn})^\perp}\right\| - \int_{\Gamma_\mathrm{S}} g\left\|u_{(\mathrm{Mn})^\perp}\right\| \geq \int_{\Gamma_\mathrm{S}} h\mathrm{Mn}\cdot(v-u),$$

for all $v \in \mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d)$. Moreover one has

$$-\left\langle \left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}, w\right\rangle_{\mathrm{H}^{-1/2}_{00}(\Gamma_\mathrm{S},\mathbb{R}^d)\times\mathrm{H}^{1/2}_{00}(\Gamma_\mathrm{S},\mathbb{R}^d)} \leq \left(\|g\|_{\mathrm{L}^2(\Gamma_\mathrm{S})}+\|h\|_{\mathrm{L}^2(\Gamma_\mathrm{S})}\right)\|w\|_{\mathrm{L}^2(\Gamma_\mathrm{S},\mathbb{R}^d)},$$

for all $w \in \mathrm{H}^{1/2}_{00}(\Gamma_\mathrm{S},\mathbb{R}^d)$. From Proposition 2.1.7 and 2.1.8, it follows that $\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n} \in \mathrm{L}^2(\Gamma_\mathrm{S},\mathbb{R}^d)$ and then

$$\int_{\Gamma_\mathrm{S}}\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\cdot(v-u)+\int_{\Gamma_\mathrm{S}} g\left\|v_{(\mathrm{Mn})^\perp}\right\|-\int_{\Gamma_\mathrm{S}} g\left\|u_{(\mathrm{Mn})^\perp}\right\| \geq \int_{\Gamma_\mathrm{S}} h\mathrm{Mn}\cdot(v-u),$$

for all $v \in \mathrm{H}^{1/2}_{00}(\Gamma_\mathrm{S},\mathbb{R}^d)$, that is also

$$\int_{\Gamma_\mathrm{S}}\left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\cdot\mathrm{Mn}\right)\mathrm{Mn}\cdot(v-u)+\int_{\Gamma_\mathrm{S}}\left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\right)_{(\mathrm{Mn})^\perp}\cdot(v-u)+\int_{\Gamma_\mathrm{S}} g\left\|v_{(\mathrm{Mn})^\perp}\right\|$$
$$-\int_{\Gamma_\mathrm{S}} g\left\|u_{(\mathrm{Mn})^\perp}\right\| \geq \int_{\Gamma_\mathrm{S}} h\mathrm{Mn}\cdot(v-u), \quad (4.8)$$

for all $v \in \mathrm{H}^{1/2}_{00}(\Gamma_\mathrm{S},\mathbb{R}^d)$, thus for all $v \in \mathrm{L}^2(\Gamma_\mathrm{S},\mathbb{R}^d)$ by density. Let us consider the function $w = u \pm \psi\mathrm{Mn} \in \mathrm{L}^2(\Gamma_\mathrm{S},\mathbb{R}^d)$, for any function $\psi \in \mathrm{L}^2(\Gamma_\mathrm{S})$. Then one gets in Inequality (4.8),

$$\int_{\Gamma_\mathrm{S}}\psi\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\cdot\mathrm{Mn}=\int_{\Gamma_\mathrm{S}} h\psi, \qquad \forall\psi\in\mathrm{L}^2(\Gamma_\mathrm{S}),$$

thus $\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\cdot\mathrm{Mn} = h$ in $\mathrm{L}^2(\Gamma_\mathrm{S})$. Inequality (4.8) becomes

$$\int_{\Gamma_\mathrm{S}}\left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\right)_{(\mathrm{Mn})^\perp}\cdot(v-u)+\int_{\Gamma_\mathrm{S}} g\left\|v_{(\mathrm{Mn})^\perp}\right\|-\int_{\Gamma_\mathrm{S}} g\left\|u_{(\mathrm{Mn})^\perp}\right\| \geq 0, \quad (4.9)$$

Let $s \in \Gamma_\mathrm{S}$ be a Lebesgue point of the functions $||\left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\right)_{(\mathrm{Mn})^\perp}||^2 \in \mathrm{L}^2(\Gamma_\mathrm{S})$ and also of $g||\left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\right)_{(\mathrm{Mn})^\perp}|| \in \mathrm{L}^1(\Gamma_\mathrm{S})$, which is moreover in $\mathrm{int}_\Gamma(\Gamma_\mathrm{S})$. Then, if we consider the function $v = u-\psi \in \mathrm{L}^2(\Gamma_\mathrm{S},\mathbb{R}^d)$, where $\psi \in \mathrm{L}^2(\Gamma_\mathrm{S},\mathbb{R}^d)$ is defined by

$$\psi := \begin{cases} \left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\right)_{(\mathrm{Mn})^\perp} & \text{on } B_\Gamma(s,\varepsilon), \\ 0 & \text{on } \Gamma_\mathrm{S}\backslash B_\Gamma(s,\varepsilon), \end{cases}$$

for $\varepsilon > 0$ such that $B_\Gamma(s,\varepsilon) \subset \Gamma_\mathrm{S}$. Thus, one gets in Inequality (4.9)

$$\frac{1}{|B_\Gamma(s,\varepsilon)|}\int_{B_\Gamma(s,\varepsilon)}\left\|\left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\right)_{(\mathrm{Mn})^\perp}\right\|^2 \leq \frac{1}{|B_\Gamma(s,\varepsilon)|}\int_{B_\Gamma(s,\varepsilon)} g\left\|\left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\right)_{(\mathrm{Mn})^\perp}\right\|,$$

thus $\left\|\left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\right)_{(\mathrm{Mn})^\perp}(s)\right\|^2 \leq g(s)\left\|\left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\right)_{(\mathrm{Mn})^\perp}(s)\right\|$ by letting $\varepsilon \to 0^+$, and then one has $\left\|\left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\right)_{(\mathrm{Mn})^\perp}(s)\right\| \leq g(s)$. Since almost every point of $\Gamma_\mathrm{S}$ is in $\mathrm{int}_\Gamma(\Gamma_\mathrm{S})$ and is a Lesbegue point of $||\left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\right)_{(\mathrm{Mn})^\perp}||^2 \in \mathrm{L}^2(\Gamma_\mathrm{S})$ and of $g||\left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\right)_{(\mathrm{Mn})^\perp}|| \in \mathrm{L}^1(\Gamma_\mathrm{S})$, one deduces that

$$\left\|\left(\left(z\mathrm{A}\left[\nabla u\mathrm{B}\right]\mathrm{B}^\top\right)\mathrm{n}\right)_{(\mathrm{Mn})^\perp}\right\| \leq g,$$

*a.e.* on $\Gamma_{\mathrm{S}}$. Moreover, if we consider $v = 0$ in Inequality (4.9), one gets

$$\int_{\Gamma_{\mathrm{S}}} \left(\left(z\mathrm{A}\left[\nabla u \mathrm{B}\right]\mathrm{B}^{\top}\right)\mathrm{n}\right)_{(\mathrm{Mn})^{\perp}} \cdot u_{(\mathrm{Mn})^{\perp}} + g\left\|u_{(\mathrm{Mn})^{\perp}}\right\| \leq 0,$$

thus it follows that

$$\left(\left(z\mathrm{A}\left[\nabla u \mathrm{B}\right]\mathrm{B}^{\top}\right)\mathrm{n}\right)_{(\mathrm{Mn})^{\perp}} \cdot u_{(\mathrm{Mn})^{\perp}} + g\left\|u_{(\mathrm{Mn})^{\perp}}\right\| = 0,$$

*a.e.* on $\Gamma_{\mathrm{S}}$. This concludes the proof. □

From definition of the proximal operator (see Definition 3.1.10), one deduces the following existence/uniqueness result.

**Proposition 4.2.22.** *The general Tresca friction problem* (GTP) *admits a unique solution* $u \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ *given by*

$$u = \mathrm{prox}_{\phi}(F),$$

*where* $F \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ *is the unique solution to the general Dirichlet-Neumann problem*

$$\begin{cases} -\mathrm{div}\left(z\mathrm{A}\left[(\nabla F)\mathrm{B}\right]\mathrm{B}^{\top}\right) = f & \text{in } \Omega, \\ F = 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \left(z\mathrm{A}\left[(\nabla F)\mathrm{B}\right]\mathrm{B}^{\top}\right)\mathrm{n} = h\mathrm{Mn} & \text{on } \Gamma_{\mathrm{S}}, \end{cases}$$

*and where* $\mathrm{prox}_{\phi}$ *stands for the proximal operator associated with the Tresca friction functional* $\phi$ *defined by*

$$\begin{array}{rccl} \phi: & \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d) & \longrightarrow & \mathbb{R} \\ & v & \longmapsto & \phi(v) := \displaystyle\int_{\Gamma_{\mathrm{S}}} g\left\|v_{(\mathrm{Mn})^{\perp}}\right\|, \end{array}$$

*considered on the Hilbert space* $(\mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d), \langle\cdot,\cdot\rangle_{z,\mathrm{A},\mathrm{B}})$.

*Proof.* Let $F \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ be the solution to the above general Dirichlet-Neumann problem. Then $u \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ is solution to the general Tresca friction problem (GTP) if and only if

$$\langle u, v-u\rangle_{z,\mathrm{A},\mathrm{B}} + \int_{\Gamma_{\mathrm{S}}} g\left\|v_{(\mathrm{Mn})^{\perp}}\right\| - \int_{\Gamma_{\mathrm{S}}} g\left\|u_{(\mathrm{Mn})^{\perp}}\right\| \geq \langle F, v-u\rangle_{z,\mathrm{A},\mathrm{B}}\, \forall v \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d),$$

that is, if and only if

$$F - u \in \partial\phi(u),$$

that is, if and only if

$$u = \mathrm{prox}_{\phi}(F),$$

which concludes the proof. □

**Remark 4.2.23.** The assumption that almost every point of $\Gamma_{\mathrm{S}}$ is in $\mathrm{int}_{\Gamma}(\Gamma_{\mathrm{S}})$ is only used to prove that a weak solution to the general Tresca friction problem (GTP) is also a (strong) solution, more precisely to get the Tresca friction law on $\Gamma_{\mathrm{S}}$. Of course, some sets do not satisfy this assumption, for instance the well-known Smith–Volterra–Cantor set (see, e.g, [5, Example 6.15 Section 6 Chapter

1]). Nevertheless it is trivially satisfied in most of standard cases found in practice. Furthermore, if this assumption is not satisfied, one can also prove that the weak solution to the general Tresca friction problem (GTP) is a (strong) solution by adding the assumption that $g \in \mathrm{L}^{\infty}(\Gamma_{\mathrm{S}})$, and by using the isometry between the dual of $(\mathrm{L}^1(\Gamma_{\mathrm{S}},\mathbb{R}^d), \|\cdot\|_{\mathrm{L}^1(\Gamma_{\mathrm{S}},\mathbb{R}^d)_g})$ and $\mathrm{L}^{\infty}(\Gamma_{\mathrm{S}},\mathbb{R}^d)$ (with its standard norm $\|\cdot\|_{\mathrm{L}^{\infty}(\Gamma_{\mathrm{S}},\mathbb{R}^d)}$) where $\|\cdot\|_{\mathrm{L}^1(\Gamma_{\mathrm{S}},\mathbb{R}^d)_g}$ is the norm defined by

$$\begin{array}{rrcl} \|\cdot\|_{\mathrm{L}^1(\Gamma_{\mathrm{S}},\mathbb{R}^d)_g} : & \mathrm{L}^1(\Gamma_{\mathrm{S}},\mathbb{R}^d) & \longrightarrow & \mathbb{R} \\ & v & \longmapsto & \displaystyle\int_{\Gamma_{\mathrm{S}}} g\,\|v\|\,. \end{array}$$

We refer to [31, Chapitre 3] for a similar proof.

Part II

# Sensitivity analysis with respect to the data

In this second part, we investigate the sensitivity analysis of boundary value problems involving the Signorini unilateral conditions or the Tresca friction law, with respect to the data. More precisely, in Chapter 5 we focus on the scalar model of those problems. We start with the scalar Signorini problem in Section 5.1 and with the scalar Tresca friction problem in Section 5.2. Then, in Chapter 6, we focus on the linear elastic model: the Signorini problem in Section 6.1 and the Tresca friction problem in Section 6.2. Using our methodology based on convex and variational analysis, we characterize the solution to the Signorini (resp. Tresca friction) problem using the proximal operator associated with the corresponding Signorini indicator functional (resp. the nonsmooth convex Tresca friction functional). Then, by invoking the notion of twice epi-differentiability, we prove the differentiability of the solution to the parameterized Signorini (resp. Tresca friction) problem, and we characterize its derivative as the solution to a boundary value problem involving Signorini unilateral conditions. Those characterizations are the main results of this part and, to the best of our knowledge, have never been obtained in the literature.

Let us emphasize that in Subsections 5.1.2, 5.2.3 and Sections 6.1, 6.2, we perform the sensitivity analysis of a general Signorini (resp. Tresca friction) problem with additional perturbed data, the aim being to take into account a shape perturbation case. Indeed, as we saw in the Introduction, if the shape is perturbed in the classical Signorini (resp. Tresca friction) problem (see Remark 4.1.1 and 4.2.1), those general problems will appear naturally after an appropriate change of variables in the variational formulation of the classical Signorini (resp. Tresca friction) problem (see Chapters 8 and 9 for details). These results will be used in Part III in order to investigate shape optimization problems. Finally, in Chapter 7, numerical simulations are provided in order to illustrate the results obtained in Subsection 5.2.2 (note that we have chosen this subsection arbitrarily, other numerical simulations could also be computed for the results obtained in the other sections and subsections).

Chapter 5

# Sensitivity analysis of boundary value problems in the scalar model

In this chapter, $d \in \mathbb{N}^*$ is a positive integer. For all $t \geq 0$, let us consider $f_t \in \mathrm{L}^2(\Omega)$, $g_t \in \mathrm{L}^2(\Gamma)$, $k_t \in \mathrm{L}^\infty(\Omega)$ and $\mathrm{M}_t \in \mathrm{L}^\infty(\Omega, \mathbb{R}^{d\times d})$, with $k_t$ and $\mathrm{M}_t$ satisfying the assumptions described at the beginning of Section 4.1. Moreover, we assume that $\mathrm{M}_0 = \mathrm{I}$ and $k_0 = 1$ *a.e.* on $\Omega$, and that:

(H1) the map $t \in \mathbb{R}_+ \mapsto f_t \in \mathrm{L}^2(\Omega)$ is differentiable at $t = 0$, with its derivative denoted by $f_0' \in \mathrm{L}^2(\Omega)$;

(H2) the map $t \in \mathbb{R}_+ \mapsto \mathrm{M}_t \in \mathrm{L}^\infty(\Omega, \mathbb{R}^{d\times d})$ is differentiable at $t = 0$, with its derivative denoted by $\mathrm{M}_0' \in \mathrm{L}^\infty(\Omega, \mathbb{R}^{d\times d})$;

(H3) the map $t \in \mathbb{R}_+ \mapsto k_t \in \mathrm{L}^\infty(\Omega)$ is differentiable at $t = 0$, with its derivative denoted by $k_0' \in \mathrm{L}^\infty(\Omega)$.

In the sequel, the classical scalar Signorini (resp. Tresca friction) problem is called the scalar Signorini (resp. Tresca friction) problem (see Remark 4.1.1).

In Section 5.1, the sensitivity analysis of boundary value problems involving the scalar Signorini unilateral conditions is performed. In Subsection 5.1.1, a scalar Signorini problem is considered and we perturb the source term and the scalar Signorini unilateral conditions. Then in Subsection 5.1.2, a general scalar Signorini problem is considered with additional perturbed data, which cause a new difficulty compared to the Subsection 5.1.1: the scalar product is perturbed. In order to apply our methodology, in particular to use the twice epi-differentiabiliy and Proposition 3.2.10, we characterize the solution to the (general) scalar Signorini problem using the proximal operator instead of the projection operator.

In Section 5.2, we focus on boundary value problems involving the scalar Tresca friction law. More precisely, in Subsection 5.2.1 we perform the sensitivity analysis of the scalar Tresca friction problem with respect to the source term. Therefore the friction term in the Tresca friction law is not perturbed and the classical notion of twice epi-differentiability can be used (see Definition 3.2.5). In Subsection 5.2.2 we add an additional difficulty: the friction term is perturbed, thus the extended notion of twice epi-differentiability depending on a parameter is required (see Definition 3.2.11). Finally, the general scalar Tresca friction law is considered in Subsection 5.2.3, where the source term, the friction term and additional data are perturbed, which cause a perturbation of the scalar product.

## 5.1 Boundary value problems involving scalar Signorini unilateral conditions

### 5.1.1 The scalar Signorini problem

Let us consider the parameterized scalar Signorini problem

$$\begin{cases} -\Delta u_t + u_t = f_t & \text{in } \Omega, \\ u_t \leq 0,\ \partial_{\mathrm{n}} u_t \leq g_t \text{ and } u_t\,(\partial_{\mathrm{n}} u_t - g_t) = 0 & \text{on } \Gamma, \end{cases} \tag{SSP$_t$}$$

for all $t \geq 0$. From Subsection 4.1.2 (consider $\mathrm{M}(x) = \mathrm{I}$ and $k(x) = 1$ for almost all $x \in \Omega$), for all $t \geq 0$, there exists a unique weak solution $u_t \in \mathcal{K}^1_0(\Omega)$ to (SSP$_t$) which satisfies

$$\int_\Omega \nabla u_t \cdot \nabla(v - u_t) + \int_\Omega u_t\,(v - u_t) \geq \int_\Omega f_t\,(v - u_t) + \int_\Gamma g_t\,(v - u_t)\,, \qquad \forall v \in \mathcal{K}^1_0(\Omega),$$

where $\mathcal{K}^1_0(\Omega) := \{v \in \mathrm{H}^1(\Omega) \mid v \leq 0 \ \textit{a.e.} \text{ on } \Gamma\}$, and it can be expressed as

$$u_t = \mathrm{proj}_{\mathcal{K}^1_0(\Omega)}(F_t),$$

where $F_t \in \mathrm{H}^1(\Omega)$ is the unique solution to the parameterized scalar Neumann problem

$$\begin{cases} -\Delta F_t + F_t = f_t & \text{in } \Omega, \\ \partial_{\mathrm{n}} F_t = g_t & \text{on } \Gamma. \end{cases} \tag{5.1}$$

In order to use our methodology, we use the proximal operator to characterize $u_t$, that is (see Remark 3.1.11)

$$u_t = \mathrm{prox}_{\iota_{\mathcal{K}^1_0(\Omega)}}(F_t),$$

where $\mathrm{prox}_{\iota_{\mathcal{K}^1_0(\Omega)}}$ is the proximal operator associated with the Signorini indicator function $\iota_{\mathcal{K}^1_0(\Omega)}$ considered on the Hilbert space $(\mathrm{H}^1(\Omega), \langle\cdot,\cdot\rangle_{\mathrm{H}^1(\Omega)})$.

Since, for all $t \geq 0$, $u_t$ is related to $F_t$ by the proximal operator, let us start with the sensitivity analysis of the parameterized scalar Neumann problem (5.1), which will be useful for the sensitivity analysis of the parameterized scalar Signorini problem (SSP$_t$). The following proposition is easily proved using Hypothesis (H1), the linearity of the scalar Neumann problem and Proposition 4.1.5.

**Lemma 5.1.1.** *Let $F_t \in \mathrm{H}^1(\Omega)$ be the unique solution to the parameterized scalar Neumann problem* (5.1)*, for all $t \geq 0$. Assume that the map $t \in \mathbb{R}_+ \mapsto g_t \in \mathrm{L}^2(\Gamma)$ is differentiable at $t = 0$, with its derivative denoted by $g'_0 \in \mathrm{L}^2(\Gamma)$. Then the map $t \in \mathbb{R}_+ \mapsto F_t \in \mathrm{H}^1(\Omega)$ is differentiable at $t = 0$ and its derivative, denoted by $F'_0 \in \mathrm{H}^1(\Omega)$, is the unique solution to the scalar Neumann problem given by*

$$\begin{cases} -\Delta F'_0 + F'_0 = f'_0 & \textit{in } \Omega, \\ \partial_{\mathrm{n}} F'_0 = g'_0 & \textit{on } \Gamma. \end{cases} \tag{5.2}$$

Now, in order to differentiate the map $t \in \mathbb{R}_+ \mapsto u_t \in \mathrm{H}^1(\Omega)$ at $t = 0$, we will use Proposition 3.2.10

which characterizes the derivative of a map given by a proximal operator using the notion of twice epi-differentiability (see Definition 3.2.5). To this aim the twice epi-differentiability of the Signorini indicator function $\iota_{\mathcal{K}^1_0(\Omega)}$ has to be investigated. From Proposition 3.2.9, we need the polyhedricity (see Definition 3.1.5) of the subset $\mathcal{K}^1_0(\Omega)$. This has been already proved in [58] and requires some concepts from convex analysis and capacity theory reminded in Sections 2.2 and 3.1.

**Lemma 5.1.2.** *The nonempty closed convex subset $\mathcal{K}^1_0(\Omega)$ of $\mathrm{H}^1(\Omega)$ is polyhedric at $u_0 \in \mathcal{K}^1_0(\Omega)$ for $F_0 - u_0 \in \mathrm{N}_{\mathcal{K}^1_0(\Omega)}(u_0)$ and, one has*

$$\mathrm{T}_{\mathrm{N}_{\mathcal{K}^1_0(\Omega)}}(u_0) \cap (\mathbb{R}\,(F_0 - u_0))^{\perp} = \left\{ v \in \mathrm{H}^1(\Omega) \mid v \leq 0 \text{ q.e. on } \Gamma^{u_0=0} \text{ and } \langle F_0 - u_0, v\rangle_{\mathrm{H}^1(\Omega)} = 0 \right\},$$

*where $\Gamma^{u_0=0} := \{s \in \Gamma \mid u_0(s) = 0\}$, "q.e." means quasi everywhere, $\mathrm{N}_{\mathcal{K}^1_0(\Omega)}(u_0)$ (resp. $\mathrm{T}_{\mathrm{N}_{\mathcal{K}^1_0(\Omega)}}(u_0)$) is the normal cone (resp. tangent cone) to $\mathcal{K}^1_0(\Omega)$ (resp. $\mathrm{N}_{\mathcal{K}^1_0(\Omega)}$) at $u_0$.*

Using the previous Lemma, one can deduce from Proposition 3.2.9 the following corollary.

**Corollary 5.1.3.** *The Signorini indicator function $\iota_{\mathcal{K}^1_0(\Omega)}$ is twice epi-differentiable at $u_0 \in \mathcal{K}^1_0(\Omega)$ for $F_0 - u_0 \in \mathrm{N}_{\mathcal{K}^1_0(\Omega)}(u_0)$ and*

$$\mathrm{d}^2_e \iota_{\mathcal{K}^1_0(\Omega)}(u_0|F_0 - u_0) = \iota_{\mathrm{T}_{\mathrm{N}_{\mathcal{K}^1_0(\Omega)}}(u_0) \cap (\mathbb{R}(F_0 - u_0))^{\perp}}.$$

The twice epi-differentiability of the Signorini indicator function allows us to apply Proposition 3.2.10 in order to prove the differentiability of the map $t \in \mathbb{R}_+ \mapsto u_t \in \mathrm{H}^1(\Omega)$.

**Theorem 5.1.4.** *Let $u_t \in \mathrm{H}^1(\Omega)$ be the unique solution to the parameterized scalar Signorini problem ($\mathrm{SSP}_t$) for all $t \geq 0$. Assume that the map $t \in \mathbb{R}_+ \mapsto g_t \in \mathrm{L}^2(\Gamma)$ is differentiable at $t = 0$, with its derivative denoted by $g'_0 \in \mathrm{L}^2(\Gamma)$. Then the map $t \in \mathbb{R}_+ \mapsto u_t \in \mathrm{H}^1(\Omega)$ is differentiable at $t = 0$, and its derivative denoted by $u'_0 \in \mathrm{T}_{\mathrm{N}_{\mathcal{K}^1_0(\Omega)}}(u_0) \cap (\mathbb{R}\,(F_0 - u_0))^{\perp}$ is the unique solution to the variational inequality*

$$\langle u'_0, v - u'_0\rangle_{\mathrm{H}^1(\Omega)} \geq \int_\Omega f'_0\,(v - u'_0) + \int_\Gamma g'_0\,(v - u'_0), \qquad \forall v \in \mathrm{T}_{\mathrm{N}_{\mathcal{K}^1_0(\Omega)}}(u_0) \cap (\mathbb{R}\,(F_0 - u_0))^{\perp}.$$

*Proof.* Using Hypothesis (H1) and the assumption that the map $t \in \mathbb{R}_+ \mapsto g_t \in \mathrm{L}^2(\Gamma)$ is differentiable at $t = 0$, we know from Proposition 5.1.1, that the map $t \in \mathbb{R}_+ \mapsto F_t \in \mathrm{H}^1(\Omega)$ is differentiable at $t = 0$, with its derivative $F'_0 \in \mathrm{H}^1(\Omega)$ being the unique solution to the scalar Neumann problem (5.2). Thus, from Corollary 5.1.3, one can apply Proposition 3.2.10 to deduce that the map $t \in \mathbb{R}_+ \mapsto u_t \in \mathrm{H}^1(\Omega)$ is differentiable at $t = 0$, and its derivative $u'_0 \in \mathrm{H}^1(\Omega)$ satisfies

$$u'_0 = \mathrm{prox}_{\mathrm{d}^2_e \iota_{\mathcal{K}^1_0(\Omega)}(u_0|F_0 - u_0)}(F'_0),$$

which, from the definition of the proximal operator (see Proposition 3.1.10), leads to

$$\langle u'_0, v - u'_0\rangle_{\mathrm{H}^1(\Omega)} \geq \int_\Omega f'_0\,(v - u'_0) + \int_\Gamma g'_0\,(v - u'_0), \qquad \forall v \in \mathrm{T}_{\mathrm{N}_{\mathcal{K}^1_0(\Omega)}}(u_0) \cap (\mathbb{R}\,(F_0 - u_0))^{\perp}.$$

□

In [58], the same result is obtained using the conical differentiability of the projection operator. Since $\mathcal{K}_0^1(\Omega)$ is polyhedric at $u_0 \in \mathcal{K}_0^1(\Omega)$ for $F_0 - u_0 \in \mathrm{N}_{\mathcal{K}_0^1(\Omega)}(u_0)$, then from Mignot's theorem the projection operator on $\mathcal{K}_0^1(\Omega)$ is conically differentiable at $u_0$ for $F_0 - u_0$, and its conical derivative is given by $\mathrm{proj}_{\mathrm{T}_{\mathrm{N}_{\mathcal{K}_0^1(\Omega)}}(u_0)\cap(\mathbb{R}(F_0-u_0))^\perp}(F_0')$, which is exactly $\mathrm{prox}_{\iota_{\mathrm{T}_{\mathcal{K}_0^1(\Omega)}(u_0)\cap(\mathbb{R}(F_0-u_0))^\perp}}(F_0')$. Nevertheless, to the best of our knowledge, no one notices that it was possible to improve this result under additional assumptions, in order to characterize $u_0'$ as solution to a boundary value problem. Indeed, as mentioned in [19, 47], it is possible to replaced *q.e.* in the subset $\mathrm{T}_{\mathrm{N}_{\mathcal{K}_0^1(\Omega)}}(u_0) \cap (\mathbb{R}\,(F_0 - u_0))^\perp$ by *a.e.* under some hypotheses like, for instance, if $\Gamma^{u_0=0} = \overline{\mathrm{int}_\Gamma(\Gamma^{u_0=0})}$. Moreover, if we assume some regularity on $\partial_\mathrm{n} u_0$, then it allows us to characterize $u_0'$ as weak solution to a scalar Signorini problem.

**Corollary 5.1.5.** *Consider the framework of Theorem 5.1.4, and assume that* $\partial_\mathrm{n} u_0 \in \mathrm{L}^2(\Gamma)$ *and also* $\Gamma^{u_0=0} = \overline{\mathrm{int}_\Gamma(\Gamma^{u_0=0})}$. *Then* $u_0' \in \mathrm{H}^1(\Omega)$ *is the unique weak solution to the scalar Signorini problem*

$$\begin{cases} -\Delta u_0' + u_0' = f_0' & \text{in } \Omega, \\ \partial_\mathrm{n} u_0' = g_0' & \text{on } \Gamma_\mathrm{N}^{u_0,g_0}, \\ u_0' = 0 & \text{on } \Gamma_\mathrm{D}^{u_0,g_0}, \\ u_0' \leq 0,\ \partial_\mathrm{n} u_0' \leq g_0' \text{ and } u_0'\,(\partial_\mathrm{n} u_0' - g_0') = 0 & \text{on } \Gamma_{\mathrm{S}-}^{u_0,g_0}, \end{cases} \tag{5.3}$$

*where* $\Gamma$ *is decomposed, up to a null set, as* $\Gamma_\mathrm{N}^{u_0,g_0} \cup \Gamma_\mathrm{D}^{u_0,g_0} \cup \Gamma_{\mathrm{S}-}^{u_0,g_0}$, *where*

$$\begin{aligned} \Gamma_\mathrm{N}^{u_0,g_0} &:= \{s \in \Gamma \mid u_0(s) \neq 0\}, \\ \Gamma_\mathrm{D}^{u_0,g_0} &:= \{s \in \Gamma \mid u_0(s) = 0 \text{ and } \partial_\mathrm{n} u_0(s) < g_0(s)\}, \\ \Gamma_{\mathrm{S}-}^{u_0,g_0} &:= \{s \in \Gamma \mid u_0(s) = 0 \text{ and } \partial_\mathrm{n} u_0(s) = g_0(s)\}. \end{aligned}$$

*Proof.* Since $\mathrm{H}^{1/2}(\Gamma)$ is a Dirichlet space (see Example 2.2.6), then for all $v \in \mathrm{H}^{1/2}(\Gamma)$, $v$ admits a unique quasi-continuous representative for the *q.e* equivalence relation (see Proposition 2.2.4), thus it follows that (see [19, Remark 3.12 p.13] for details)

$$\mathrm{T}_{\mathrm{N}_{\mathcal{K}_0^1(\Omega)}}(u_0) \cap (\mathbb{R}\,(F_0 - u_0))^\perp = \left\{ v \in \mathrm{H}^1(\Omega) \mid v \leq 0 \ a.e. \text{ on } \Gamma^{u_0=0} \text{ and } \langle F_0 - u_0, v\rangle_{\mathrm{H}^1(\Omega)} = 0 \right\}.$$

Moreover, since $\partial_\mathrm{n} u_0 \in \mathrm{L}^2(\Gamma)$, then $u_0$ is a (strong) solution to the scalar Signorini problem ($\mathrm{SSP}_t$) for the parameter $t = 0$ (see Proposition 4.1.9 since $\Gamma$ is obviously consistent), thus from the Signorini unilateral conditions, one has

$$\langle F_0 - u_0, v\rangle_{\mathrm{H}^1(\Omega)} = \int_\Gamma (g_0 - \partial_\mathrm{n} u_0)\, v = \int_{\Gamma^{u_0=0}} (g_0 - \partial_\mathrm{n} u_0)\, v = \int_{\Gamma_\mathrm{D}^{u_0,g_0}} (g_0 - \partial_\mathrm{n} u_0)\, v = 0,$$

for all $v \in \mathrm{T}_{\mathrm{N}_{\mathcal{K}_0^1(\Omega)}}(u_0) \cap (\mathbb{R}\,(F_0 - u_0))^\perp$, and since $(g_0 - \partial_\mathrm{n} u_0)v \leq 0$ and $\partial_\mathrm{n} u_0 < g_0$ *a.e.* on $\Gamma_\mathrm{D}^{u_0,g_0}$, one deduces that $v = 0$ *a.e.* on $\Gamma_\mathrm{D}^{u_0,g_0}$, for all $v \in \mathrm{T}_{\mathrm{N}_{\mathcal{K}_0^1(\Omega)}}(u_0) \cap (\mathbb{R}\,(F_0 - u_0))^\perp$, which concludes the proof from Subsection 4.1.2. □

**Remark 5.1.6.** Consider the framework of Corollary 5.1.5. Note that, for instance, the assumption $\partial_{\mathrm{n}} u_0 \in \mathrm{L}^2(\Gamma)$ is satisfied if $u_0 \in \mathrm{H}^2(\Omega)$, which is true if $\Gamma$ is sufficiently regular (see [14, Chapter 1, Theorem I.10 p.43]). Moreover $u_0'$ is the unique weak solution to the scalar Signorini problem (5.3), but it is not necessarily a strong solution. However, in the case where $\partial_{\mathrm{n}} u_0' \in \mathrm{L}^2(\Gamma)$ and the decomposition $\Gamma_{\mathrm{N}}^{u_0,g_0} \cup \Gamma_{\mathrm{D}}^{u_0,g_0} \cup \Gamma_{\mathrm{S-}}^{u_0,g_0}$ is consistent (see Definition 4.1.8), then $u_0'$ is a strong solution to the scalar Signorini problem (5.3).

### 5.1.2 A general scalar Signorini problem

Let us consider the parameterized general scalar Signorini problem

$$\begin{cases} -\mathrm{div}(\mathrm{M}_t \nabla u_t) + k_t u_t = f_t & \text{in } \Omega, \\ u_t \leq 0,\ \mathrm{M}_t \nabla u_t \cdot \mathrm{n} \leq g_t \text{ and } u_t\left(\mathrm{M}_t \nabla u_t \cdot \mathrm{n} - g_t\right) = 0 & \text{on } \Gamma, \end{cases} \tag{GSSP$_t$}$$

for all $t \geq 0$. From Subsection 4.1.2, there exists a unique weak solution $u_t \in \mathcal{K}_0^1(\Omega)$ to (GSSP$_t$) which satisfies

$$\int_\Omega \mathrm{M}_t \nabla u_t \cdot \nabla(v - u_t) + \int_\Omega k_t u_t\,(v-u_t) \geq \int_\Omega f_t\,(v-u_t) + \int_\Gamma g_t\,(v-u_t)\,, \qquad \forall v \in \mathcal{K}_0^1(\Omega),$$

where $\mathcal{K}_0^1(\Omega) := \left\{ v \in \mathrm{H}^1(\Omega) \mid v \leq 0 \textit{ a.e.} \text{ on } \Gamma \right\}$. In order to use our methodology, we express the solution $u_t$ using the proximal operator as (see Remark 3.1.11)

$$u_t = \mathrm{prox}_{\iota_{\mathcal{K}_0^1(\Omega)}}(F_t),$$

where $F_t \in \mathrm{H}^1(\Omega)$ is the unique solution to the parameterized general scalar Neumann problem

$$\begin{cases} -\mathrm{div}(\mathrm{M}_t \nabla F_t) + k_t F_t = f_t & \text{in } \Omega, \\ \mathrm{M}_t \nabla F \cdot \mathrm{n} = g_t & \text{on } \Gamma, \end{cases} \tag{5.4}$$

and $\mathrm{prox}_{\iota_{\mathcal{K}_0^1(\Omega)}}$ is the proximal operator associated with the Signorini indicator function $\iota_{\mathcal{K}_0^1(\Omega)}$ considered on the perturbed Hilbert space $(\mathrm{H}^1(\Omega), \langle\cdot,\cdot\rangle_{\mathrm{M}_t,k_t})$ (see the notations in Section 4.1). One might believe that Proposition 3.2.10 could allow to compute the derivative of the map $t \in \mathbb{R}_+ \mapsto u_t \in \mathrm{H}^1(\Omega)$ at $t = 0$, as we did in Subsection 5.1.1. However, we face here to a scalar product $\langle\cdot,\cdot\rangle_{\mathrm{M}_t,k_t}$ that is $t$-dependent and we need to overcome this difficulty as follows. Let us write $\mathrm{M}_t = \mathrm{I} + (\mathrm{M}_t - \mathrm{I})$ and $k_t = 1 + (k_t - 1)$ to get

$$\langle u_t, v - u_t\rangle_{\mathrm{H}^1(\Omega)} \geq \int_\Omega f_t(v-u_t) + \int_\Gamma g_t\,(v-u_t) - \int_\Omega (\mathrm{M}_t - \mathrm{I})\,\nabla u_t \cdot \nabla(v-u_t) - \int_\Omega (k_t - 1)\,u_t\,(v-u_t)\,,$$

for all $v \in \mathcal{K}_0^1(\Omega)$, and thus

$$u_t = \mathrm{prox}_{\iota_{\mathcal{K}_0^1(\Omega)}}(E_t),$$

where $E_t \in \mathrm{H}^1(\Omega)$ stands for the unique solution to the parameterized variational equality given by

$$\langle E_t, v\rangle_{\mathrm{H}^1(\Omega)} = \int_\Omega f_t v + \int_\Gamma g_t v - \int_\Omega (\mathrm{M}_t - \mathrm{I})\,\nabla u_t \cdot \nabla v - \int_\Omega (k_t - 1)\,u_t v, \qquad \forall v \in \mathrm{H}^1(\Omega), \tag{5.5}$$

and where $\mathrm{prox}_{\iota_{\mathcal{K}^1_0(\Omega)}}$ is now considered on the Hilbert space $(\mathrm{H}^1(\Omega), \langle\cdot,\cdot\rangle_{\mathrm{H}^1(\Omega)})$ whose scalar product is $t$-independent.

**Remark 5.1.7.** Note that the existence/uniqueness of the solution $E_t \in \mathrm{H}^1(\Omega)$ to the parameterized variational equality (5.5) can be easily derived from the Riesz representation theorem, and that $E_0 \in \mathrm{H}^1(\Omega)$ coincides with the solution $F_0 \in \mathrm{H}^1(\Omega)$ to the parameterized general scalar Neumann problem (5.4) for the parameter $t = 0$. Furthermore note that, if $\mathrm{div}\left((\mathrm{M}_t - \mathrm{I})\,\nabla u_t\right) \in \mathrm{L}^2(\Omega)$, then the above parameterized variational equality (5.5) corresponds exactly to the weak variational formulation of the parameterized scalar Neumann problem given by

$$\begin{cases} -\Delta E_t + E_t = f_t - (k_t - 1)\,u_t + \mathrm{div}\left((\mathrm{M}_t - \mathrm{I})\,\nabla u_t\right) & \text{in } \Omega, \\ \partial_{\mathrm{n}} E_t = g_t - (\mathrm{M}_t - \mathrm{I})\,\nabla u_t \cdot \mathrm{n} & \text{on } \Gamma. \end{cases}$$

For instance, the condition $\mathrm{div}\left((\mathrm{M}_t - \mathrm{I})\,\nabla u_t\right) \mathrm{L}^2(\Omega)$ is satisfied when $\mathrm{M}_t \in \mathrm{W}^{1,\infty}(\Omega, \mathbb{R}^{d\times d})$ and $u_t \in \mathrm{H}^2(\Omega)$.

Now, using Hypothesis (H1), (H2) and (H3), our next step is to differentiate the map $t \in \mathbb{R}_+ \mapsto E_t \in \mathrm{H}^1(\Omega)$ at $t = 0$.

**Lemma 5.1.8.** *Let $E_t \in \mathrm{H}^1(\Omega)$ be the unique solution to the parameterized variational equality* (5.5)*, for all $t \geq 0$. Assume that the map $t \in \mathbb{R}_+ \mapsto g_t \in \mathrm{L}^2(\Gamma)$ is differentiable at $t = 0$, with its derivative denoted by $g_0' \in \mathrm{L}^2(\Gamma)$. Then the map $t \in \mathbb{R}_+ \mapsto E_t \in \mathrm{H}^1(\Omega)$ is differentiable at $t = 0$ and its derivative, denoted by $E_0' \in \mathrm{H}^1(\Omega)$, is the unique solution to the variational equality given by*

$$\langle E_0', v\rangle_{\mathrm{H}^1(\Omega)} = \int_\Omega f_0' v + \int_\Gamma g_0' v - \int_\Omega \mathrm{M}_0' \nabla u_0 \cdot \nabla v - \int_\Omega k_0' u_0 v, \qquad \forall v \in \mathrm{H}^1(\Omega). \tag{5.6}$$

*Proof.* Using the Riesz representation theorem, we denote by $Z \in \mathrm{H}^1(\Omega)$ the unique solution to the variational equality (5.6). From linearity we get that

$$\begin{aligned} \left\| \frac{E_t - E_0}{t} - Z \right\|_{\mathrm{H}^1(\Omega)} \leq{}& C\left( \left\| \frac{f_t - f_0}{t} - f_0' \right\|_{\mathrm{L}^2(\Omega)} + \left\| \frac{g_t - g_0}{t} - g_0' \right\|_{\mathrm{L}^2(\Gamma)} \right) \\ &+ C\left( \|\mathrm{M}_0'\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^{d\times d})} + \|k_0'\|_{\mathrm{L}^\infty(\Omega)} \right) \|u_t - u_0\|_{\mathrm{H}^1(\Omega)} \\ &+ C\left( \left\| \frac{\mathrm{M}_t - \mathrm{I}}{t} - \mathrm{M}_0' \right\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^{d\times d})} + \left\| \frac{k_t - 1}{t} - k_0' \right\|_{\mathrm{L}^\infty(\Omega)} \right) \|u_t\|_{\mathrm{H}^1(\Omega)}, \end{aligned}$$

for all $t > 0$, where $C > 0$ is a constant which depends on $\Omega$. Therefore, to conclude the proof we only need to prove the continuity of the map $t \in \mathbb{R}_+ \mapsto u_t \in \mathrm{H}^1(\Omega)$ at $t = 0$. To this aim let us take $v = u_0$ in the weak variational formulation of $u_t$ and $v = u_t$ in the weak variational formulation of $u_0$ to get

$$\begin{aligned} \|u_t - u_0\|_{\mathrm{H}^1(\Omega)} \leq{}& C\left( \|\mathrm{M}_t - \mathrm{I}\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^{d\times d})} + \|k_t - 1\|_{\mathrm{L}^\infty(\Omega)} \right) \|u_t\|_{\mathrm{H}^1(\Omega)} \\ &+ C\left( \|g_t - g_0\|_{\mathrm{L}^2(\Gamma)} + \|f_t - f_0\|_{\mathrm{L}^2(\Omega)} \right), \end{aligned}$$

for all $t \geq 0$. Therefore, to conclude the proof we need to prove that the map $t \in \mathbb{R}_+ \mapsto \|u_t\|_{\mathrm{H}^1(\Omega)} \in \mathbb{R}$ is bounded for $t \geq 0$ sufficiently small. For this purpose, let us take $v = 0$ in the weak variational formulation of $u_t$ to get that

$$\|u_t\|_{\mathrm{H}^1(\Omega)} \leq C \left( \|\mathrm{M}_t - \mathrm{I}\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^{d\times d})} + \|k_t - 1\|_{\mathrm{L}^\infty(\Omega)} \right) \|u_t\|_{\mathrm{H}^1(\Omega)} + C \left( \|g_t\|_{\mathrm{L}^2(\Gamma)} + \|f_t\|_{\mathrm{L}^2(\Omega)} \right),$$

for all $t \geq 0$, and thus

$$\|u_t\|_{\mathrm{H}^1(\Omega)} \leq \frac{C \left( \|f_t\|_{\mathrm{L}^2(\Omega)} + \|g_t\|_{\mathrm{L}^2(\Gamma)} \right)}{1 - C \left( \|\mathrm{M}_t - \mathrm{I}\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^{d\times d})} + \|k_t - 1\|_{\mathrm{L}^\infty(\Omega)} \right)},$$

for all $t \geq 0$ sufficiently small. The assumptions (H1), (H2), (H3) and that the map $t \in \mathbb{R}_+ \mapsto g_t \in \mathrm{L}^2(\Gamma)$ is differentiable at $t = 0$, imply the continuity of the corresponding maps. The proof is complete. ☐

Using the twice epi-differentiability of the Signorini indicator function $\iota_{\mathcal{K}^1_0(\Omega)}$ (see Corollary 5.1.3), one can prove in the same way as Theorem 5.1.4, that the map $t \in \mathbb{R}_+ \mapsto u_t \in \mathrm{H}^1(\Omega)$ is differentiable at $t = 0$.

**Theorem 5.1.9.** *Let $u_t \in \mathrm{H}^1(\Omega)$ be the unique solution to the parameterized general scalar Signorini problem* ($\mathrm{GSSP}_t$)*, for all $t \geq 0$. Assume that the map $t \in \mathbb{R}_+ \mapsto g_t \in \mathrm{L}^2(\Gamma)$ is differentiable at $t = 0$, with its derivative denoted by $g'_0 \in \mathrm{L}^2(\Gamma)$. Then the map $t \in \mathbb{R}_+ \mapsto u_t \in \mathrm{H}^1(\Omega)$ is differentiable at $t = 0$, and its derivative denoted by $u'_0 \in \mathrm{T}_{\mathrm{N}_{\mathcal{K}^1_0(\Omega)}}(u_0) \cap (\mathbb{R}\,(F_0 - u_0))^\perp$ is the unique solution to the variational inequality*

$$\begin{aligned} \langle u'_0, v - u'_0 \rangle_{\mathrm{H}^1(\Omega)} \geq & \int_\Omega f'_0\,(v - u'_0) + \int_\Gamma g'_0\,(v - u'_0) \\ & - \int_\Omega \mathrm{M}'_0 \nabla u_0 \cdot \nabla(v - u'_0) - \int_\Omega k'_0 u_0\,(v - u'_0), \qquad \forall v \in \mathrm{T}_{\mathrm{N}_{\mathcal{K}^1_0(\Omega)}}(u_0) \cap (\mathbb{R}\,(F_0 - u_0))^\perp, \end{aligned} \tag{5.7}$$

*where*

$$\mathrm{T}_{\mathrm{N}_{\mathcal{K}^1_0(\Omega)}}(u_0) \cap (\mathbb{R}\,(F_0 - u_0))^\perp = \left\{ v \in \mathrm{H}^1(\Omega) \mid v \leq 0 \text{ q.e. on } \Gamma^{u_0=0} \text{ and } \langle F_0 - u_0, v \rangle_{\mathrm{H}^1(\Omega)} = 0 \right\},$$

*and* $\Gamma^{u_0=0} := \{ s \in \Gamma \mid u_0(s) = 0 \}$.

With some additional assumptions, we can prove that $u'_0$ is the unique weak solution to a scalar Signorini problem.

**Corollary 5.1.10.** *Consider the framework of Theorem 5.1.9 and assume that:*

1. $\mathrm{div}(\mathrm{M}'_0 \nabla u_0) \in \mathrm{L}^2(\Omega)$ *with* $\mathrm{M}'_0 \nabla u_0 \cdot \mathrm{n} \in \mathrm{L}^2(\Gamma)$*, and also* $\partial_{\mathrm{n}} u_0 \in \mathrm{L}^2(\Gamma)$*;*
2. $\Gamma^{u_0=0} = \overline{\mathrm{int}_\Gamma(\Gamma^{u_0=0})}$.

*Then* $u_0' \in \mathrm{H}^1(\Omega)$ *is the unique weak solution to the scalar Signorini problem*

$$\begin{cases} -\Delta u_0' + u_0' = f_0' - k_0' u_0 + \mathrm{div}\,(\mathrm{M}_0' \nabla u_0) & \text{in } \Omega, \\ \partial_{\mathrm{n}} u_0' = h & \text{on } \Gamma_{\mathrm{N}}^{u_0,g_0}, \\ u_0' = 0 & \text{on } \Gamma_{\mathrm{D}}^{u_0,g_0}, \\ u_0' \leq 0,\ \partial_{\mathrm{n}} u_0' \leq h \text{ and } u_0'\,(\partial_{\mathrm{n}} u_0' - h) = 0 & \text{on } \Gamma_{\mathrm{S-}}^{u_0,g_0}, \end{cases} \tag{5.8}$$

*where* $h := g_0' - \mathrm{M}_0' \nabla u_0 \cdot \mathrm{n} \in \mathrm{L}^2(\Gamma)$, *and* $\Gamma$ *is decomposed, up to a null set, as* $\Gamma_{\mathrm{N}}^{u_0,g_0} \cup \Gamma_{\mathrm{D}}^{u_0,g_0} \cup \Gamma_{\mathrm{S-}}^{u_0,g_0}$, *where*

$$\begin{aligned} \Gamma_{\mathrm{N}}^{u_0,g_0} &:= \{s \in \Gamma \mid u_0(s) \neq 0\}, \\ \Gamma_{\mathrm{D}}^{u_0,g_0} &:= \{s \in \Gamma \mid u_0(s) = 0 \text{ and } \partial_{\mathrm{n}} u_0(s) < g_0(s)\}, \\ \Gamma_{\mathrm{S-}}^{u_0,g_0} &:= \{s \in \Gamma \mid u_0(s) = 0 \text{ and } \partial_{\mathrm{n}} u_0(s) = g_0(s)\}. \end{aligned}$$

*Proof.* From Hypothesis 1, one can apply divergence formula (see Proposition 2.1.1) in Inequality (5.7). The rest of the proof is similar to Corollary 5.1.5. ☐

**Remark 5.1.11.** Consider the framework of Corollary 5.1.10. Note that Hypothesis 1 is, for instance, satisfied if $u_0 \in \mathrm{H}^2(\Omega)$ and $\mathrm{M}_0' \in \mathrm{W}^{1,\infty}(\Omega, \mathbb{R}^{d\times d})$. Note that $u_0'$ is the unique weak solution to the scalar Signorini problem (5.8), but is not necessarily a strong solution. Nevertheless, in the case where $\partial_{\mathrm{n}} u_0' \in \mathrm{L}^2(\Gamma)$ and the decomposition $\Gamma_{\mathrm{N}}^{u_0,g_0} \cup \Gamma_{\mathrm{D}}^{u_0,g_0} \cup \Gamma_{\mathrm{S-}}^{u_0,g_0}$ is consistent, then $u_0'$ is a strong solution to the scalar Signorini problem (5.8).

## 5.2 Boundary value problems involving the scalar Tresca friction law

In this section, we assume that $g_t > 0$ *a.e.* on $\Gamma$, for all $t \geq 0$.

### 5.2.1 The scalar Tresca friction problem with perturbation of the source term only

In this first subsection, we do not perturb the scalar Tresca friction law, and we assume that, for all $t \geq 0$, $g_t = 1$ *a.e.* on $\Gamma$. Let us consider the parameterized scalar Tresca friction problem

$$\begin{cases} -\Delta u_t + u_t = f_t & \text{in } \Omega, \\ |\partial_{\mathrm{n}} u_t| \leq 1 \text{ and } u_t \partial_{\mathrm{n}} u_t + |u_t| = 0 & \text{on } \Gamma, \end{cases} \tag{STP1$_t$}$$

for all $t \geq 0$. From Subsection 4.1.3 (consider $\mathrm{M}(x) = \mathrm{I}$ and $k(x) = 1$ for almost all $x \in \Omega$), for all $t \geq 0$, there exists a unique weak solution $u_t \in \mathrm{H}^1(\Omega)$ to (STP1$_t$) which satisfies

$$\int_\Omega \nabla u_t \cdot \nabla(v - u_t) + \int_\Omega u_t\,(v - u_t) + \int_\Gamma |v| - \int_\Gamma |u_t| \geq \int_\Omega f_t\,(v - u_t), \qquad \forall v \in \mathrm{H}^1(\Omega),$$

and it can be expressed as

$$u_t = \mathrm{prox}_\phi(F_t),$$

where $F_t \in \mathrm{H}^1(\Omega)$ is the unique solution to the parameterized scalar Neumann problem

$$\begin{cases} -\Delta F_t + F_t = f_t & \text{in } \Omega, \\ \partial_{\mathrm{n}} F_t = 0 & \text{on } \Gamma, \end{cases} \tag{5.9}$$

and where $\mathrm{prox}_\phi$ stands for the proximal operator associated with the Tresca friction functional given by

$$\begin{array}{rccl} \phi : & \mathrm{H}^1(\Omega) & \longrightarrow & \mathbb{R} \\ & v & \longmapsto & \phi(v) := \displaystyle\int_\Gamma |v|, \end{array}$$

considered on the Hilbert space $(\mathrm{H}^1(\Omega), \langle \cdot, \cdot \rangle_{\mathrm{H}^1(\Omega)}$.

In the same way as Subsection 5.1.1, using Hypothesis (H1) we have the following lemma.

**Lemma 5.2.1.** *Let $F_t \in \mathrm{H}^1(\Omega)$ be the unique solution to the parameterized scalar Neumann problem* (5.9)*, for all $t \geq 0$. Then the map $t \in \mathbb{R}_+ \mapsto F_t \in \mathrm{H}^1(\Omega)$ is differentiable at $t = 0$ and its derivative, denoted by $F_0' \in \mathrm{H}^1(\Omega)$, is the unique solution to the scalar Neumann problem given by*

$$\begin{cases} -\Delta F_0' + F_0' = f_0' & \textit{in } \Omega, \\ \partial_{\mathrm{n}} F_0' = 0 & \textit{on } \Gamma. \end{cases} \tag{5.10}$$

In order to use Proposition 3.2.10 and then to deduce the derivative of the map $t \in \mathbb{R}_+ \mapsto u_t \in \mathrm{H}^1(\Omega)$ at $t = 0$, let us prepare the background for the twice epi-differentiability (see Definition 3.2.5) of the Tresca friction functional $\phi$. More specifically let us start with the characterization of the convex subdifferential of $\phi$. To this aim, we introduce an auxiliary problem defined, for all $u \in \mathrm{H}^1(\Omega)$, by

$$\begin{cases} -\Delta w + w = 0 & \text{in } \Omega, \\ \partial_{\mathrm{n}} w(s) \in \partial|\cdot|(u(s)) & \text{on } \Gamma, \end{cases} \tag{AP1$_u$}$$

where, for almost all $s \in \Gamma$, $\partial|\cdot|(u(s))$ stands for the convex subdifferential of the classical absolute value map $|\cdot| : \mathbb{R} \to \mathbb{R}$ at $u(s) \in \mathbb{R}$. For a given $u \in \mathrm{H}^1(\Omega)$, a solution to this problem (AP1$_u$) is a function $w \in \mathrm{H}^1(\Omega)$ which satisfies $-\Delta w + w = 0$ in $\mathcal{D}'(\Omega)$ and such that $\partial_{\mathrm{n}} w \in \mathrm{L}^2(\Gamma)$ with $\partial_{\mathrm{n}} w(s) \in \partial|\cdot|(u(s))$ for almost all $s \in \Gamma$.

**Lemma 5.2.2.** *Let $u \in \mathrm{H}^1(\Omega)$. Then*

$$\partial\phi(u) = \textit{the set of solutions to Problem } (\mathrm{AP1}_u).$$

*Proof.* Let $u \in \mathrm{H}^1(\Omega)$ and let us prove the two inclusions. First, let $w \in \mathrm{H}^1(\Omega)$ be a solution to Problem (AP1$_u$). Then $w \in \mathrm{H}^1(\Omega)$, $\partial_{\mathrm{n}} w \in \mathrm{L}^2(\Gamma)$ and $\partial_{\mathrm{n}} w(s) \in \partial|\cdot|(u(s))$ for almost all $s \in \Gamma$. Hence one has

$$\partial_{\mathrm{n}} w(s)(v(s) - u(s)) \leq (|v(s)| - |u(s)|),$$

for all $v \in \mathrm{H}^1(\Omega)$ and for almost all $s \in \Gamma$. It follows that

$$\int_\Gamma \partial_{\mathrm{n}} w(v - u) \leq \int_\Gamma (|v| - |u|),$$

for all $v \in \mathrm{H}^1(\Omega)$. Moreover $-\Delta w + w = 0$ in $\mathcal{D}'(\Omega)$, thus it holds $-\Delta w + w = 0$ in $\mathrm{L}^2(\Omega)$. Hence, from Green formula (see Proposition 2.1.2), one gets since $\partial_{\mathrm{n}} w \in \mathrm{L}^2(\Gamma)$,

$$\int_\Omega \nabla w \cdot \nabla(v-u) + \int_\Omega w\,(v-u) = \int_\Gamma \partial_{\mathrm{n}} w\,(v-u)\,,$$

for all $v \in \mathrm{H}^1(\Omega)$. Therefore one deduces

$$\int_\Omega \nabla w \cdot \nabla(v-u) + \int_\Omega w\,(v-u) \leq \int_\Gamma (|v|-|u|),$$

for all $v \in \mathrm{H}^1(\Omega)$, that is

$$\langle w, v-u \rangle_{\mathrm{H}^1(\Omega)} \leq \phi(v) - \phi(u),$$

for all $v \in \mathrm{H}^1(\Omega)$. Thus $w \in \partial\phi(u)$ and the first inclusion is proved. Conversely let $w \in \partial\phi(u)$. One has

$$\int_\Omega \nabla w \cdot \nabla(v-u) + \int_\Omega w\,(v-u) \leq \int_\Gamma (|v|-|u|), \tag{5.11}$$

for all $v \in \mathrm{H}^1(\Omega)$. Considering the function $v = u \pm \varphi \in \mathrm{H}^1(\Omega)$ with any function $\varphi \in \mathcal{D}(\Omega)$, one deduces from Inequality (5.11) that $-\Delta w + w = 0$ in $\mathcal{D}'(\Omega)$, and also $-\Delta w + w = 0$ in $\mathrm{L}^2(\Omega)$. Hence, from Green formula (see Proposition 2.1.2) and Inequality (5.11), it follows that

$$\langle \partial_{\mathrm{n}} w, v-u \rangle_{\mathrm{H}^{-1/2}(\Gamma)\times\mathrm{H}^{1/2}(\Gamma)} \leq \int_\Gamma (|v|-|u|),$$

for all $v \in \mathrm{H}^1(\Omega)$, and thus also for all $v \in \mathrm{H}^{1/2}(\Gamma)$. Now let us consider the function $v = u + \varphi \in \mathrm{H}^{1/2}(\Gamma)$ for any $\varphi \in \mathrm{H}^{1/2}(\Gamma)$. One gets

$$\left| \langle \partial_{\mathrm{n}} w, \varphi \rangle_{\mathrm{H}^{-1/2}(\Gamma)\times\mathrm{H}^{1/2}(\Gamma)} \right| \leq \int_\Gamma |\varphi| \leq |\Gamma|^{\frac{1}{2}} \|\varphi\|_{\mathrm{L}^2(\Gamma)}\,.$$

From Proposition 2.1.7 and 2.1.8, one deduces that $\partial_{\mathrm{n}} w \in \mathrm{L}^2(\Gamma)$ and also that

$$\int_\Gamma \partial_{\mathrm{n}} w\,(v-u) \leq \int_\Gamma (|v|-|u|), \tag{5.12}$$

for all $v \in \mathrm{H}^{1/2}(\Gamma)$ and, by density, for all $v \in \mathrm{L}^2(\Gamma)$. Let $s \in \Gamma$ be a Lebesgue point of $\partial_{\mathrm{n}} w \in \mathrm{L}^2(\Gamma)$, of the product $u\partial_{\mathrm{n}} w \in \mathrm{L}^1(\Gamma)$ and of $|u| \in \mathrm{L}^2(\Gamma)$. Let us consider the function $v \in \mathrm{L}^2(\Gamma)$ defined by

$$v := \begin{cases} x & \text{on } B_\Gamma(s,\varepsilon), \\ u & \text{on } \Gamma \backslash B_\Gamma(s,\varepsilon), \end{cases}$$

with $x \in \mathbb{R}$ and $\varepsilon > 0$ such that $B_\Gamma(s,\varepsilon) \subset \Gamma$. Then one has from Inequality (5.12) that

$$\frac{1}{|B_\Gamma(s,\varepsilon)|}\int_{B_\Gamma(s,\varepsilon)} \partial_{\mathrm{n}} w(x-u) \leq \frac{1}{|B_\Gamma(s,\varepsilon)|}\int_{B_\Gamma(s,\varepsilon)} |x| - \frac{1}{|B_\Gamma(s,\varepsilon)|}\int_{B_\Gamma(s,\varepsilon)} |u|,$$

thus $\partial_{\mathrm{n}} w(s)(x-u(s)) \leq |x| - |u(s)|$ by letting $\varepsilon \to 0^+$. This inequality is true for any $x \in \mathbb{R}$, therefore $\partial_{\mathrm{n}} w(s) \in \partial|\cdot|(u(s))$. Moreover, since almost every point of $\Gamma$ is a Lesbegue point of $\partial_{\mathrm{n}} w \in \mathrm{L}^2(\Gamma)$, $u\partial_{\mathrm{n}} w \in \mathrm{L}^1(\Gamma)$ and of $|u| \in \mathrm{L}^2(\Gamma)$, one deduces

$$\partial_{\mathrm{n}} w(s) \in \partial|\cdot|(u(s)),$$

for almost all $s \in \Gamma$, and this proves the second inclusion. □

The twice epi-differentiability (see Definition 3.2.5) is defined using the second-order difference quotient functions. Therefore let us compute the following second-order difference quotient functions of $\phi$ at $u \in \mathrm{H}^1(\Omega)$ for $w \in \partial\phi(u)$ defined by

$$\begin{array}{rccl} \delta_t^2 \phi(u|w): & \mathrm{H}^1(\Omega) & \longrightarrow & \mathbb{R} \\ & v & \longmapsto & \delta_t^2 \phi(u|w)(v) := \dfrac{\phi(u+tv) - \phi(u) - t\left\langle w, v\right\rangle_{\mathrm{H}^1(\Omega)}}{t^2}, \end{array}$$

for all $t > 0$.

**Proposition 5.2.3.** *For all $t > 0$, $u \in \mathrm{H}^1(\Omega)$ and $w \in \partial\phi(u)$, it holds that*

$$\delta_t^2 \phi(u|w)(v) = \int_\Gamma \delta_t^2 \left|\cdot\right| (u(s)|\partial_{\mathrm{n}} w(s))(v(s))\mathrm{d}s, \tag{5.13}$$

*for all $v \in \mathrm{H}^1(\Omega)$, where, for almost all $s \in \Gamma$, $\delta_t^2 \left|\cdot\right| (u(s)|\partial_{\mathrm{n}} w(s))$ stands for the second-order difference quotient function of the absolute value map $|\cdot|$ at $u(s) \in \mathbb{R}$ for $\partial_{\mathrm{n}} w(s) \in \partial|\cdot|(u(s))$.*

*Proof.* Let $t > 0$, $u \in \mathrm{H}^1(\Omega)$ and $w \in \partial\phi(u)$. From Lemma 5.2.2 and Green formula (see Proposition 2.1.2), one deduces

$$\left\langle w, v\right\rangle_{\mathrm{H}^1(\Omega)} = \left\langle \partial_{\mathrm{n}} w, v\right\rangle_{\mathrm{H}^{-1/2}(\Gamma)\times\mathrm{H}^{1/2}(\Gamma)},$$

for all $v \in \mathrm{H}^1(\Omega)$. Moreover, similarly to Lemma 5.2.2, one gets

$$\left\langle w, v\right\rangle_{\mathrm{H}^1(\Omega)} = \int_\Gamma v\partial_{\mathrm{n}} w,$$

for all $v \in \mathrm{H}^1(\Omega)$. Thus it follows that

$$\delta_t^2 \phi(u|w)(v) = \int_\Gamma \frac{|u(s)+tv(s)| - |u(s)| - tv(s)\partial_{\mathrm{n}} w(s)}{t^2}\mathrm{d}s,$$

for all $v \in \mathrm{H}^1(\Omega)$. Furthermore, since $\partial_{\mathrm{n}} w(s) \in \partial|\cdot|(u(s))$ for almost all $s \in \Gamma$, one deduces that

$$\delta_t^2 \phi(u|w)(v) = \int_\Gamma \delta_t^2 \left|\cdot\right| (u(s)|\partial_{\mathrm{n}} w(s))(v(s))\mathrm{d}s,$$

for all $v \in \mathrm{H}^1(\Omega)$, which concludes the proof. □

From the above proposition we can notice that the twice epi-differentiability of the Tresca friction functional is strongly related to the twice epi-differentiability of the absolute value map $|\cdot|$. Therefore the computation of the second-order epi-derivative of $|\cdot|$ is the next step.

**Lemma 5.2.4.** The absolute value map $|\cdot| : \mathbb{R} \to \mathbb{R}$ is twice epi-differentiable at any $x \in \mathbb{R}$ for any $y \in \partial|\cdot|(x)$, and its second-order epi-derivative is given by $\mathrm{d}_e^2|\cdot|(x|y) = \iota_{\mathrm{K}_{x,y}}$, where $\mathrm{K}_{x,y}$ is the nonempty closed convex subset of $\mathbb{R}$ defined by

$$\mathrm{K}_{x,y} := \begin{cases} \mathbb{R} & \text{if } x \neq 0, \\ \mathbb{R}_- & \text{if } x = 0 \text{ and } y = -1, \\ \mathbb{R}_+ & \text{if } x = 0 \text{ and } y = 1, \\ \{0\} & \text{if } x = 0 \text{ and } y \in (-1,1), \end{cases}$$

and where $\iota_{\mathrm{K}_{x,y}}$ stands for the indicator function of $\mathrm{K}_{x,y}$.

*Proof.* Since $|\cdot| = h_{[-1,1]}$, where $h_{[-1,1]}$ is the support function of $[-1,1]$, then one can apply Proposition 3.2.8 to get, for $x = 0$ and for all $y \in \partial|\cdot|(0) = [-1,1]$,

$$\mathrm{d}_e^2 \left|\cdot\right| (0|y) = \iota_{\mathrm{N}_{[-1,1]}(y)},$$

where $\mathrm{N}_{[-1,1]}(y)$ is the normal cone to $[-1,1]$ at $y$. One can easily prove that

$$\mathrm{N}_{[-1,1]}(y) := \begin{cases} \mathbb{R}_- & \text{if } y = -1, \\ \mathbb{R}_+ & \text{if } y = 1, \\ \{0\} & \text{if } y \in (-1,1). \end{cases}$$

In the case where $x \neq 0$, then $\partial|\cdot|(x) = \{\frac{x}{|x|}\}$, and $|\cdot|$ is twice Fréchet differentiable at $x$ with its twice Fréchet differential given by

$$\mathrm{D}^2 \left|\cdot\right| (x) = 0.$$

Thus one gets (see Remark 3.2.6),

$$\mathrm{d}_e^2 \left|\cdot\right| (x|\frac{x}{|x|})(z) = 0,$$

for all $z \in \mathbb{R}$, which concludes the proof. □

**Remark 5.2.5.** Since $|\cdot|$ is twice epi-differentiable, then from Proposition 5.2.3, if $\phi$ is twice epi-differentiable at $u \in \mathrm{H}^1(\Omega)$ for $w \in \partial\phi(u)$, one can naturally expect that its second order epi-derivative satisfies

$$\mathrm{d}_e^2\phi(u|w)(v) = \int_\Gamma \mathrm{d}_e^2 \left|\cdot\right| (u(s)|\partial_{\mathrm{n}} w(s))(v(s))\mathrm{d}s,$$

for all $v \in \mathrm{H}^1(\Omega)$, which corresponds to the inversion of symbols ME-lim and $\int_\Gamma$ in Equality (5.13) on the $\mathrm{H}^1(\Omega)$-space. Nevertheless, to the best of our knowledge, this inversion is an open question. Therefore we do not know, in general, if the Tresca friction functional is indeed twice epi-differentiable at $u$ for $w$. Nevertheless, in Appendix A, we prove it in several particular cases corresponding to practical situations.

From the previous results and some additional assumptions detailed below, we are now in a position to characterize the derivative of the solution to the parameterized Tresca friction problem (STP1$_t$).

**Theorem 5.2.6.** *Let $u_t \in \mathrm{H}^1(\Omega)$ be the unique solution to the parameterized scalar Tresca friction problem* ($\mathrm{STP1}_t$) *for all $t \geq 0$. Assume that the Tresca friction functional $\phi$ is twice epi-differentiable at $u_0$ for $F_0 - u_0 \in \partial\phi(u_0)$, with*

$$\mathrm{d}_e^2\phi(u_0|F_0-u_0)(v) = \int_\Gamma \mathrm{d}_e^2 \left|\cdot\right| (u_0(s)|\partial_\mathrm{n}(F_0-u_0)(s))(v(s))\mathrm{d}s,$$

*for all $v \in \mathrm{H}^1(\Omega)$, where $F_0$ is the unique solution to the parameterized scalar Neumann problem* (5.9) *for the parameter $t = 0$. Then the map $t \in \mathbb{R}_+ \mapsto u_t \in \mathrm{H}^1(\Omega)$ is differentiable at $t = 0$, and its derivative denoted by $u_0' \in \mathrm{H}^1(\Omega)$ is the unique weak solution to the Signorini problem*

$$\begin{cases} -\Delta u_0' + u_0' = f_0' & \text{in } \Omega, \\ \partial_\mathrm{n} u_0' = 0 & \text{on } \Gamma_\mathrm{N}^{u_0}, \\ u_0' = 0 & \text{on } \Gamma_\mathrm{D}^{u_0}, \\ u_0' \leq 0,\ \partial_\mathrm{n}u_0' \leq 0 \text{ and } u_0'\partial_\mathrm{n}u_0' = 0 & \text{on } \Gamma_{\mathrm{S}-}^{u_0}, \\ u_0' \geq 0,\ \partial_\mathrm{n}u_0' \geq 0 \text{ and } u_0'\partial_\mathrm{n}u_0' = 0 & \text{on } \Gamma_{\mathrm{S}+}^{u_0}, \end{cases} \tag{5.14}$$

*where $\Gamma$ is decomposed, up to a null set, as $\Gamma_\mathrm{N}^{u_0} \cup \Gamma_\mathrm{D}^{u_0} \cup \Gamma_{\mathrm{S}-}^{u_0} \cup \Gamma_{\mathrm{S}+}^{u_0}$ with*

$$\begin{aligned} \Gamma_\mathrm{N}^{u_0} &:= \left\{ s \in \Gamma \mid u_0(s) \neq 0 \right\}, \\ \Gamma_\mathrm{D}^{u_0} &:= \left\{ s \in \Gamma \mid u_0(s) = 0 \text{ and } \partial_\mathrm{n}u_0(s) \in (-1,1) \right\}, \\ \Gamma_{\mathrm{S}-}^{u_0} &:= \left\{ s \in \Gamma \mid u_0(s) = 0 \text{ and } \partial_\mathrm{n}u_0(s) = 1 \right\}, \\ \Gamma_{\mathrm{S}+}^{u_0} &:= \left\{ s \in \Gamma \mid u_0(s) = 0 \text{ and } \partial_\mathrm{n}u_0(s) = -1 \right\}. \end{aligned}$$

*Proof.* From the assumption and Lemma 5.2.4, it follows that

$$\mathrm{d}_e^2\phi(u_0|F_0-u_0)(v) = \int_\Gamma \iota_{\mathrm{K}_{u_0(s),\partial_\mathrm{n}(F_0-u_0)(s)}}(v(s))\mathrm{d}s,$$

for all $v \in \mathrm{H}^1(\Omega)$. Therefore, one deduces that

$$\mathrm{d}_e^2\phi(u_0|F_0-u_0)(v) = \iota_{\mathcal{K}_{u_0,\partial_\mathrm{n}(F_0-u_0)}}(v),$$

for all $v \in \mathrm{H}^1(\Omega)$, where $\mathcal{K}_{u_0,\partial_\mathrm{n}(F_0-u_0)}$ is the nonempty closed convex subset of $\mathrm{H}^1(\Omega)$ defined by

$$\mathcal{K}_{u_0,\partial_\mathrm{n}(F_0-u_0)} := \left\{ v \in \mathrm{H}^1(\Omega) \mid v(s) \in \mathrm{K}_{u_0(s),\partial_\mathrm{n}(F_0-u_0)(s)} \text{ for almost all } s \in \Gamma \right\}.$$

Moreover, since $\partial_\mathrm{n}F_0 = 0$ *a.e.* on $\Gamma$, one gets

$$\mathcal{K}_{u_0,\partial_\mathrm{n}(F_0-u_0)} = \left\{ v \in \mathrm{H}^1(\Omega) \mid v = 0 \ \textit{a.e.} \text{ on } \Gamma_\mathrm{D}^{u_0},\ v \leq 0 \ \textit{a.e.} \text{ on } \Gamma_{\mathrm{S}-}^{u_0},\ v \geq 0 \ \textit{a.e.} \text{ on } \Gamma_{\mathrm{S}+}^{u_0} \right\},$$

where subsets $\Gamma_\mathrm{N}^{u_0}$, $\Gamma_\mathrm{D}^{u_0}$, $\Gamma_{\mathrm{S}-}^{u_0}$, $\Gamma_{\mathrm{S}+}^{u_0}$ are defined in Theorem 5.2.6. Moreover, from Hypothesis (H1) we know from Lemma 5.2.1 that the map $t \in \mathbb{R}_+ \mapsto F_t \in \mathrm{H}^1(\Omega)$ is differentiable at $t = 0$, with its derivative $F_0' \in \mathrm{H}^1(\Omega)$ being the unique solution to the scalar Neumann problem (5.10). Thus, using Proposition 3.2.10 one deduces that the map $t \in \mathbb{R}_+ \mapsto u_t \in \mathrm{H}^1(\Omega)$ is differentiable at $t = 0$, and its derivative $u_0' \in \mathrm{H}^1(\Omega)$ satisfies

$$u_0' = \mathrm{prox}_{\mathrm{d}_e^2\phi(u_0|F_0-u_0)}(F_0') = \mathrm{prox}_{\iota_{\mathcal{K}_{u_0,\partial_\mathrm{n}(F_0-u_0)}}}(F_0'),$$

which is also (see Remark 3.1.11)

$$u_0' = \mathrm{proj}_{\mathcal{K}_{u_0,\partial_\mathrm{n}(F_0-u_0)}}(F_0').$$

From Subsection 4.1.2, one deduces that $u_0'$ is the unique weak solution to the scalar Signorini problem (5.14). □

### 5.2.2 The scalar Tresca friction problem with perturbation of the source term and of the friction term

In this subsection, the scalar Tresca friction law is perturbed. We recall that, for all $t \geq 0$, $g_t > 0$ *a.e.* on $\Gamma$. Let us consider the parameterized scalar Tresca friction problem

$$\begin{cases} -\Delta u_t + u_t = f_t & \text{in } \Omega, \\ |\partial_\mathrm{n} u_t| \leq g_t \text{ and } u_t \partial_\mathrm{n} u_t + g_t |u_t| = 0 & \text{on } \Gamma, \end{cases} \tag{STP2$_t$}$$

for all $t \geq 0$. From Subsection 4.1.3, for all $t \geq 0$, there exists a unique weak solution $u_t \in \mathrm{H}^1(\Omega)$ to (STP2$_t$) which satisfies

$$\int_\Omega \nabla u_t \cdot \nabla (v - u_t) + \int_\Omega u_t \,(v - u_t) + \int_\Gamma g_t\,|v| - \int_\Gamma g_t\,|u_t| \geq \int_\Omega f_t\,(v - u_t)\,, \qquad \forall v \in \mathrm{H}^1(\Omega).$$

The Tresca friction functional, defined in Proposition 4.1.16, now depends on the parameter $t \geq 0$. Precisely we are led to define the parameterized Tresca friction functional given by

$$\begin{array}{rccl} \Phi: & \mathbb{R}_+ \times \mathrm{H}^1(\Omega) & \longrightarrow & \mathbb{R} \\ & (t, v) & \longmapsto & \Phi(t, v) := \displaystyle\int_\Gamma g_t |v|. \end{array} \tag{5.15}$$

Note that, for all $t \geq 0$, $\Phi(t, \cdot)$ is a proper lower semi-continuous convex function on $\mathrm{H}^1(\Omega)$ and, the unique solution to the parameterized scalar Tresca friction problem (STP2$_t$) is given by

$$u_t = \mathrm{prox}_{\Phi(t,\cdot)}(F_t), \tag{5.16}$$

where $F_t \in \mathrm{H}^1(\Omega)$ is the unique solution to the parameterized scalar Neumann problem

$$\begin{cases} -\Delta F_t + F_t = f_t & \text{in } \Omega, \\ \partial_\mathrm{n} F_t = 0 & \text{on } \Gamma, \end{cases} \tag{5.17}$$

and where $\mathrm{prox}_{\Phi(t,\cdot)}$ stands for the proximal operator associated with $\Phi(t, \cdot)$ considered on the Hilbert space $(\mathrm{H}^1(\Omega), \langle \cdot, \cdot \rangle_{\mathrm{H}^1(\Omega)})$.

As we can see in Equality (5.16), the proximal operator $\mathrm{prox}_{\Phi(t,\cdot)}$ depends on the parameter $t \geq 0$. This leads us to use Proposition 3.2.15 which characterizes the derivative of a map given by a parameterized proximal operator, using the notion of twice epi-differentiability depending on a parameter (see Definition 3.2.11). Let us underline that this is an important difference with the previous Subsection 5.2.1, where the proximal operator was associated to a functional that did not depend on the parameter $t \geq 0$, therefore the classical notion of twice epi-differentiability (see Definition 3.2.5) introduced by R.T. Rockafellar (see [68]) was sufficient.

In order to investigated the twice epi-differentiability of the parameterized Tresca friction functional defined in (5.15), let us start with the characterization of the convex subdifferential of $\Phi(0,\cdot)$. To this aim, we introduce an auxiliary problem defined, for all $u \in \mathrm{H}^1(\Omega)$, by

$$\begin{cases} -\Delta w + w = 0 & \text{in } \Omega, \\ \partial_{\mathrm{n}} w(s) \in g_0(s)\partial|\cdot|(u(s)) & \text{on } \Gamma. \end{cases} \tag{AP2$_u$}$$

For a given $u \in \mathrm{H}^1(\Omega)$, a solution to this problem (AP2$_u$) is a function $w \in \mathrm{H}^1(\Omega)$ which satisfies $-\Delta w + w = 0$ in $\mathcal{D}'(\Omega)$ and such that $\partial_{\mathrm{n}} w \in \mathrm{L}^2(\Gamma)$ with $\partial_{\mathrm{n}} w(s) \in g_0(s)\partial|\cdot|(u(s))$ for almost all $s \in \Gamma$.

The proof of the following lemma can be proved in the same way as Lemma 5.2.2

**Lemma 5.2.7.** *Let $u \in \mathrm{H}^1(\Omega)$. Then*

$$\partial\Phi(0,\cdot)(u) = \textit{the set of solutions to Problem } (\mathrm{AP2}_u).$$

The twice epi-differentiability depending on a parameter is defined using the second-order difference quotient functions (see Definition 3.2.11). Therefore let us compute the following second-order difference quotient functions of $\Phi$ at $u \in \mathrm{H}^1(\Omega)$ for $w \in \partial\Phi(0,\cdot)(u)$ defined by

$$\begin{array}{rccl} \Delta_t^2\Phi(u|w): & \mathrm{H}^1(\Omega) & \longrightarrow & \mathbb{R} \\ & v & \longmapsto & \Delta_t^2\Phi(u|w)(v) := \dfrac{\Phi(t,u+tv) - \Phi(t,u) - t\langle w, v\rangle_{\mathrm{H}^1(\Omega)}}{t^2}, \end{array}$$

for all $t > 0$.

**Proposition 5.2.8.** *For all $t > 0$, $u \in \mathrm{H}^1(\Omega)$ and $w \in \partial\Phi(0,\cdot)(u)$, it holds that*

$$\Delta_t^2\Phi(u|w)(v) = \int_\Gamma \Delta_t^2 G(s)(u(s)|\partial_{\mathrm{n}} w(s))(v(s))\mathrm{d}s,$$

*for all $v \in \mathrm{H}^1(\Omega)$, where, for almost all $s \in \Gamma$, $\Delta_t^2 G(s)(u(s)|\partial_{\mathrm{n}} w(s))$ stands for the second-order difference quotient function of $G(s)$ at $u(s) \in \mathbb{R}$ for $\partial_{\mathrm{n}} w(s) \in g_0(s)\partial|\cdot|(u(s))$, with $G(s)$ defined by*

$$\begin{array}{rccl} G(s): & \mathbb{R}_+ \times \mathbb{R} & \longrightarrow & \mathbb{R} \\ & (t,x) & \longmapsto & G(s)(t,x) := g_t(s)|x|. \end{array}$$

**Remark 5.2.9.** Note that, for almost all $s \in \Gamma$ and all $t \geq 0$, $G(s)(t,\cdot) := g_t(s)|\cdot|$ is a proper lower semi-continuous convex function on $\mathbb{R}$. Moreover, since $g_0 > 0$ *a.e.* on $\Gamma$, it follows that

$$\partial \left[G(s)(0,\cdot)\right](x) = g_0(s)\partial|\cdot|(x),$$

for all $x \in \mathbb{R}$ and for almost all $s \in \Gamma$.

*Proof of Proposition 5.2.8.* Let $t > 0$, $u \in \mathrm{H}^1(\Omega)$ and $w \in \partial\Phi(0,\cdot)(u)$. From Lemma 5.2.7 and Green formula (see Proposition 2.1.2), one deduces that

$$\langle w, v\rangle_{\mathrm{H}^1(\Omega)} = \langle \partial_{\mathrm{n}} w, v\rangle_{\mathrm{H}^{-1/2}(\Gamma)\times\mathrm{H}^{1/2}(\Gamma)} = \int_\Gamma v\partial_{\mathrm{n}} w,$$

for all $v \in \mathrm{H}^1(\Omega)$. Thus it follows that

$$\Delta_t^2\Phi(u|w)(v) = \int_\Gamma \frac{g_t(s)|u(s)+tv(s)| - g_t(s)|u(s)| - tv(s)\partial_{\mathrm{n}} w(s)}{t^2}\mathrm{d}s,$$

for all $v \in \mathrm{H}^1(\Omega)$. Furthermore, since $\partial_{\mathrm{n}} w(s) \in g_0(s)\partial|\cdot|(u(s))$ for almost all $s \in \Gamma$, one deduces that

$$\Delta_t^2\Phi(u|w)(v) = \int_\Gamma \Delta_t^2 G(s)(u(s)|\partial_{\mathrm{n}} w(s))(v(s))\mathrm{d}s,$$

for all $v \in \mathrm{H}^1(\Omega)$, which concludes the proof. □

From the above proposition we note that the twice epi-differentiability of the parameterized Tresca friction functional is strongly related to the twice epi-differentiability of the parameterized function $G(s)$ for almost all $s \in \Gamma$. Therefore the computation of the second-order epi-derivative of $G(s)$ for almost all $s \in \Gamma$ is the next step.

**Proposition 5.2.10.** *Assume that, for almost all $s \in \Gamma$, the map $t \in \mathbb{R}_+ \mapsto g_t(s) \in \mathbb{R}_+$ is differentiable at $t = 0$, with its derivative denoted by $g_0'(s)$. Then, for almost all $s \in \Gamma$, the map $G(s)$ is twice epi-differentiable at any $x \in \mathbb{R}$ and for all $y \in g_0(s)\partial|\cdot|(x)$ with*

$$\mathrm{D}_e^2 G(s)(x|y)(z) = \iota_{\mathrm{K}_{x,\frac{y}{g_0(s)}}}(z) + g_0'(s)\frac{y}{g_0(s)}z,$$

*for all $z \in \mathbb{R}$, where $\iota_{\mathrm{K}_{x,\frac{y}{g_0(s)}}}$ stands for the indicator function of the set $\mathrm{K}_{x,\frac{y}{g_0(s)}}$ (see Lemma 5.2.4).*

*Proof.* We use the same notations as in Definitions 3.2.5 and 3.2.11. Let $x \in \mathbb{R}$. Then, for almost all $s \in \Gamma$, for all $y \in g_0(s)\partial|\cdot|(x)$ and all $z \in \mathbb{R}$, one has

$$\Delta_t^2 G(s)(x|y)(z) = \frac{g_t(s)|x+tz| - g_t(s)|x| - tyz}{t^2} = g_t(s)\frac{|x+tz| - |x| - t\frac{y}{g_0(s)}z}{t^2} + \frac{(g_t(s)-g_0(s))\,y}{tg_0(s)}z,$$

that is

$$\Delta_t^2 G(s)(x|y)(z) = g_t(s)\delta_t^2|\cdot|\left(x|\frac{y}{g_0(s)}\right)(z) + \frac{(g_t(s)-g_0(s))\,y}{tg_0(s)}z,$$

with $\frac{y}{g_0(s)} \in \partial|\cdot|(x)$, and where $\delta_t^2|\cdot|(x|\frac{y}{g_0(s)})$ is the second-order difference quotient function of $|\cdot|$ at $x$ for $\frac{y}{g_0(s)}$ (see Definition 3.2.5 since $|\cdot|$ is $t$-independent function). Using the characterization of

Mosco epi-convergence (see Proposition 3.2.4) and Lemma 5.2.4, it follows that the map $G(s)$ is twice epi-differentiable at $x$ for $y$ with

$$\mathrm{D}_e^2 G(s)(x|y)(z) = \iota_{\mathrm{K}_{x,\frac{y}{g_0(s)}}}(z) + g_0'(s)\frac{y}{g_0(s)}z,$$

for all $z \in \mathbb{R}$, and the proof is complete. □

**Remark 5.2.11.** The perturbation of the friction term $g_t$ in the parameterized scalar Tresca friction problem (STP2$_t$) (which is not considered in Subsection 5.2.1) generates an additional term in the expression of the second-order epi-derivative of $G(s)$, for almost all $s \in \Gamma$, at all $x \in \mathbb{R}$ for all $y \in g_0(s)\partial|\cdot|(x)$, given by the function $z \in \mathbb{R} \mapsto g_0'(s)\frac{y}{g_0(s)}z \in \mathbb{R}$.

**Theorem 5.2.12.** *Let $u_t \in \mathrm{H}^1(\Omega)$ be the unique solution to the parameterized scalar Tresca friction problem* (STP2$_t$) *for all $t \geq 0$. Assume that:*

1. *for almost all $s \in \Gamma$, the map $t \in \mathbb{R}_+ \mapsto g_t(s) \in \mathbb{R}_+$ is differentiable at $t = 0$, with its derivative denoted by $g_0'(s)$, and also $g_0' \in \mathrm{L}^2(\Gamma)$;*
2. *the parameterized Tresca friction functional $\Phi$ defined in* (5.15) *is twice epi-differentiable at $u_0$ for $F_0 - u_0 \in \partial\Phi(0,\cdot)(u_0)$, with*

$$\mathrm{D}_e^2\Phi(u_0|F_0-u_0)(v) = \int_\Gamma \mathrm{D}_e^2 G(s)(u_0(s)|\partial_{\mathrm{n}}(F_0-u_0)(s))(v(s))\mathrm{d}s,$$

*for all $v \in \mathrm{H}^1(\Omega)$, where $F_0$ is the unique solution to the parameterized scalar Neumann problem* (5.17) *for the parameter $t = 0$.*

*Then the map $t \in \mathbb{R}_+ \mapsto u_t \in \mathrm{H}^1(\Omega)$ is differentiable at $t = 0$, and its derivative denoted by $u_0' \in \mathrm{H}^1(\Omega)$ is the unique weak solution to the scalar Signorini problem*

$$\begin{cases} -\Delta u_0' + u_0' = f_0' & \text{in } \Omega, \\ \partial_{\mathrm{n}} u_0' = g_0'\frac{\partial_{\mathrm{n}} u_0}{g_0} & \text{on } \Gamma_{\mathrm{N}}^{u_0,g_0} \\ u_0' = 0 & \text{on } \Gamma_{\mathrm{D}}^{u_0,g_0}, \\ u_0' \leq 0,\ \partial_{\mathrm{n}} u_0' \leq g_0'\frac{\partial_{\mathrm{n}} u_0}{g_0} \text{ and } u_0'\left(\partial_{\mathrm{n}} u_0' - g_0'\frac{\partial_{\mathrm{n}} u_0}{g_0}\right) = 0 & \text{on } \Gamma_{\mathrm{S-}}^{u_0,g_0}, \\ u_0' \geq 0,\ \partial_{\mathrm{n}} u_0' \geq g_0'\frac{\partial_{\mathrm{n}} u_0}{g_0} \text{ and } u_0'\left(\partial_{\mathrm{n}} u_0' - g_0'\frac{\partial_{\mathrm{n}} u_0}{g_0}\right) = 0 & \text{on } \Gamma_{\mathrm{S+}}^{u_0,g_0}, \end{cases} \tag{5.18}$$

*where $\Gamma$ is decomposed, up to a null set, as $\Gamma_{\mathrm{N}}^{u_0,g_0} \cup \Gamma_{\mathrm{D}}^{u_0,g_0} \cup \Gamma_{\mathrm{S-}}^{u_0,g_0} \cup \Gamma_{\mathrm{S+}}^{u_0,g_0}$ with*

$$\begin{aligned} \Gamma_{\mathrm{N}}^{u_0,g_0} &:= \{s \in \Gamma \mid u_0(s) \neq 0\}, \\ \Gamma_{\mathrm{D}}^{u_0,g_0} &:= \{s \in \Gamma \mid u_0(s) = 0 \text{ and } \partial_{\mathrm{n}} u_0(s) \in (-g_0(s), g_0(s))\}, \\ \Gamma_{\mathrm{S-}}^{u_0,g_0} &:= \{s \in \Gamma \mid u_0(s) = 0 \text{ and } \partial_{\mathrm{n}} u_0(s) = g_0(s)\}, \\ \Gamma_{\mathrm{S+}}^{u_0,g_0} &:= \{s \in \Gamma \mid u_0(s) = 0 \text{ and } \partial_{\mathrm{n}} u_0(s) = -g_0(s)\}. \end{aligned}$$

**Remark 5.2.13.** The problem (5.18) in Theorem 5.2.12 is a well-posed problem since

$$\left|\frac{\partial_{\mathrm{n}} u_0(s)}{g_0(s)}\right| \leq 1,$$

for almost all $s \in \Gamma$, and hence $g_0' \frac{\partial_n u_0}{g_0} \in L^2(\Gamma)$ since $g_0' \in L^2(\Gamma)$.

*Proof of Theorem 5.2.12.* From the assumptions and Proposition 5.2.10, it follows that

$$D_e^2\Phi(u_0|F_0 - u_0)(v) = \iota_{\mathcal{K}_{u_0, \frac{\partial_n(F_0-u_0)}{g_0}}}(v) + \int_\Gamma g_0'(s) \frac{\partial_n(F_0 - u_0)(s)}{g_0(s)} v(s)\mathrm{d}s,$$

for all $v \in H^1(\Omega)$, where $\mathcal{K}_{u_0, \frac{\partial_n(F_0-u_0)}{g_0}}$ is the nonempty closed convex subset of $H^1(\Omega)$ defined by

$$\mathcal{K}_{u_0, \frac{\partial_n(F_0-u_0)}{g_0}} = \left\{ v \in H^1(\Omega) \mid v = 0 \text{ } a.e. \text{ on } \Gamma_D^{u_0,g_0},\ v \leq 0 \text{ } a.e. \text{ on } \Gamma_{S-}^{u_0,g_0},\ v \geq 0 \text{ } a.e. \text{ on } \Gamma_{S+}^{u_0,g_0} \right\},$$

where subsets $\Gamma_N^{u_0,g_0}$, $\Gamma_D^{u_0,g_0}$, $\Gamma_{S-}^{u_0,g_0}$ and $\Gamma_{S+}^{u_0,g_0}$ are defined in Theorem 5.2.12. One can easily prove that $D_e^2\Phi(u_0|F_0 - u_0)$ is a proper lower semi-continuous convex function on $H^1(\Omega)$. Moreover, from Hypothesis (H1), we know that the map $t \in \mathbb{R}_+ \mapsto F_t \in H^1(\Omega)$ is differentiable at $t = 0$, with its derivative $F_0' \in H^1(\Omega)$ being the unique solution to the scalar Neumann problem

$$\begin{cases} -\Delta F_0' + F_0' = f_0' & \text{in } \Omega, \\ \partial_n F_0' = 0 & \text{on } \Gamma. \end{cases}$$

Thus, using (5.16) and Proposition 3.2.15, the map $t \in \mathbb{R}_+ \mapsto u_t \in H^1(\Omega)$ is differentiable at $t = 0$, and its derivative $u_0' \in H^1(\Omega)$ satisfies

$$u_0' = \mathrm{prox}_{D_e^2\Phi(u_0|F_0-u_0)}(F_0'),$$

which, from the definition of the proximal operator (see Definition 3.1.10) leads to

$$F_0' - u_0' \in \partial D_e^2\Phi(u_0|F_0 - u_0)(u_0'),$$

which means that

$$\langle F_0' - u_0', v - u_0' \rangle_{H^1(\Omega)} \leq D_e^2\Phi(u_0|F_0 - u_0)(v) - D_e^2\Phi(u_0|F_0 - u_0)(u_0'),$$

for all $v \in H^1(\Omega)$. Hence we get that

$$\int_\Omega \nabla\left(F_0' - u_0'\right) \cdot \nabla(v - u_0') + \int_\Omega \left(F_0' - u_0'\right)(v - u_0') \\ \leq \iota_{\mathcal{K}_{u_0, \frac{\partial_n(F_0-u_0)}{g_0}}}(v) - \iota_{\mathcal{K}_{u_0, \frac{\partial_n(F_0-u_0)}{g_0}}}(u_0') + \int_\Gamma g_0' \frac{\partial_n(F_0 - u_0)}{g_0}(v - u_0'),$$

for all $v \in H^1(\Omega)$. Moreover, since $\partial_n F_0 = 0$ *a.e.* on $\Gamma$, it follows that

$$\int_\Omega \nabla u_0' \cdot \nabla(v - u_0') + \int_\Omega u_0' \left(v - u_0'\right) \geq \iota_{\mathcal{K}_{u_0, \frac{\partial_n(F_0-u_0)}{g_0}}}(u_0') - \iota_{\mathcal{K}_{u_0, \frac{\partial_n(F_0-u_0)}{g_0}}}(v) \\ + \int_\Omega f_0'(v - u_0') + \int_\Gamma g_0' \frac{\partial_n u_0}{g_0}(v - u_0'),$$

for all $v \in \mathrm{H}^1(\Omega)$. Hence $u_0' \in \mathcal{K}_{u_0, \frac{\partial_\mathrm{n}(F_0 - u_0)}{g_0}}$ and

$$\int_\Omega \nabla u_0' \cdot \nabla(v - u_0') + \int_\Omega u_0' \,(v - u_0') \geq \int_\Omega f_0'(v - u_0') + \int_\Gamma g_0' \frac{\partial_\mathrm{n} u_0}{g_0}(v - u_0'),$$

for all $v \in \mathcal{K}_{u_0, \frac{\partial_\mathrm{n}(F_0 - u_0)}{g_0}}$, which concludes the proof from Subsection 4.1.2. □

### 5.2.3 Perturbation of a general scalar Tresca friction problem

Let us consider the parameterized general scalar Tresca friction problem

$$\begin{cases} -\mathrm{div}(\mathrm{M}_t \nabla u_t) + k_t u_t = f_t & \text{in } \Omega, \\ |\mathrm{M}_t \nabla u_t \cdot \mathrm{n}| \leq g_t \text{ and } u_t \mathrm{M} \nabla u_t \cdot \mathrm{n} + g_t \,|u_t| = 0 & \text{on } \Gamma, \end{cases} \tag{GSTP$_t$}$$

for all $t \geq 0$. From Subsection 4.1.3, for all $t \geq 0$, there exists a unique weak solution $u_t \in \mathrm{H}^1(\Omega)$ to (GSTP$_t$) which satisfies

$$\int_\Omega \mathrm{M}_t \nabla u_t \cdot \nabla(v - u_t) + \int_\Omega k_t u_t (v - u_t) + \int_\Gamma g_t |v| - \int_\Gamma g_t |u_t| \geq \int_\Omega f_t (v - u_t), \qquad \forall v \in \mathrm{H}^1(\Omega),$$

and it can be expressed as

$$u_t = \mathrm{prox}_{\Phi(t,\cdot)}(F_t),$$

where $F_t \in \mathrm{H}^1(\Omega)$ is the unique solution to the parameterized general scalar Neumann problem

$$\begin{cases} -\mathrm{div}(\mathrm{M}_t \nabla F_t) + k_t F_t = f_t & \text{in } \Omega, \\ \mathrm{M}_t \nabla F_t \cdot \mathrm{n} = 0 & \text{on } \Gamma, \end{cases} \tag{5.19}$$

and where $\mathrm{prox}_{\Phi(t,\cdot)}$ stands for the proximal operator associated with $\Phi(t,\cdot)$ (already defined in the previous subsection in (5.15)) considered on the perturbed Hilbert space $(\mathrm{H}^1(\Omega), \langle\cdot,\cdot\rangle_{\mathrm{M}_t,k_t})$ (see notations in Section 4.1). Since the scalar product $\langle\cdot,\cdot\rangle_{\mathrm{M}_t,k_t}$ depends on $t$, we follow the same strategy already described in Subsection 5.1.2, to deduce that

$$u_t = \mathrm{prox}_{\Phi(t,\cdot)}(E_t),$$

where $E_t \in \mathrm{H}^1(\Omega)$ is the unique solution to the parameterized variational equality given by

$$\langle E_t, v\rangle_{\mathrm{H}^1(\Omega)} = \int_\Omega f_t v - \int_\Omega (\mathrm{M}_t - \mathrm{I}) \nabla u_t \cdot \nabla v - \int_\Omega (k_t - 1)\, u_t v, \qquad \forall v \in \mathrm{H}^1(\Omega),$$

and where $\mathrm{prox}_{\Phi(t,\cdot)}$ is now considered on the Hilbert space $(\mathrm{H}^1(\Omega), \langle\cdot,\cdot\rangle_{\mathrm{H}^1(\Omega)})$ whose scalar product is $t$-independent. Note that $E_0 \in \mathrm{H}^1(\Omega)$ coincides with the solution $F_0 \in \mathrm{H}^1(\Omega)$ to the parameterized general scalar Neumann problem (5.19) for the parameter $t = 0$. Since we have already investigated the twice epi-differentiability of $\Phi$ on the Hilbert space $(\mathrm{H}^1(\Omega), \langle\cdot,\cdot\rangle_{\mathrm{H}^1(\Omega)})$ in Subsection 5.2.2, and the differentiability of the map $t \in \mathbb{R}_+ \mapsto E_t \in \mathrm{H}^1(\Omega)$ at $t = 0$ in Subsection 5.1.2, one can obtain the following theorem whose the proof is similar to Theorem 5.2.12.

**Theorem 5.2.14.** *Let $u_t \in \mathrm{H}^1(\Omega)$ be the unique solution to the parameterized general scalar Tresca friction problem* $(\mathrm{GSTP}_t)$ *for all $t \geq 0$. Assume that:*

1. *for almost all $s \in \Gamma$, the map $t \in \mathbb{R}_+ \mapsto g_t(s) \in \mathbb{R}_+$ is differentiable at $t = 0$, with its derivative denoted by $g_0'(s)$, and also $g_0' \in \mathrm{L}^2(\Gamma)$;*
2. *the parameterized Tresca friction functional $\Phi$ is twice epi-differentiable at $u_0$ for $F_0 - u_0 \in \partial\Phi(0,\cdot)(u_0)$, with*

$$\mathrm{D}_e^2\Phi(u_0|F_0-u_0)(v) = \int_\Gamma \mathrm{D}_e^2 G(s)(u_0(s)|\partial_\mathrm{n}\left(F_0-u_0\right)(s))(v(s))\mathrm{d}s,$$

*for all $v \in \mathrm{H}^1(\Omega)$, where $F_0$ is the unique solution to the parameterized general scalar Neumann problem* (5.19) *for the parameter $t = 0$.*

*Then the map $t \in \mathbb{R}_+ \mapsto u_t \in \mathrm{H}^1(\Omega)$ is differentiable at $t = 0$, and its derivative denoted by $u_0' \in \mathrm{H}^1(\Omega)$ is the unique solution to the variational inequality*

$$\begin{aligned}\langle u_0', v-u_0'\rangle_{\mathrm{H}^1(\Omega)} \geq \int_\Omega f_0'\,(v-u_0') - \int_\Omega \mathrm{M}_0'\nabla u_0\cdot\nabla(v-u_0') - \int_\Omega k_0' u_0\,(v-u_0') \\ + \int_\Gamma g_0'\frac{\partial_\mathrm{n}u_0}{g_0}(v-u_0'), \qquad \forall v \in \mathcal{K}_{u_0,\frac{\partial_\mathrm{n}(F_0-u_0)}{g_0}},\end{aligned} \tag{5.20}$$

*where*

$$\mathcal{K}_{u_0,\frac{\partial_\mathrm{n}(F_0-u_0)}{g_0}} := \left\{v\in\mathrm{H}^1(\Omega) \mid v=0 \text{ a.e. on } \Gamma_\mathrm{D}^{u_0,g_0},\ v\leq 0 \text{ a.e. on } \Gamma_{\mathrm{S}-}^{u_0,g_0},\ v \geq 0 \text{ a.e. on } \Gamma_{\mathrm{S}+}^{u_0,g_0}\right\},$$

*is a nonempty closed convex subset of $\mathrm{H}^1(\Omega)$, and $\Gamma$ is decomposed, up to a null set, as $\Gamma_\mathrm{N}^{u_0,g_0}\cup\Gamma_\mathrm{D}^{u_0,g_0}\cup\Gamma_{\mathrm{S}-}^{u_0,g_0}\cup\Gamma_{\mathrm{S}+}^{u_0,g_0}$ with*

$$\begin{aligned}\Gamma_\mathrm{N}^{u_0,g_0} &:= \left\{s\in\Gamma \mid u_0(s)\neq 0\right\},\\ \Gamma_\mathrm{D}^{u_0,g_0} &:= \left\{s\in\Gamma \mid u_0(s)=0 \text{ and } \partial_\mathrm{n}u_0(s)\in(-g_0(s),g_0(s))\right\},\\ \Gamma_{\mathrm{S}-}^{u_0,g_0} &:= \left\{s\in\Gamma \mid u_0(s)=0 \text{ and } \partial_\mathrm{n}u_0(s)=g_0(s)\right\},\\ \Gamma_{\mathrm{S}+}^{u_0,g_0} &:= \left\{s\in\Gamma \mid u_0(s)=0 \text{ and } \partial_\mathrm{n}u_0(s)=-g_0(s)\right\}.\end{aligned}$$

With some additional assumptions, one can characterize $u_0' \in \mathrm{H}^1(\Omega)$ as solution to a scalar Signorini problem.

**Corollary 5.2.15.** *Consider the framework of Theorem 5.2.14, and assume that* $\mathrm{div}(\mathrm{M}_0'\nabla u_0) \in \mathrm{L}^2(\Omega)$ *and $\mathrm{M}_0'\nabla u_0\cdot\mathrm{n} \in \mathrm{L}^2(\Gamma)$. Then $u_0' \in \mathrm{H}^1(\Omega)$ is the unique weak solution to the scalar Signorini problem*

$$\left\{\begin{aligned} -\Delta u_0' + u_0' &= f_0' - k_0'u_0 + \mathrm{div}\left(\mathrm{M}_0'\nabla u_0\right) && \text{in } \Omega,\\ \partial_\mathrm{n}u_0' &= h && \text{on } \Gamma_\mathrm{N}^{u_0,g_0}\\ u_0' &= 0 && \text{on } \Gamma_\mathrm{D}^{u_0,g_0},\\ u_0'\leq 0,\ \partial_\mathrm{n}u_0'\leq h \text{ and } u_0'\left(\partial_\mathrm{n}u_0'-h\right) &= 0 && \text{on } \Gamma_{\mathrm{S}-}^{u_0,g_0},\\ u_0'\geq 0,\ \partial_\mathrm{n}u_0'\geq h \text{ and } u_0'\left(\partial_\mathrm{n}u_0'-h\right) &= 0 && \text{on } \Gamma_{\mathrm{S}+}^{u_0,g_0},\end{aligned}\right.$$

*where $h := g_0'\frac{\partial_\mathrm{n}u_0}{g_0} - \mathrm{M}_0'\nabla u_0\cdot\mathrm{n} \in \mathrm{L}^2(\Gamma)$.*

*Proof.* Since $\mathrm{div}(\mathrm{M}_0'\nabla u_0) \in \mathrm{L}^2(\Omega)$ one can apply divergence formula (see Proposition 2.1.1) in Inequality (5.20) to deduce that

$$\langle u_0', v-u_0'\rangle_{\mathrm{H}^1(\Omega)} \geq \int_\Omega f_0'\,(v-u_0') + \int_\Omega \mathrm{div}(\mathrm{M}_0'\nabla u_0)\,(v-u_0') - \langle \mathrm{M}_0'\nabla u_0\cdot \mathrm{n}, v-u_0'\rangle_{\mathrm{H}^{-1/2}(\Gamma)\times\mathrm{H}^{1/2}(\Gamma)}$$
$$- \int_\Omega k_0' u_0\,(v-u_0') + \int_\Gamma g_0'\frac{\partial_{\mathrm{n}} u_0}{g_0}(v-u_0'), \qquad \forall v \in \mathcal{K}_{u_0,\frac{\partial_{\mathrm{n}}(F_0-u_0)}{g_0}},$$

and since $\mathrm{M}_0'\nabla u_0\cdot\mathrm{n} \in \mathrm{L}^2(\Gamma)$, we can conclude the proof from Subsection 4.1.2. ☐

**Remark 5.2.16.** Note that the assumptions in Corollary 5.2.15 are, for instance, satisfied if $u_0 \in \mathrm{H}^2(\Omega)$ and $\mathrm{M}_0' \in \mathrm{W}^{1,\infty}(\Omega,\mathbb{R}^d)$. Note that $u_0$ is in $\mathrm{H}^2(\Omega)$ if $\Gamma$ is sufficiently regular (see [14, Chapter 1, Theorem I.10 p.43])

CHAPTER 6

# SENSITIVITY ANALYSIS OF BOUNDARY VALUE PROBLEMS IN THE LINEAR ELASTIC MODEL

---

In this chapter we consider $d = 2$ or $d = 3$, and we assume that $\Omega$ is an elastic solid satisfying the linear elastic model (see Chapter 1 for details) and that $\Gamma$ is decomposed as $\Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{S}}$, where $\Gamma_{\mathrm{D}}$ and $\Gamma_{\mathrm{S}}$ are two measurable pairwise disjoint subsets of $\Gamma$, such that $\Gamma_{\mathrm{D}}$ and $\Gamma_{\mathrm{S}}$ have a positive measure. Consider also, for all $t \geq 0$, $f_t \in \mathrm{L}^2(\Omega, \mathbb{R}^d)$, $g_t \in \mathrm{L}^2(\Gamma)$, $z_t \in \mathrm{L}^\infty(\Omega)$, $\mathrm{B}_t \in \mathrm{L}^\infty(\Omega, \mathbb{R}^{d\times d})$, $\mathrm{M}_t \in \mathrm{C}^{1,\infty}(\overline{\Omega}, \mathrm{GL}_d(\mathbb{R}))$ such that ${\mathrm{M}_t}^{-1} \in \mathrm{C}^{1,\infty}(\overline{\Omega}, \mathrm{GL}_d(\mathbb{R}))$, where ${\mathrm{M}_t}^{-1}$ is defined as the map $x \in \overline{\Omega} \mapsto \mathrm{M}_t(x)^{-1}\mathrm{GL}_d(\mathbb{R}))$. In the same way as Section 4.2, we assume that, for all $t \geq 0$,

$$\int_\Omega z_t \mathrm{A}\left[(\nabla v_1)\mathrm{B}_t\right]\mathrm{B}_t^\top : \nabla v_2 \geq C\int_\Omega \mathrm{Ae}(v_1) : \mathrm{e}(v_2), \qquad \forall v_1, v_2 \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d),$$

where $C > 0$ is a constant. We recall that, from the assumptions on the stiffness tensor A (see (1.1)), one has $\mathrm{Ae}(v) = \mathrm{A}\nabla v$, for all $v \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$, and that if for some $v \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$, $\mathrm{Ae}(v)\mathrm{n} \in \mathrm{L}^2(\Gamma_{\mathrm{S}}, \mathbb{R}^d)$, then we denote $\sigma_{\mathrm{n}}(v) := \mathrm{Ae}(v)\mathrm{n}\cdot\mathrm{n} \in \mathrm{L}^2(\Gamma_{\mathrm{S}})$ and $\sigma_\tau(v) := \mathrm{Ae}(v)\mathrm{n} - \sigma_{\mathrm{n}}(v)\mathrm{n} \in \mathrm{L}^2(\Gamma_{\mathrm{S}}, \mathbb{R}^d)$. We also recall that, for a matrix $\mathrm{M} \in \mathbb{R}^{d\times d}$, we denote by (when it is necessary to avoid any confusion) $\mathrm{A}[\mathrm{M}]$ the stiffness tensor A applied to M, and that we denote $v_{(\mathrm{Mn})^\perp} := v - (v\cdot\frac{\mathrm{Mn}}{||\mathrm{Mn}||^2})\mathrm{Mn}$, for any $v \in \mathrm{L}^2(\Gamma_{\mathrm{S}}, \mathbb{R}^d)$. Moreover we assume that:

(H4) the map $t \in \mathbb{R}_+ \mapsto f_t \in \mathrm{L}^2(\Omega, \mathbb{R}^d)$ is differentiable at $t = 0$, with its derivative denoted by $f_0' \in \mathrm{L}^2(\Omega, \mathbb{R}^d)$;

(H5) the map $t \in \mathbb{R}_+ \mapsto z_t \in \mathrm{L}^\infty(\Omega)$ is differentiable at $t = 0$, with its derivative denoted by $z_0' \in \mathrm{L}^\infty(\Omega)$;

(H6) the map $t \in \mathbb{R}_+ \mapsto \mathrm{M}_t \in \mathrm{C}^{1,\infty}(\overline{\Omega}, \mathbb{R}^{d\times d})$ is differentiable at $t = 0$, with its derivative denoted by $\mathrm{M}_0' \in \mathrm{C}^{1,\infty}(\overline{\Omega}, \mathbb{R}^{d\times d})$;

(H7) the map $t \in \mathbb{R}_+ \mapsto \mathrm{B}_t \in \mathrm{L}^\infty(\Omega, \mathbb{R}^{d\times d})$ is differentiable at $t = 0$, with its derivative denoted by $\mathrm{B}_0' \in \mathrm{L}^\infty(\Omega, \mathbb{R}^{d\times d})$.

In the sequel, the classical Signorini (resp. Tresca friction) problem is called the Signorini (resp. Tresca friction) problem (see Remark 4.2.1).

In this chapter, we perform directly the sensitivity analysis of a general Signorini problem in Section 6.1 using the notion of twice epi-differentiability (see Definition 3.2.5). In Section 6.2, we perform the sensitivity analysis of two general Tresca friction problems, both with perturbation of the Tresca friction law, thus the extended notion of twice epi-differentiability depending on a parameter is required (see Definition 3.2.11). More precisely, in Section 6.2.1 the unit normal vector to the boundary is not perturbed (i.e. we assume that $\mathrm{M}_t = \mathrm{I}$ on $\overline{\Omega}$, for all $t \geq 0$), while in Section 6.2.2 we perturb it but only in the two-dimensional case. Note that if the shape $\Omega$ is perturbed, then so is the unit normal vector n. Therefore, the perturbation of the normal n is important if we want to take into account a shape perturbation case in the Signorini (resp. Tresca friction) problem, since the normal n can appear in the corresponding variational formulation.

In the sequel, to simplify the following computations, we assume that $\mathrm{M}_0 = \mathrm{I}$ on $\overline{\Omega}$, $\mathrm{B}_0 = \mathrm{I}$ and $z_0 = 1$ *a.e.* on $\Omega$.

## 6.1 A general Signorini problem

Consider the parameterized general Signorini problem given by

$$\left\{ \begin{aligned} -\mathrm{div}\left(z_t \mathrm{A}\left[\nabla u_t \mathrm{B}_t\right] \mathrm{B}_t^\top\right) &= f_t && \text{in } \Omega, \\ u_t &= 0 && \text{on } \Gamma_{\mathrm{D}}, \\ \left(\left(z_t \mathrm{A}\left[\nabla u_t \mathrm{B}_t\right] \mathrm{B}_t^\top\right) \mathrm{n}\right)_{\left(\mathrm{M}_t^{-1^\top} \mathrm{n}\right)^\perp} &= 0 && \text{on } \Gamma_{\mathrm{S}}, \\ u_t \cdot \mathrm{M}_t^{-1^\top}\mathrm{n} \leq 0, \ \left(z_t \mathrm{A}\left[\nabla u_t \mathrm{B}_t\right] \mathrm{B}_t^\top\right) \mathrm{n} \cdot \frac{\mathrm{M}_t^{-1^\top}\mathrm{n}}{||\mathrm{M}_t^{-1^\top}\mathrm{n}||^2} \leq g_t \text{ and} & && \\ u_t \cdot \mathrm{M}_t^{-1^\top}\mathrm{n}\left(\left(z_t \mathrm{A}\left[\nabla u_t \mathrm{B}_t\right] \mathrm{B}_t^\top\right) \mathrm{n} \cdot \frac{\mathrm{M}_t^{-1^\top}\mathrm{n}}{||\mathrm{M}_t^{-1^\top}\mathrm{n}||^2} - g_t\right) &= 0 && \text{on } \Gamma_{\mathrm{S}}, \end{aligned} \right. \tag{GSP$_t$}$$

for all $t \geq 0$. From Subsection 4.2.2, for all $t \geq 0$, there exists a unique weak solution $u_t \in \mathcal{K}_t^1(\Omega, \mathbb{R}^d)$ to (GSP$_t$) which satisfies

$$\int_\Omega z_t \mathrm{A}\left[\nabla u_t \mathrm{B}_t\right] \mathrm{B}_t^\top : \nabla(v - u_t) \geq \int_\Omega f_t \cdot (v - u_t) + \int_{\Gamma_{\mathrm{S}}} g_t \mathrm{M}_t^{-1^\top}\mathrm{n} \cdot (v - u_t), \qquad \forall v \in \mathcal{K}_t^1(\Omega, \mathbb{R}^d), \tag{6.1}$$

where $\mathcal{K}_t^1(\Omega, \mathbb{R}^d) := \{v \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d) \mid v \cdot \mathrm{M}_t^{-1^\top}\mathrm{n} \leq 0 \text{ } a.e. \text{ on } \Gamma_{\mathrm{S}}\}$ and, in order to use our methodology, we express it using the proximal operator as (see Remark 3.1.11)

$$u_t = \mathrm{prox}_{\iota_{\mathcal{K}_t^1(\Omega,\mathbb{R}^d)}}(F_t),$$

where $F_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ is the unique solution to the parameterized general Dirichlet-Neumann problem

$$\left\{ \begin{aligned} -\mathrm{div}\left(z_t \mathrm{A}\left[(\nabla F_t) \mathrm{B}_t\right] \mathrm{B}_t^\top\right) &= f_t && \text{in } \Omega, \\ F_t &= 0 && \text{on } \Gamma_{\mathrm{D}}, \\ \left(z_t \mathrm{A}\left[(\nabla F_t) \mathrm{B}_t\right] \mathrm{B}_t^\top\right) \mathrm{n} &= g_t \mathrm{M}_t^{-1^\top}\mathrm{n} && \text{on } \Gamma_{\mathrm{S}}, \end{aligned} \right. \tag{6.2}$$

and where $\mathrm{prox}_{\iota_{\mathcal{K}_t^1(\Omega,\mathbb{R}^d)}}$ stands for the proximal operator associated with the Signorini indicator function $\iota_{\mathcal{K}_t^1(\Omega,\mathbb{R}^d)}$ considered on the perturbed Hilbert space $(\mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d), \langle \cdot, \cdot \rangle_{z_t,\mathrm{A},\mathrm{B}_t})$ (see the notations in

Section 4.2). Here, the main difficulty is that the indicator function $\iota_{\mathcal{K}^1_t(\Omega,\mathbb{R}^d)}$ depends on the parameter $t$, thus it would required the extended notion of twice epi-differentiability depending on a parameter in order to apply the Proposition 3.2.15, like we did in Subsection 5.2.2 and 5.2.3. Nevertheless, it is not necessary since $\mathrm{M}_t \in \mathrm{C}^{1,\infty}(\overline{\Omega}, \mathrm{GL}_d(\mathbb{R}))$ and, for all $v \in \mathcal{K}^1_t(\Omega,\mathbb{R}^d)$, one has

$$\mathrm{M}_t^{-1} v \in \mathcal{K}^1_0(\Omega,\mathbb{R}^d),$$

and conversely, for all $v \in \mathcal{K}^1_0(\Omega,\mathbb{R}^d)$,

$$\mathrm{M}_t v \in \mathcal{K}^1_t(\Omega,\mathbb{R}^d),$$

where $\mathcal{K}^1_0(\Omega,\mathbb{R}^d) = \left\{v \in \mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d) \mid v_\mathrm{n} \leq 0 \text{ } a.e. \text{ on } \Gamma_\mathrm{S}\right\}$. Thus, from Inequality (6.1), one proves that $\overline{u}_t := \mathrm{M}_t^{-1} u_t \in \mathcal{K}^1_0(\Omega,\mathbb{R}^d)$ satisfies

$$\int_\Omega z_t \mathrm{A}\left[\nabla(\mathrm{M}_t\overline{u}_t)\mathrm{B}_t\right]\mathrm{B}_t^\top : \nabla(\mathrm{M}_t\left(v-\overline{u}_t\right)) \geq \int_\Omega \mathrm{M}_t^\top f_t \cdot (v-\overline{u}_t) + \int_{\Gamma_\mathrm{S}} g_t\left(v_\mathrm{n} - \overline{u}_{t\mathrm{n}}\right), \tag{6.3}$$

for all $v \in \mathcal{K}^1_0(\Omega,\mathbb{R}^d)$. Thus, using the characterization of the proximal operator (see Definition 3.1.10), we can expressed $\overline{u}_t$ as

$$\overline{u}_t = \mathrm{prox}_{\iota_{\mathcal{K}^1_0(\Omega,\mathbb{R}^d)}}(G_t),$$

where $G_t \in \mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d)$ is the unique solution to the parameterized variational equality

$$\int_\Omega z_t \mathrm{A}\left[\nabla(\mathrm{M}_t G_t)\mathrm{B}_t\right]\mathrm{B}_t^\top : \nabla(\mathrm{M}_t v) = \int_\Omega \mathrm{M}_t^\top f_t \cdot v + \int_{\Gamma_\mathrm{S}} g_t v_\mathrm{n}, \qquad \forall v \in \mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d),$$

and $\mathrm{prox}_{\iota_{\mathcal{K}^1_0(\Omega,\mathbb{R}^d)}}$ is the proximal operator associated with the Signorini indicator function $\iota_{\mathcal{K}^1_0(\Omega,\mathbb{R}^d)}$ considered on the perturbed Hilbert space $(\mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d), \langle\cdot,\cdot\rangle_{z_t,\mathrm{A},\mathrm{B}_t,\mathrm{M}_t})$, where $\langle\cdot,\cdot\rangle_{z_t,\mathrm{A},\mathrm{B}_t,\mathrm{M}_t}$ is the scalar product defined by

$$(v_1,v_2) \in \left(\mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d)\right)^2 \mapsto \int_\Omega z_t \mathrm{A}\left[\nabla(\mathrm{M}_t v_1)\mathrm{B}_t\right]\mathrm{B}_t^\top : \nabla(\mathrm{M}_t v_2) \in \mathbb{R}.$$

The previous difficulty is solved since the Signorini indicator function $\iota_{\mathcal{K}^1_0(\Omega,\mathbb{R}^d)}$ does not depend on the parameter $t \geq 0$. Nevertheless, we face here to a perturbed Hilbert space due to the scalar product $\langle\cdot,\cdot\rangle_{z_t,\mathrm{A},\mathrm{B}_t,\mathrm{M}_t}$ that is $t$-dependent, thus we could not apply Proposition 3.2.10. To overcome this difficulty let us rewrite Inequality (6.3) by adding to both members $\langle\overline{u}_t, v-\overline{u}_t\rangle_\mathrm{A}$, to deduce that

$$\begin{aligned}\langle\overline{u}_t, v-\overline{u}_t\rangle_\mathrm{A} \geq{}& \int_\Omega \mathrm{M}_t^\top f_t \cdot (v-\overline{u}_t) + \int_{\Gamma_\mathrm{S}} g_t\left(v_\mathrm{n}-\overline{u}_{t\mathrm{n}}\right)\\ &- \int_\Omega z_t \mathrm{A}\left[\nabla(\mathrm{M}_t\overline{u}_t)\mathrm{B}_t\right]\mathrm{B}_t^\top : \nabla(\mathrm{M}_t\left(v-\overline{u}_t\right)) + \int_\Omega \mathrm{A}\nabla\overline{u}_t : \nabla(v-\overline{u}_t),\end{aligned}$$

for all $v \in \mathcal{K}^1_0(\Omega,\mathbb{R}^d)$. Thus $\overline{u}_t$ is also expressed as

$$\overline{u}_t = \mathrm{prox}_{\iota_{\mathcal{K}^1_0(\Omega,\mathbb{R}^d)}}(E_t),$$

where $E_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ stands for the unique solution to the parameterized variational equality

$$\langle E_t, v\rangle_{\mathrm{A}} \;=\; \int_\Omega \mathrm{M}_t^\top f_t \cdot v + \int_{\Gamma_\mathrm{S}} g_t v_\mathrm{n} - \int_\Omega z_t \mathrm{A}\left[\nabla(\mathrm{M}_t\overline{u}_t)\mathrm{B}_t\right]\mathrm{B}_t^\top \;:\; \nabla(\mathrm{M}_t v) + \int_\Omega \mathrm{A}\nabla\overline{u}_t \;:\; \nabla v, \quad (6.4)$$

for all $v \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ and for all $t \geq 0$, where $\mathrm{prox}_{\iota_{\mathcal{K}^1_0(\Omega,\mathbb{R}^d)}}$ is the proximal operator associated with the Signorini indicator function $\iota_{\mathcal{K}^1_0(\Omega,\mathbb{R}^d)}$ considered on the Hilbert space $(\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d),\langle\cdot,\cdot\rangle_{\mathrm{A}})$, which is $t$-independent. Now, using Hypothesis (H4), (H5), (H6), (H7), let us prove that, under some assumption, the map $t \in \mathbb{R}_+ \mapsto E_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ is differentiable at $t = 0$.

**Lemma 6.1.1.** *Let $E_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ be the unique solution to the parameterized variational equality (6.4), for all $t \geq 0$. Assume that the map $t \in \mathbb{R}_+ \mapsto g_t \in \mathrm{L}^2(\Gamma_\mathrm{S})$ is differentiable at $t = 0$, with its derivative denoted by $g'_0 \in \mathrm{L}^2(\Gamma_\mathrm{S})$. Then the map $t \in \mathbb{R}_+ \mapsto E_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ is differentiable at $t = 0$ and its derivative, denoted by $E'_0 \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$, is the unique solution to the variational equality given by*

$$\begin{aligned}\langle E'_0, v\rangle_{\mathrm{A}} = \int_\Omega \left(\mathrm{M}_0'^{\top} f_0 + f'_0\right)\cdot v + \int_{\Gamma_\mathrm{S}} g'_0 v_\mathrm{n} - \int_\Omega \mathrm{A}\nabla u_0 : \nabla(\mathrm{M}'_0 v)&\\ - \int_\Omega \left(\mathrm{A}\left[\nabla u_0\right]\mathrm{B}_0'^{\top} + \mathrm{A}\left[\nabla u_0 \mathrm{B}'_0\right] + \mathrm{A}\nabla(\mathrm{M}'_0 u_0) + z'_0\mathrm{A}\nabla u_0\right) : \nabla v,& \quad (6.5)\end{aligned}$$

*for all $v \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$.*

*Proof.* From Hypothesis (H4), (H6), the map $t \in \mathbb{R}_+ \mapsto \mathrm{M}_t^\top f_t \in \mathrm{L}^2(\Omega,\mathbb{R}^d)$ is differentiable at $t = 0$, with its derivative given by $\mathrm{M}_0'^{\top} f_0 + f'_0 \in \mathrm{L}^2(\Omega,\mathbb{R}^d)$. Using the Riesz representation theorem, we denote by $Z \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ the unique solution to the above variational inequality (6.5). From linearity and using differentiability results (H4), (H5), (H6), (H7) and that the map $t \in \mathbb{R}_+ \mapsto g_t \in \mathrm{L}^2(\Gamma_\mathrm{S})$ is differentiable at $t = 0$, one gets

$$\begin{aligned}\left\|\frac{E_t - E_0}{t} - Z\right\|_{\mathrm{A}} \leq C\Bigg(\left\|\frac{\mathrm{M}_t^\top f_t - f_0}{t} - \left(\mathrm{M}_0'^{\top} f_0 + f'_0\right)\right\|_{\mathrm{L}^2(\Omega,\mathbb{R}^d)} + \left\|\frac{g_t - g_0}{t} - g'_0\right\|_{\mathrm{L}^2(\Gamma_\mathrm{S})}&\\ + \left\|\overline{u}_t - \overline{u}_0\right\|_{\mathrm{A}} + \frac{o(t)}{t}\left\|\overline{u}_t\right\|_{\mathrm{A}}\Bigg),&\end{aligned}$$

for all $t \geq 0$ sufficiently small, where $C > 0$ is a constant which depends only on the data and not on $t$, and where $o$ stands for the standard Bachmann–Landau notation, with $\frac{|o(t)|}{t} \to 0$ when $t \to 0^+$. Therefore, to conclude the proof, we only need to prove the continuity of the map $t \in \mathbb{R}_+ \mapsto \overline{u}_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ at $t = 0$. To this aim, let us take $v = \overline{u}_0$ in the variational formulation of $\overline{u}_t$ and $v = \overline{u}_t$ in the variational formulation of $\overline{u}_0$ to get that

$$\left\|\overline{u}_t - \overline{u}_0\right\|_{\mathrm{A}} \leq C\left(\left\|\mathrm{M}_t^\top f_t - f_0\right\|_{\mathrm{L}^2(\Omega,\mathbb{R}^d)} + \left\|g_t - g_0\right\|_{\mathrm{L}^2(\Gamma_\mathrm{S})} + \left\|\overline{u}_t\right\|_{\mathrm{A}}(t + o(t))\right),$$

for all $t \geq 0$ sufficiently small. Then, to conclude the proof, we need to prove that the map $t \in \mathbb{R}_+ \mapsto \left\|\overline{u}_t\right\|_{\mathrm{A}} \in \mathbb{R}$ is bounded for $t \geq 0$ sufficiently small. Let us take $v = 0$ in the variational formulation of $\overline{u}_t$ to get that

$$\|\overline{u}_t\|_{\mathrm{A}} \leq C\left(\left\|\mathrm{M}_t^\top f_t\right\|_{\mathrm{L}^2(\Omega,\mathbb{R}^d)} + \|g_t\|_{\mathrm{L}^2(\Gamma_{\mathrm{S}})}\right) + C\,\|\overline{u}_t\|_{\mathrm{A}}\,(t+o(t))\,,$$

for all $t \geq 0$ sufficiently small. Thus one deduces

$$\|\overline{u}_t\|_{\mathrm{A}} \leq \frac{C\left(\left\|\mathrm{M}_t^\top f_t\right\|_{\mathrm{L}^2(\Omega,\mathbb{R}^d)} + \|g_t\|_{\mathrm{L}^2(\Gamma_{\mathrm{S}})}\right)}{1 - C\,(t+o(t))},$$

for all $t \geq 0$ sufficiently small, and using the continuity of the map $t \in \mathbb{R}_+ \mapsto \mathrm{M}_t^\top f_t \in \mathrm{L}^2(\Omega,\mathbb{R}^d)$ and of the map $t \in \mathbb{R}_+ \mapsto g_t \in \mathrm{L}^2(\Gamma_{\mathrm{S}})$, the proof is complete. □

Note that $\overline{u}_0$ (resp. $E_0$) coincides with the solution $u_0$ (resp. $F_0$) to the parameterized general Signorini problem ($\mathrm{GSP}_t$) (resp. to the parameterized general Dirichlet-Neumann problem (6.2)) for the parameter $t = 0$. Since we have expressed $\overline{u}_t = \mathrm{prox}_{\iota_{\mathcal{K}_0^1(\Omega,\mathbb{R}^d)}}(E_t)$ and characterized in Lemma 6.1.1 the derivative of the map $t \in \mathbb{R}_+ \mapsto E_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ at $t = 0$, we now need to prove that the Signorini indicator function $\iota_{\mathcal{K}_0^1(\Omega,\mathbb{R}^d)}$ is twice epi-differentiable at $u_0 \in \mathcal{K}_0^1(\Omega,\mathbb{R}^d)$ for $F_0 - u_0 \in \mathrm{N}_{\mathcal{K}_0^1(\Omega,\mathbb{R}^d)}(u_0)$. Thus from Proposition 3.2.9, we need the polyhedricity of the subset $\mathcal{K}_0^1(\Omega,\mathbb{R}^d)$ at $u_0 \in \mathcal{K}_0^1(\Omega,\mathbb{R}^d)$ for $F_0 - u_0 \in \mathrm{N}_{\mathcal{K}_0^1(\Omega,\mathbb{R}^d)}(u_0)$ on the Hilbert space $(\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d), \langle\cdot,\cdot\rangle_{\mathrm{A}}))$. This result has already been proved in [57, Lemma 5.2.9 p.116].

**Lemma 6.1.2.** *The nonempty closed convex subset $\mathcal{K}_0^1(\Omega,\mathbb{R}^d)$ of $\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ is polyhedric at $u_0 \in \mathcal{K}_0^1(\Omega,\mathbb{R}^d)$ for $F_0 - u_0 \in \mathrm{N}_{\mathcal{K}_0^1(\Omega,\mathbb{R}^d)}(u_0)$ and one has*

$$\mathrm{T}_{\mathrm{N}_{\mathcal{K}_0^1(\Omega,\mathbb{R}^d)}}(u_0) \cap (\mathbb{R}\,(F_0 - u_0))^\perp = \left\{v \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d) \mid v_{\mathrm{n}} \leq 0 \text{ q.e. on } \Gamma_{\mathrm{S}}^{u_{0\mathrm{n}}=0} \text{ and } \langle F_0 - u_0, v\rangle_{\mathrm{A}} = 0\right\},$$

*where $\Gamma_{\mathrm{S}}^{u_{0\mathrm{n}}=0} := \{s \in \Gamma_{\mathrm{S}} \mid u_{0\mathrm{n}}(s) = 0\}$.*

We can now prove that the map $t \in \mathbb{R}_+ \mapsto \overline{u}_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ is differentiable at $t = 0$ under some assumptions.

**Theorem 6.1.3.** *Let $\overline{u}_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ be the unique solution to the parameterized variational inequality (6.3), for all $t \geq 0$. Assume that the map $t \in \mathbb{R}_+ \mapsto g_t \in \mathrm{L}^2(\Gamma_{\mathrm{S}})$ is differentiable at $t = 0$, with its derivative denoted by $g'_0 \in \mathrm{L}^2(\Gamma_{\mathrm{S}})$. Then the map $t \in \mathbb{R}_+ \mapsto \overline{u}_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ is differentiable at $t = 0$, and its derivative denoted by $\overline{u}'_0 \in \mathrm{T}_{\mathrm{N}_{\mathcal{K}_0^1(\Omega,\mathbb{R}^d)}}(u_0) \cap (\mathbb{R}\,(F_0 - u_0))^\perp$ is the unique solution to the variational inequality*

$$\langle \overline{u}'_0, v - \overline{u}'_0\rangle_{\mathrm{A}} \geq \int_\Omega f'_0 \cdot (v - \overline{u}'_0) + \int_{\Gamma_{\mathrm{S}}} g'_0\,(v_{\mathrm{n}} - \overline{u}'_{0\mathrm{n}}) - \langle \mathrm{Ae}(u_0)\mathrm{n}, \mathrm{M}'_0\,(v - \overline{u}'_0)\rangle_{\mathrm{H}^{-1/2}(\Gamma,\mathbb{R}^d)\times\mathrm{H}^{1/2}(\Gamma,\mathbb{R}^d)}$$
$$- \int_\Omega \left(\mathrm{A}\,[\nabla u_0]\,{\mathrm{B}'_0}^\top + \mathrm{A}\,[\nabla u_0 \mathrm{B}'_0] + \mathrm{A}\nabla(\mathrm{M}'_0 u_0) + z'_0 \mathrm{A}\nabla u_0\right) : \nabla(v - \overline{u}'_0), \quad (6.6)$$

*for all $v \in \mathrm{T}_{\mathrm{N}_{\mathcal{K}_0^1(\Omega,\mathbb{R}^d)}}(u_0) \cap (\mathbb{R}\,(F_0 - u_0))^\perp$.*

*Proof.* Using Lemma 6.1.2, one deduces from Proposition 3.2.9 that the Signorini indicator function $\iota_{\mathcal{K}^1_0(\Omega,\mathbb{R}^d)}$ is twice epi-differentiable at $u_0 \in \mathcal{K}^1_0(\Omega,\mathbb{R}^d)$ for $F_0 - u_0 \in \mathrm{N}_{\mathcal{K}^1_0(\Omega,\mathbb{R}^d)}(u_0)$ and

$$\mathrm{d}^2_e \iota_{\mathcal{K}^1_0(\Omega,\mathbb{R}^d)}(u_0|F_0-u_0) = \iota_{\mathrm{T}_{\mathrm{N}_{\mathcal{K}^1_0(\Omega,\mathbb{R}^d)}}(u_0)\cap(\mathbb{R}(F_0-u_0))^{\perp}}.$$

Thus, from Lemma 6.1.1 one can use Proposition 3.2.10 to deduce that the map $t\in\mathbb{R}_+ \mapsto \overline{u}_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ is differentiable at $t=0$, and its derivative $\overline{u}'_0 \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ satisfies

$$\overline{u}'_0 = \mathrm{prox}_{\mathrm{d}^2_e\iota_{\mathcal{K}^1_0(\Omega,\mathbb{R}^d)}(u_0|F_0-u_0)}(E'_0),$$

where $E'_0 \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ is solution to the variational equality (6.5). Thus, one gets

$$\langle \overline{u}'_0, v-\overline{u}'_0\rangle_{\mathrm{A}} \geq \langle E'_0, v-\overline{u}'_0\rangle_{\mathrm{A}}, \qquad \forall v \in \mathrm{T}_{\mathrm{N}_{\mathcal{K}^1_0(\Omega,\mathbb{R}^d)}}(u_0)\cap\left(\mathbb{R}\left(F_0-u_0\right)\right)^{\perp}.$$

The proof is complete since $\mathrm{div}\left(\mathrm{Ae}(u_0)\right) = -f_0 \in \mathrm{L}^2(\Omega,\mathbb{R}^d)$. □

Since $u_t = \mathrm{M}_t\overline{u}_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ it is now possible to characterize the derivative of the weak solution to the parameterized general Signorini problem ($\mathrm{GSP}_t$).

**Theorem 6.1.4.** *Consider the framework of Theorem 6.1.3. Then the map $t\in\mathbb{R}_+\mapsto u_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ is differentiable at $t=0$ and its derivative, denoted by $u'_0 \in \mathrm{T}_{\mathrm{N}_{\mathcal{K}^1_0(\Omega,\mathbb{R}^d)}}(u_0)\cap\left(\mathbb{R}\left(F_0-u_0\right)\right)^{\perp} + \mathrm{M}'_0u_0$, is the unique solution to the variational inequality*

$$\begin{aligned}\int_\Omega \mathrm{A}\nabla u'_0 : \nabla(v-u'_0) \geq \int_\Omega f'_0\cdot(v-u'_0) + \int_{\Gamma_{\mathrm{S}}} g'_0\left(v_{\mathrm{n}}-u'_{0\mathrm{n}}\right) \\ - \left\langle \mathrm{Ae}(u_0)\mathrm{n}, \mathrm{M}'_0\left(v-u'_0\right)\right\rangle_{\mathrm{H}^{-1/2}(\Gamma,\mathbb{R}^d)\times\mathrm{H}^{1/2}(\Gamma,\mathbb{R}^d)} \\ - \int_\Omega \left(\mathrm{A}\left[\nabla u_0\right]{\mathrm{B}'_0}^{\top} + \mathrm{A}\left[\nabla u_0\mathrm{B}'_0\right] + z'_0\mathrm{A}\nabla u_0\right) : \nabla(v-u'_0),\end{aligned} \tag{6.7}$$

*for all $v \in \mathrm{T}_{\mathrm{N}_{\mathcal{K}^1_0(\Omega,\mathbb{R}^d)}}(u_0)\cap\left(\mathbb{R}\left(F_0-u_0\right)\right)^{\perp} + \mathrm{M}'_0u_0$, where*

$$\begin{aligned}&\mathrm{T}_{\mathrm{N}_{\mathcal{K}^1_0(\Omega,\mathbb{R}^d)}}(u_0)\cap\left(\mathbb{R}\left(F_0-u_0\right)\right)^{\perp} + \mathrm{M}'_0u_0 = \\ &\qquad\left\{v\in\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d) \mid (v-\mathrm{M}'_0u_0)_{\mathrm{n}} \leq 0 \text{ q.e. on } \Gamma_{\mathrm{S}}^{u_{0\mathrm{n}}=0} \text{ and } \langle F_0-u_0, v-\mathrm{M}'_0u_0\rangle_{\mathrm{A}} = 0\right\}.\end{aligned}$$

*Proof.* Since $u_t = \mathrm{M}_t\overline{u}_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$, then $u'_0 = \overline{u}'_0 + \mathrm{M}'_0u_0 \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$. Thus, using the variational inequality satisfied by $\overline{u}'_0$ (see (6.6)), one gets

$$\begin{aligned}\langle u'_0-\mathrm{M}'_0u_0, v-u'_0+\mathrm{M}'_0u_0\rangle_{\mathrm{A}} \geq \int_\Omega f'_0\cdot(v-u'_0+\mathrm{M}'_0u_0) + \int_{\Gamma_{\mathrm{S}}} g'_0\left(v_{\mathrm{n}}-u'_{0\mathrm{n}}+(\mathrm{M}'_0u_0)_{\mathrm{n}}\right) \\ - \left\langle \mathrm{Ae}(u_0)\mathrm{n}, \mathrm{M}'_0\left(v-u'_0+\mathrm{M}'_0u_0\right)\right\rangle_{\mathrm{H}^{-1/2}(\Gamma,\mathbb{R}^d)\times\mathrm{H}^{1/2}(\Gamma,\mathbb{R}^d)} \\ - \int_\Omega \left(\mathrm{A}\left[\nabla u_0\right]{\mathrm{B}'_0}^{\top} + \mathrm{A}\left[\nabla u_0\mathrm{B}'_0\right] + \mathrm{A}\nabla(\mathrm{M}'_0u_0) + z'_0\mathrm{A}\nabla u_0\right) : \nabla(v-u'_0+\mathrm{M}'_0u_0),\end{aligned}$$

for all $v \in \mathrm{T}_{\mathrm{N}_{\mathcal{K}^1_0(\Omega,\mathbb{R}^d)}}(u_0) \cap (\mathbb{R}\,(F_0 - u_0))^{\perp}$. This is also

$$\begin{aligned}\langle u_0' - \mathrm{M}_0' u_0, v - u_0' \rangle_{\mathrm{A}} \geq \int_\Omega f_0' \cdot (v - u_0') + \int_{\Gamma_{\mathrm{S}}} g_0' \,(v_{\mathrm{n}} - u'_{0\mathrm{n}}) \\ - \langle \mathrm{Ae}(u_0)\mathrm{n}, \mathrm{M}_0'\,(v - u_0') \rangle_{\mathrm{H}^{-1/2}(\Gamma,\mathbb{R}^d)\times \mathrm{H}^{1/2}(\Gamma,\mathbb{R}^d)} \\ - \int_\Omega \left( \mathrm{A}\,[\nabla u_0]\,{\mathrm{B}_0'}^{\top} + \mathrm{A}\,[\nabla u_0 \mathrm{B}_0'] + \mathrm{A}\nabla(\mathrm{M}_0' u_0) + z_0' \mathrm{A}\nabla u_0 \right) : \nabla(v - u_0')\end{aligned}$$

for all $v \in \mathrm{T}_{\mathrm{N}_{\mathcal{K}^1_0(\Omega,\mathbb{R}^d)}}(u_0) \cap (\mathbb{R}\,(F_0 - u_0))^{\perp} + \mathrm{M}_0' u_0$, where

$$\begin{aligned}\mathrm{T}_{\mathrm{N}_{\mathcal{K}^1_0(\Omega,\mathbb{R}^d)}}(u_0) \cap (\mathbb{R}\,(F_0 - u_0))^{\perp} + \mathrm{M}_0' u_0 = \\ \left\{ v \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d) \mid (v - \mathrm{M}_0' u_0)_{\mathrm{n}} \leq 0 \text{ } q.e. \text{ on } \Gamma_{\mathrm{S}}^{u_{0\mathrm{n}}=0} \text{ and } \langle F_0 - u_0, v - \mathrm{M}_0' u_0 \rangle_{\mathrm{A}} = 0 \right\}.\end{aligned}$$

This concludes the proof. □

In the same way as for the Theorem 5.1.4, is it possible to improve Theorem 6.1.4 under additional assumptions. Indeed, as mentioned in [19, 47], it is possible to replace *q.e.* in the set $\mathrm{T}_{\mathrm{N}_{\mathcal{K}^1_0(\Omega,\mathbb{R}^d)}}(u_0) \cap (\mathbb{R}\,(F_0 - u_0))^{\perp}$ by *a.e.* under some hypotheses, like, for instance, if $\Gamma_{\mathrm{S}}^{\mathrm{u}_{0\mathrm{n}}=0} = \overline{\mathrm{int}_{\Gamma_{\mathrm{S}}}(\Gamma_{\mathrm{S}}^{\mathrm{u}_{0\mathrm{n}}=0})}$. Moreover, if we assume that the decomposition $\Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{S}}$ of $\Gamma$ is consistent (see Definition 4.2.8 with $\Gamma_{\mathrm{S}-} := \Gamma_{\mathrm{S}}$, $w = 0$ and $\mathrm{M} = \mathrm{I}$) and some regularity assumptions, then we can characterize $u_0'$ as weak solution to a Signorini problem.

**Corollary 6.1.5.** *Consider the framework of Theorem 6.1.4 with the additional assumptions:*

1. $\mathrm{div}(\mathrm{A}\,[\nabla u_0]\,{\mathrm{B}_0'}^{\top} + \mathrm{A}\,[\nabla u_0 \mathrm{B}_0'] + z_0' \mathrm{A}\nabla u_0) \in \mathrm{L}^2(\Omega,\mathbb{R}^d)$ *with also* $(\mathrm{A}\,[\nabla u_0]\,{\mathrm{B}_0'}^{\top} + \mathrm{A}\,[\nabla u_0 \mathrm{B}_0'] + z_0' \mathrm{A}\nabla u_0)\mathrm{n} \in \mathrm{L}^2(\Gamma,\mathbb{R}^d)$ *and* $\mathrm{Ae}(u_0)\mathrm{n} \in \mathrm{L}^2(\Gamma,\mathbb{R}^d)$*;*
2. *the decomposition* $\Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{S}}$ *of* $\Gamma$ *is consistent;*
3. $\Gamma_{\mathrm{S}}^{\mathrm{u}_{0\mathrm{n}}=0} = \overline{\mathrm{int}_{\Gamma_{\mathrm{S}}}(\Gamma_{\mathrm{S}}^{\mathrm{u}_{0\mathrm{n}}=0})}$.

*Then* $u_0' \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ *is the unique weak solution to the Signorini problem*

$$\begin{cases} -\mathrm{div}\,(\mathrm{Ae}(u_0')) = \tilde{f} & \text{in } \Omega, \\ u_0' = 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \sigma_\tau(u_0') = h_\tau & \text{on } \Gamma_{\mathrm{S}}, \\ \sigma_{\mathrm{n}}(u_0') = g_0' + h_{\mathrm{n}} & \text{on } \Gamma_{\mathrm{S_N}}^{u_0,g_0}, \\ (u_0' - \mathrm{M}_0' u_0)_{\mathrm{n}} = 0 & \text{on } \Gamma_{\mathrm{S_D}}^{u_0,g_0}, \\ (u_0' - \mathrm{M}_0' u_0)_{\mathrm{n}} \leq 0,\ \sigma_{\mathrm{n}}(u_0') \leq g_0' + h_{\mathrm{n}} \text{ and } (u_0' - \mathrm{M}_0' u_0)_{\mathrm{n}}\,(\sigma_{\mathrm{n}}(u_0') - g_0' - h_{\mathrm{n}}) = 0 & \text{on } \Gamma_{\mathrm{S-}}^{u_0,g_0}, \end{cases}$$

*where*

$$\tilde{f} := f_0' + \mathrm{div}\left( \mathrm{A}\,[\nabla u_0]\,{\mathrm{B}_0'}^{\top} + \mathrm{A}\,[\nabla u_0 \mathrm{B}_0'] + z_0' \mathrm{A}\nabla u_0 \right) \in \mathrm{L}^2(\Omega,\mathbb{R}^d),$$

*and*

$$h := -\left( \mathrm{A}\,[\nabla u_0]\,{\mathrm{B}_0'}^{\top} + \mathrm{A}\,[\nabla u_0 \mathrm{B}_0'] + z_0' \mathrm{A}\nabla u_0 \right)\mathrm{n} - {\mathrm{M}_0'}^{\top}\,(\mathrm{Ae}(u_0)\mathrm{n}) \in \mathrm{L}^2(\Gamma,\mathbb{R}^d),$$

*and where* $\Gamma_{\mathrm{S}}$ *is decomposed, up to a null set, as* $\Gamma_{\mathrm{S_N}}^{u_0,g_0} \cup \Gamma_{\mathrm{S_D}}^{u_0,g_0} \cup \Gamma_{\mathrm{S-}}^{u_0,g_0}$*, with*

$$\begin{aligned}
\Gamma_{\mathrm{S_N}}^{u_0,g_0} &:= \{s \in \Gamma_{\mathrm{S}} \mid u_{0\mathrm{n}}(s) \neq 0\}, \\
\Gamma_{\mathrm{S_D}}^{u_0,g_0} &:= \{s \in \Gamma_{\mathrm{S}} \mid u_{0\mathrm{n}}(s) = 0 \text{ and } \sigma_{\mathrm{n}}(u_0)(s) < g_0(s)\}, \\
\Gamma_{\mathrm{S-}}^{u_0,g_0} &:= \{s \in \Gamma_{\mathrm{S}} \mid u_{0\mathrm{n}}(s) = 0 \text{ and } \sigma_{\mathrm{n}}(u_0)(s) = g_0(s)\}.
\end{aligned}$$

*Proof.* From Hypothesis 1 one can apply divergence formula (see Proposition 2.1.1) in Inequality (6.7), with $(\mathrm{A}\,[\nabla u_0]\,\mathrm{B}_0'^{\top} + \mathrm{A}\,[\nabla u_0 \mathrm{B}_0'] + z_0' \mathrm{A} \nabla u_0) \in \mathrm{L}^2(\Gamma, \mathbb{R}^d)$. Moreover, since $\mathrm{A}\mathrm{e}(u_0)\mathrm{n} \in \mathrm{L}^2(\Gamma, \mathbb{R}^d)$ and the decomposition $\Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{S}}$ of $\Gamma$ is consistent, then $u_0$ is a (strong) solution to the parameterized general Signorini problem (GSP$_t$) for the parameter $t = 0$ (see Proposition 4.2.9). Thus the rest of the proof is similar to Corollary 5.1.5 since $\Gamma_{\mathrm{S}}^{\mathrm{u_{0n}}=0} = \overline{\mathrm{int}_{\Gamma_{\mathrm{S}}}(\Gamma_{\mathrm{S}}^{\mathrm{u_{0n}}=0})}$ and $\mathcal{H} := \{v_{\mathrm{n}} \in \mathrm{H}^{1/2}(\Gamma_{\mathrm{S}}, \mathbb{R}) \mid v \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)\}$ is a Dirichlet space (see Example 2.2.7 and [19, Remark 3.12 p.13] for more details). □

**Remark 6.1.6.** Note that Hypothesis 1 is, for instance, satisfied if $u_0 \in \mathrm{H}^2(\Omega, \mathbb{R}^d)$, $\mathrm{B}_0' \in \mathrm{W}^{1,\infty}(\Omega, \mathbb{R}^d)$ and $z_0' \in \mathrm{W}^{1,\infty}(\Omega)$.

## 6.2 A general Tresca friction problem

In this section we assume that $\Gamma$ is of class $\mathcal{C}^1$. In that case $\mathrm{n} \in \mathcal{C}^0(\Gamma)$ which will be used to compute the convex subdifferential of the Tresca friction functional (see Lemma 6.2.1). We also assume that almost every point of $\Gamma_{\mathrm{S}}$ is in $\mathrm{int}_{\Gamma}(\Gamma_{\mathrm{S}})$ (see Remark 4.2.23) and, for all $t \geq 0$, $g_t > 0$ *a.e.* on $\Gamma_{\mathrm{S}}$ and we consider $h_t \in \mathrm{L}^2(\Gamma_{\mathrm{S}})$. This section is separated in two subsections. In Subsection 6.2.1 the unit normal vector n to the boundary is not perturbed, i.e. $\mathrm{M}_t = \mathrm{I}$ on $\overline{\Omega}$, while in Subsection 6.2.2, the normal n is perturbed but only in the two-dimensional case, since we are not able to perform the sensitivity analysis in the three-dimensional case (see Remark 6.2.13). As mentioned at the beginning of Chapter 6, the perturbation of the unit normal vector is important in order to take into account a shape perturbation case in the Tresca friction problem. Let us emphasize that, in the following subsections, the Tresca friction law is perturbed.

### 6.2.1 Without perturbation of the unit normal vector to the boundary

In this subsection, we assume that, for all $t \geq 0$, $\mathrm{M}_t = \mathrm{I}$ on $\overline{\Omega}$. Consider the parameterized general Tresca friction problem given by

$$\left\{\begin{aligned}
-\mathrm{div}\left(z_t \mathrm{A}\,[\nabla u_t \mathrm{B}_t]\,\mathrm{B}_t^{\top}\right) &= f_t && \text{in } \Omega, \\
u_t &= 0 && \text{on } \Gamma_{\mathrm{D}}, \\
\left(\left(z_t \mathrm{A}\,[\nabla u_t \mathrm{B}_t]\,\mathrm{B}_t^{\top}\right)\mathrm{n}\right)_{\mathrm{n}} &= h_t && \text{on } \Gamma_{\mathrm{S}}, \\
\left\|\left(\left(z_t \mathrm{A}\,[\nabla u_t \mathrm{B}_t]\,\mathrm{B}_t^{\top}\right)\mathrm{n}\right)_{\tau}\right\| \leq g_t \text{ and } u_{t\tau} \cdot \left(\left(z_t \mathrm{A}\,[\nabla u_t \mathrm{B}_t]\,\mathrm{B}_t^{\top}\right)\mathrm{n}\right)_{\tau} + g_t \left\|u_{t\tau}\right\| &= 0 && \text{on } \Gamma_{\mathrm{S}},
\end{aligned}\right. \tag{GTP$_t$}$$

for all $t \geq 0$. From Subsection 4.2.4, for all $t \geq 0$, there exists a unique weak solution $u_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ to (GTP$_t$) which satisfies

$$\int_{\Omega} z_t \mathrm{A}\,[\nabla u_t \mathrm{B}_t]\,\mathrm{B}_t^{\top} : \nabla(v - u_t) + \int_{\Gamma_{\mathrm{S}}} g_t \left\|v_{\tau}\right\| - \int_{\Gamma_{\mathrm{S}}} g_t \left\|u_{t\tau}\right\| \geq \int_{\Omega} f_t \cdot (v - u_t) + \int_{\Gamma_{\mathrm{S}}} h_t \left(v_{\mathrm{n}} - u_{t\mathrm{n}}\right),$$

for all $v \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$. In the same way as Subsections 5.2.2 and 5.2.3, since the Tresca friction law is perturbed, we define the parameterized Tresca friction functional given by

$$\begin{array}{rccl} \Phi : & \mathbb{R}_+ \times \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d) & \longrightarrow & \mathbb{R} \\ & (t, v) & \longmapsto & \Phi(t, v) := \displaystyle\int_{\Gamma_{\mathrm{S}}} g_t \left\| v_\tau \right\|. \end{array} \tag{6.8}$$

Thus $u_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ can be characterized as

$$u_t = \mathrm{prox}_{\Phi(t,\cdot)}(F_t),$$

where $F_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ is the unique solution to the parameterized general Dirichlet-Neumann problem

$$\left\{ \begin{array}{rcll} -\mathrm{div}\left( z_t \mathrm{A}\left[ (\nabla F_t) \mathrm{B}_t \right] \mathrm{B}_t^\top \right) & = & f_t & \text{in } \Omega, \\ F_t & = & 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \left( z_t \mathrm{A}\left[ (\nabla F_t) \mathrm{B}_t \right] \mathrm{B}_t^\top \right) \mathrm{n} & = & h_t \mathrm{n} & \text{on } \Gamma_{\mathrm{S}}, \end{array} \right. \tag{6.9}$$

where $\mathrm{prox}_{\Phi(t,\cdot)}$ stands for the proximal operator associated with the parameterized Tresca friction functional $\Phi(t, \cdot)$ considered on the perturbed Hilbert space $(\mathrm{H}^1(\Omega), \langle \cdot, \cdot \rangle_{\mathrm{z}_t, \mathrm{A}, \mathrm{B}_t})$ (see the notations in Section 4.2). Since the scalar product $\langle \cdot, \cdot \rangle_{\mathrm{z}_t, \mathrm{A}, \mathrm{B}_t}$ depends on $t$, we follow the same strategy already described in Section 6.1, to deduce that $u_t$ is also characterized as

$$u_t = \mathrm{prox}_{\Phi(t,\cdot)}(E_t),$$

where $E_t \in \mathrm{H}^1(\Omega)$ is the unique solution to the parameterized variational equality

$$\langle E_t, v \rangle_{\mathrm{A}} = \int_\Omega f_t \cdot v + \int_{\Gamma_{\mathrm{S}}} h_t v_{\mathrm{n}} - \int_\Omega z_t \mathrm{A}\left[ \nabla u_t \mathrm{B}_t \right] \mathrm{B}_t^\top : \nabla v + \int_\Omega \mathrm{A} \nabla u_t : \nabla v, \qquad \forall v \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d),$$

where $\mathrm{prox}_{\Phi(t,\cdot)}$ is now considered on the Hilbert space $(\mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d), \langle \cdot, \cdot \rangle_{\mathrm{A}})$ whose scalar product is $t$-independent. Note that $E_0 \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ coincides with the solution $F_0 \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ to the parameterized general Dirichlet-Neumann problem (6.9) for the parameter $t = 0$. Since $\Phi$ is a $t$-dependent functional, we have to use the notion of twice epi-differentiability depending on a parameter (see Definition 3.2.11) in order to apply Proposition 3.2.15. Thus let us start with the characterization of the convex subdifferential of $\Phi(0, \cdot)$. For $s \in \Gamma_{\mathrm{S}}$, we define the tangential norm map as

$$\begin{array}{rccl} \left\| \cdot_{\tau(s)} \right\| : & \mathbb{R}^d & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \left\| x_{\tau(s)} \right\|, \end{array}$$

and let us introduce an auxiliary problem defined, for all $u \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$, by

$$\left\{ \begin{array}{rl} -\mathrm{div}(\mathrm{Ae}(w)) = 0 & \text{in } \Omega, \\ w = 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \sigma_{\mathrm{n}}(w) = 0 & \text{on } \Gamma_{\mathrm{S}}, \\ \sigma_\tau(w)(s) \in g_0(s) \partial \left\| \cdot_{\tau(s)} \right\| (u(s)) & \text{on } \Gamma_{\mathrm{S}}, \end{array} \right. \tag{AP3$_u$}$$

where, for almost all $s \in \Gamma_{\mathrm{S}}$, $\partial \left\|\cdot_{\tau(s)}\right\| (u(s))$ stands for the convex subdifferential of the tangential norm map $\left\|\cdot_{\tau(s)}\right\|$ at $u(s) \in \mathbb{R}^d$. For a given $u \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$, a solution to this problem ($\mathrm{AP3}_u$) is a function $w \in \mathrm{H}^1(\Omega, \mathbb{R}^d)$ such that $-\mathrm{div}(\mathrm{Ae}(w)) = 0$ in $\mathcal{D}'(\Omega, \mathbb{R}^d)$, $w = 0$ *a.e.* on $\Gamma_{\mathrm{D}}$, $\mathrm{Ae}(w)\mathrm{n} \in \mathrm{L}^2(\Gamma_{\mathrm{S}}, \mathbb{R}^d)$ with $\sigma_{\mathrm{n}}(w) = 0$ *a.e.* on $\Gamma_{\mathrm{S}}$ and $\sigma_\tau(w)(s) \in g_0(s)\partial||\cdot_{\tau(s)}||(u(s))$ for almost all $s \in \Gamma_{\mathrm{S}}$.

**Lemma 6.2.1.** *Let* $u \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$. *Then*

$$\partial\Phi(0, \cdot)(u) = \textit{the set of solutions to Problem } (\mathrm{AP3}_u).$$

*Proof.* Let $u \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ and let us prove the two inclusions. Let $w \in \mathrm{H}^1(\Omega, \mathbb{R}^d)$ be a solution to Problem ($\mathrm{AP3}_u$). Then $w \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$, $\mathrm{Ae}(w)\mathrm{n} \in \mathrm{L}^2(\Gamma_{\mathrm{S}}, \mathbb{R}^d)$ with $\sigma_\tau(w)(s) \in g_0(s)\partial||\cdot_{\tau(s)}||(u(s))$ for almost all $s \in \Gamma_{\mathrm{S}}$. Hence one gets

$$\sigma_\tau(w)(s) \cdot (v_\tau(s) - u_\tau(s)) \leq g_0(s)(\|v_\tau(s)\| - \|u_\tau(s)\|),$$

for all $v \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ and for almost all $s \in \Gamma_{\mathrm{S}}$. It follows that

$$\int_{\Gamma_{\mathrm{S}}} \sigma_\tau(w) \cdot (v_\tau - u_\tau) \leq \int_{\Gamma_{\mathrm{S}}} g_0 \|v_\tau\| - \int_{\Gamma_{\mathrm{S}}} g_0 \|u_\tau\|,$$

for all $v \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$. Moreover $-\mathrm{div}(\mathrm{Ae}(w)) = 0$ in $\mathcal{D}'(\Omega, \mathbb{R}^d)$, thus it holds $-\mathrm{div}(\mathrm{Ae}(w)) = 0$ in $\mathrm{L}^2(\Omega, \mathbb{R}^d)$. Hence, from divergence formula (see Proposition 2.1.1), one gets

$$\langle w, v - u\rangle_{\mathrm{A}} = \langle \mathrm{Ae}(w)\mathrm{n}, v - u\rangle_{\mathrm{H}^{-1/2}(\Gamma, \mathbb{R}^d)\times \mathrm{H}^{1/2}(\Gamma, \mathbb{R}^d)},$$

for all $v \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$. Moreover, for all $v \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ one has $v \in \mathrm{H}^{1/2}_{00}(\Gamma_{\mathrm{S}}, \mathbb{R}^d)$ which can be identified to a linear subspace of $\mathrm{H}^{1/2}(\Gamma, \mathbb{R}^d)$, hence

$$\langle w, v - u\rangle_{\mathrm{A}} = \langle \mathrm{Ae}(w)\mathrm{n}, v - u\rangle_{\mathrm{H}^{-1/2}_{00}(\Gamma_{\mathrm{S}}, \mathbb{R}^d)\times \mathrm{H}^{1/2}_{00}(\Gamma_{\mathrm{S}}, \mathbb{R}^d)},$$

for all $v \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$. Since $\mathrm{Ae}(w)\mathrm{n} \in \mathrm{L}^2(\Gamma_{\mathrm{S}}, \mathbb{R}^d)$ and $\sigma_{\mathrm{n}}(w) = 0$ *a.e.* on $\Gamma_{\mathrm{S}}$, one deduces that

$$\langle w, v - u\rangle_{\mathrm{A}} = \int_{\Gamma_{\mathrm{S}}} \mathrm{Ae}(w)\mathrm{n} \cdot (v - u) = \int_{\Gamma_{\mathrm{S}}} \sigma_\tau(w) \cdot (v_\tau - u_\tau),$$

for all $v \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$. Therefore it follows that

$$\langle w, v - u\rangle_{\mathrm{A}} \leq \int_{\Gamma_{\mathrm{S}}} g_0 \|v_\tau\| - \int_{\Gamma_{\mathrm{S}}} g_0 \|u_\tau\|,$$

for all $v \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$. Thus $w \in \partial\Phi(0, \cdot)(u)$ and the first inclusion is proved. Conversely let $w \in \partial\Phi(0, \cdot)(u)$. One has

$$\langle w, v - u\rangle_{\mathrm{A}} \leq \int_{\Gamma_{\mathrm{S}}} g_0 \|v_\tau\| - \int_{\Gamma_{\mathrm{S}}} g_0 \|u_\tau\|, \tag{6.10}$$

for all $v \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$. Considering the function $v := u \pm \psi \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ with any function $\psi \in \mathcal{D}(\Omega, \mathbb{R}^d)$, one deduces from Inequality (6.10) that $-\mathrm{div}(\mathrm{Ae}(w)) = 0$ in $\mathcal{D}'(\Omega, \mathbb{R}^d)$, thus it also holds $-\mathrm{div}(\mathrm{Ae}(w)) = 0$ in $\mathrm{L}^2(\Omega, \mathbb{R}^d)$. Hence, from divergence formula and Inequality (6.10), it follows that

$$\langle \mathrm{Ae}(w)\mathrm{n}, v-u\rangle_{\mathrm{H}_{00}^{-1/2}(\Gamma_{\mathrm{S}},\mathbb{R}^d)\times\mathrm{H}_{00}^{1/2}(\Gamma_{\mathrm{S}},\mathbb{R}^d)} \leq \int_{\Gamma_{\mathrm{S}}} g_0 \|v_\tau\| - \int_{\Gamma_{\mathrm{S}}} g_0 \|u_\tau\|,$$

for all $v \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$, and thus for all $v \in \mathrm{H}^{1/2}_{00}(\Gamma_{\mathrm{S}}, \mathbb{R}^d)$. Now let us consider the function $v = u + \varphi \in \mathrm{H}^{1/2}_{00}(\Gamma_{\mathrm{S}}, \mathbb{R}^d)$ for any $\varphi \in \mathrm{H}^{1/2}_{00}(\Gamma_{\mathrm{S}}, \mathbb{R}^d)$. One gets

$$\langle \mathrm{Ae}(w)\mathrm{n}, \varphi\rangle_{\mathrm{H}_{00}^{-1/2}(\Gamma_{\mathrm{S}},\mathbb{R}^d)\times\mathrm{H}_{00}^{1/2}(\Gamma_{\mathrm{S}},\mathbb{R}^d)} \leq \int_{\Gamma_{\mathrm{S}}} g_0 \|\varphi_\tau\| \leq \|g_0\|_{\mathrm{L}^2(\Gamma_{\mathrm{S}})} \|\varphi\|_{\mathrm{L}^2(\Gamma_{\mathrm{S}},\mathbb{R}^d)},$$

for all $\varphi \in \mathrm{H}^{1/2}_{00}(\Gamma_{\mathrm{S}}, \mathbb{R}^d)$. From Proposition 2.1.7 and 2.1.8, one deduces that $\mathrm{Ae}(w)\mathrm{n} \in \mathrm{L}^2(\Gamma_{\mathrm{S}}, \mathbb{R}^d)$ and that

$$\int_{\Gamma_{\mathrm{S}}} \mathrm{Ae}(w)\mathrm{n}\cdot(v-u) = \int_{\Gamma_{\mathrm{S}}} \sigma_\tau(w)\cdot(v_\tau - u_\tau) + \int_{\Gamma_{\mathrm{S}}} \sigma_{\mathrm{n}}(w)\,(v_{\mathrm{n}} - u_{\mathrm{n}}) \leq \int_{\Gamma_{\mathrm{S}}} g_0\|v_\tau\| - \int_{\Gamma_{\mathrm{S}}} g_0 \|u_\tau\|, \quad (6.11)$$

for all $v \in \mathrm{H}^{1/2}_{00}(\Gamma_{\mathrm{S}}, \mathbb{R}^d)$, and, by density, for all $v \in \mathrm{L}^2(\Gamma_{\mathrm{S}}, \mathbb{R}^d)$. By considering the function $v = u \pm \varphi\mathrm{n} \in \mathrm{L}^2(\Gamma_{\mathrm{S}}, \mathbb{R}^d)$, for any $\varphi \in \mathrm{L}^2(\Gamma_{\mathrm{S}})$ one gets in Inequality (6.11),

$$\int_{\Gamma_{\mathrm{S}}} \sigma_{\mathrm{n}}(w)\varphi = 0.$$

Therefore $\sigma_{\mathrm{n}}(w) = 0$ *a.e.* on $\Gamma_{\mathrm{S}}$, and Inequality (6.11) becomes

$$\int_{\Gamma_{\mathrm{S}}} \sigma_\tau(w)\cdot(v_\tau - u_\tau) \leq \int_{\Gamma_{\mathrm{S}}} g_0 \|v_\tau\| - \int_{\Gamma_{\mathrm{S}}} g_0 \|u_\tau\|, \quad (6.12)$$

for all $v \in \mathrm{L}^2(\Gamma_{\mathrm{S}}, \mathbb{R}^d)$. Now let $s_0 \in \Gamma_{\mathrm{S}}$ be a Lebesgue point of $\sigma_\tau(w)\cdot u_\tau \in \mathrm{L}^1(\Gamma_{\mathrm{S}})$, $g_0 \in \mathrm{L}^2(\Gamma_{\mathrm{S}})$, $g_0\|u_\tau\| \in \mathrm{L}^1(\Gamma_{\mathrm{S}})$ and of $\sigma_\tau(w)_i \in \mathrm{L}^2(\Gamma_{\mathrm{S}})$, for all $i \in [[1,d]]$, such that $s_0 \in \mathrm{int}_\Gamma(\Gamma_{\mathrm{S}})$. Let us consider the function $v \in \mathrm{L}^2(\Gamma_{\mathrm{S}}, \mathbb{R}^d)$ defined by

$$v := \begin{cases} x & \text{on } B_\Gamma(s_0,\varepsilon), \\ u & \text{on } \Gamma_{\mathrm{S}} \backslash B_\Gamma(s_0,\varepsilon), \end{cases}$$

with $x \in \mathbb{R}^d$ and $\varepsilon > 0$ such that $B_\Gamma(s_0,\varepsilon) \subset \Gamma_{\mathrm{S}}$. Then, from Inequality (6.12), it follows that

$$\frac{1}{|B_\Gamma(s_0,\varepsilon)|}\int_{B_\Gamma(s_0,\varepsilon)} \sigma_\tau(w)\cdot(x_\tau - u_\tau) \leq \frac{1}{|B_\Gamma(s_0,\varepsilon)|}\int_{B_\Gamma(s_0,\varepsilon)} g_0\|x_\tau\| - \frac{1}{|B_\Gamma(s_0,\varepsilon)|}\int_{B_\Gamma(s_0,\varepsilon)} g_0\|u_\tau\|.$$

Since $\mathrm{n} \in \mathcal{C}^0(\Gamma_{\mathrm{S}})$, the map $s \in \Gamma_{\mathrm{S}} \mapsto \|x_{\tau(s)}\| \in \mathbb{R}_+$ is continuous, thus $s_0$ is a Lebesgue point of $g_0\|x_\tau\| \in \mathrm{L}^2(\Gamma_{\mathrm{S}})$, and then $\sigma_\tau(w)(s_0)\cdot(x_{\tau(s_0)} - u_\tau(s_0)) \leq g_0(s_0)\|x_{\tau(s_0)}\| - g_0(s_0)\|u_\tau(s_0)\|$ by letting $\varepsilon \to 0^+$. This inequality is true for any $x \in \mathbb{R}^d$, therefore $\sigma_\tau(w)(s_0) \in g_0(s_0)\partial\|\cdot_{\tau(s_0)}\|(u(s_0))$. Moreover, almost every point of $\Gamma_{\mathrm{S}}$ is in $\mathrm{int}_\Gamma(\Gamma_{\mathrm{S}})$ and is a Lesbegue point of $\sigma_\tau(w)\cdot u_\tau \in \mathrm{L}^1(\Gamma_{\mathrm{S}})$, $g_0 \in \mathrm{L}^2(\Gamma_{\mathrm{S}})$, $g_0\|u_\tau\| \in \mathrm{L}^1(\Gamma_{\mathrm{S}})$ and of $\sigma_\tau(w)_i \in \mathrm{L}^2(\Gamma_{\mathrm{S}})$, for all $i \in [[1,d]]$, hence one deduces that

$$\sigma_\tau(w)(s) \in g_0(s)\partial\left\|\cdot_{\tau(s)}\right\|(u(s)),$$

for almost all $s \in \Gamma_{\mathrm{S}}$. The second inclusion is proved. $\square$

Now let us compute the second-order difference quotient functions of $\Phi$

**Proposition 6.2.2.** *For all $t > 0$, $u \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ and $w \in \partial\Phi(0, \cdot)(u)$, it holds that*

$$\Delta_t^2 \Phi(u|w)(v) = \int_{\Gamma_{\mathrm{S}}} \Delta_t^2 H(s)(u(s)|\sigma_\tau(w)(s))(v(s))\mathrm{d}s,$$

*for all $v \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$, where, for almost all $s \in \Gamma_{\mathrm{S}}$, $\Delta_t^2 H(s)(u(s)|\sigma_\tau(w)(s))$ stands for the second-order difference quotient function of $H(s)$ at $u(s) \in \mathbb{R}^d$ for $\sigma_\tau(w)(s) \in g_0(s)\partial||\cdot_{\tau(s)}||(u(s))$, with $H(s)$ defined by*

$$\begin{array}{rccl} H(s): & \mathbb{R}_+ \times \mathbb{R}^d & \longrightarrow & \mathbb{R} \\ & (t, x) & \longmapsto & H(s)(t, x) := g_t(s) \left\| x_{\tau(s)} \right\|. \end{array}$$

**Remark 6.2.3.** Note that, for almost all $s \in \Gamma_{\mathrm{S}}$ and all $t \geq 0$, $H(s)(t, \cdot) := g_t(s) \left\| \cdot_{\tau(s)} \right\|$ is a proper lower semi-continuous convex function on $\mathbb{R}^d$. Moreover, since $g_0 > 0$ *a.e.* on $\Gamma_{\mathrm{S}}$, it follows that

$$\partial \left[H(s)(0, \cdot)\right](x) = g_0(s)\partial||\cdot_{\tau(s)}||(x),$$

for all $x \in \mathbb{R}^d$ and for almost all $s \in \Gamma_{\mathrm{S}}$.

*Proof of Proposition 6.2.2.* Let $t > 0$, $u \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ and $w \in \partial\Phi(0, \cdot)(u)$. From Lemma 6.2.1 and divergence formula (see Proposition 2.1.1), one deduces that

$$\langle w, v \rangle_{\mathrm{A}} = \int_{\Gamma_{\mathrm{S}}} \sigma_\tau(w) \cdot v,$$

for all $v \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$. Thus it follows that

$$\Delta_t^2 \Phi(u|w)(v) = \int_{\Gamma_{\mathrm{S}}} \frac{g_t(s) \left\| u_\tau(s) + t v_\tau(s) \right\| - g_t(s) \left\| u_\tau(s) \right\| - t\sigma_\tau(w)(s) \cdot v(s)}{t^2} \mathrm{d}s,$$

for all $v \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$. Moreover since $\sigma_\tau(w)(s) \in g_0(s)\partial||\cdot_{\tau(s)}||(u(s))$ for almost all $s \in \Gamma_{\mathrm{S}}$, one deduces that

$$\Delta_t^2 \Phi(u|w)(v) = \int_{\Gamma_{\mathrm{S}}} \Delta_t^2 H(s)(u(s)|\sigma_\tau(w)(s))(v(s))\mathrm{d}s,$$

for all $v \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$, which concludes the proof. $\square$

The twice epi-differentiability of the parameterized Tresca friction functional is related to the twice epi-differentiability of the parameterized function $H(s)$ for almost all $s \in \Gamma_{\mathrm{S}}$. Let us start with the computation of the twice epi-differentiability of the tangential norm map.

**Lemma 6.2.4.** *For all $s \in \Gamma_{\mathrm{S}}$, the map $\left\| \cdot_{\tau(s)} \right\|$ is twice epi-differentiable at any $x \in \mathbb{R}^d$ for any $y \in$*

$\partial\|\cdot_{\tau(s)}\|(x)$, *and its second-order epi-derivative is given by*

$$\mathrm{d}_e^2 \left\|\cdot_{\tau(s)}\right\| (x|y)(z) := \begin{cases} \frac{1}{2\|x_{\tau(s)}\|}\left(\left\|z_{\tau(s)}\right\|^2 - \left|z_{\tau(s)} \cdot \frac{x_{\tau(s)}}{\|x_{\tau(s)}\|}\right|^2\right) & \text{if } x_{\tau(s)} \neq 0, \\ \iota_{\mathrm{N}_{\overline{\mathrm{B}(0,1)}\cap(\mathbb{R}\mathrm{n}(s))^\perp}(y)}(z) & \text{if } x_{\tau(s)} = 0, \end{cases}$$

*for all* $z \in \mathbb{R}^d$*, where* $\mathrm{N}_{\overline{\mathrm{B}(0,1)}\cap(\mathbb{R}\mathrm{n}(s))^\perp}(y)$ *is the normal cone to* $\overline{\mathrm{B}(0,1)} \cap (\mathbb{R}\mathrm{n}(s))^\perp$ *at* $y$*.*

*Proof.* Let $s \in \Gamma_{\mathrm{S}}$. Let us notice that

$$\partial\|\cdot_{\tau(s)}\|(x) := \begin{cases} \left\{\frac{x_{\tau(s)}}{\|x_{\tau(s)}\|}\right\} & \text{if } x_{\tau(s)} \neq 0, \\ \overline{\mathrm{B}(0,1)} \cap (\mathbb{R}\mathrm{n(s)})^\perp & \text{if } x_{\tau(s)} = 0, \end{cases}$$

and that $\left\|\cdot_{\tau(s)}\right\| = h_{\overline{\mathrm{B}(0,1)}\cap(\mathbb{R}\mathrm{n(s)})^\perp}$, where $h_{\overline{\mathrm{B}(0,1)}\cap(\mathbb{R}\mathrm{n(s)})^\perp}$ is the support function of $\overline{\mathrm{B}(0,1)} \cap (\mathbb{R}\mathrm{n(s)})^\perp$ which is a nonempty convex closed subset of $\mathbb{R}^d$. Moreover since

$$(\overline{\mathrm{B}(0,1)} \cap (\mathbb{R}\mathrm{n(s)})^\perp)^\perp = \mathbb{R}\mathrm{n}(s),$$

one can apply Proposition 3.2.8 to get, for all $x \in \mathbb{R}\mathrm{n}(s)$ and for all $y \in \overline{\mathrm{B}(0,1)} \cap (\mathbb{R}\mathrm{n}(s))^\perp$,

$$\mathrm{d}_e^2 \left\|\cdot_{\tau(s)}\right\| (x|y) = \iota_{\mathrm{N}_{\overline{\mathrm{B}(0,1)}\cap(\mathbb{R}\mathrm{n}(s))^\perp}(y)}.$$

In the case where $x \notin \mathbb{R}\mathrm{n}(s)$ (i.e. $x_{\tau(s)} \neq 0$) then $\left\|\cdot_{\tau(s)}\right\|$ is twice Fréchet differentiable at $x$ with its twice Fréchet differential given by

$$\mathrm{D}^2 \left\|\cdot_{\tau(s)}\right\| (x)(z_1, z_2) = \frac{1}{\left\|x_{\tau(s)}\right\|}\left(z_{1_{\tau(s)}} \cdot z_{2_{\tau(s)}} - \left(x_{\tau(s)} \cdot z_{2_{\tau(s)}}\right) z_{1_{\tau(s)}} \cdot \frac{x_{\tau(s)}}{\left\|x_{\tau(s)}\right\|^2}\right), \qquad \forall\, (z_1, z_2) \in \mathbb{R}^d \times \mathbb{R}^d.$$

Thus one gets (see Remark 3.2.6),

$$\mathrm{d}_e^2 \left\|\cdot_{\tau(s)}\right\| (x|\frac{x_{\tau(s)}}{\left\|x_{\tau(s)}\right\|})(z) = \frac{1}{2}\mathrm{D}^2 \left\|\cdot_{\tau(s)}\right\| (x)(z,z) = \frac{1}{2\left\|x_{\tau(s)}\right\|}\left(\left\|z_{\tau(s)}\right\|^2 - \left|z_{\tau(s)} \cdot \frac{x_{\tau(s)}}{\left\|x_{\tau(s)}\right\|}\right|^2\right),$$

for all $z \in \mathbb{R}^d$, which concludes the proof. □

**Proposition 6.2.5.** *Assume that, for almost all* $s \in \Gamma_{\mathrm{S}}$*, the map* $t \in \mathbb{R}_+ \mapsto g_t(s) \in \mathbb{R}_+$ *is differentiable at* $t = 0$*, with its derivative denoted by* $g_0'(s)$*. Then, for almost all* $s \in \Gamma_{\mathrm{S}}$*, the map* $H(s)$ *is twice epi-differentiable at any* $x \in \mathbb{R}^d$ *for all* $y \in g_0(s)\partial||\cdot_{\tau(s)}||(x)$ *with*

$$\mathrm{D}_e^2 H(s)(x|y)(z) := \begin{cases} \frac{g_0(s)}{2\|x_{\tau(s)}\|}\left(\left\|z_{\tau(s)}\right\|^2 - \left|z_{\tau(s)} \cdot \frac{x_{\tau(s)}}{\|x_{\tau(s)}\|}\right|^2\right) + g_0'(s)\frac{x_{\tau(s)}}{\|x_{\tau(s)}\|} \cdot z & \text{if } x_{\tau(s)} \neq 0, \\ \iota_{\mathrm{N}_{\overline{\mathrm{B}(0,1)}\cap(\mathbb{R}\mathrm{n}(s))^\perp}(\frac{y}{g_0(s)})}(z) + g_0'(s)\frac{y}{g_0(s)} \cdot z & \text{if } x_{\tau(s)} = 0, \end{cases}$$

*for all* $z \in \mathbb{R}^d$.

*Proof.* We use the same notations as in Definitions 3.2.5 and 3.2.11. Let $x \in \mathbb{R}^d$. Then, for almost all $s \in \Gamma_{\mathrm{S}}$, for all $y \in g_0(s)\partial||\cdot_{\tau(s)}||(x)$ and all $z \in \mathbb{R}^d$, one has

$$\Delta_t^2 H(s)(x|y)(z) = \frac{g_t(s)\left\|x_{\tau(s)} + tz_{\tau(s)}\right\| - g_t(s)\left\|x_{\tau(s)}\right\| - ty\cdot z}{t^2}$$
$$= g_t(s)\frac{\left\|x_{\tau(s)} + tz_{\tau(s)}\right\| - \left\|x_{\tau(s)}\right\| - t\frac{y}{g_0(s)}\cdot z}{t^2} + \frac{(g_t(s) - g_0(s))}{tg_0(s)}y\cdot z,$$

that is

$$\Delta_t^2 H(s)(x|y)(z) = g_t(s)\delta_t^2||\cdot_{\tau(s)}||\left(x|\frac{y}{g_0(s)}\right)(z) + \frac{(g_t(s) - g_0(s))}{tg_0(s)}y\cdot z,$$

with $\frac{y}{g_0(s)} \in \partial||\cdot_{\tau(s)}||(x)$, and where $\delta_t^2||\cdot_{\tau(s)}||(x|\frac{y}{g_0(s)})$ is the second-order difference quotient function of $||\cdot_{\tau(s)}||$ at $x$ for $\frac{y}{g_0(s)}$ (see Definition 3.2.5 since $||\cdot_{\tau(s)}||$ is $t$-independent function). Using the characterization of Mosco epi-convergence (see Proposition 3.2.4) and Lemma 6.2.4, one gets

$$\mathrm{D}_e^2 H(s)(x|y)(z) = g_0(s)\mathrm{d}_e^2\left\|\cdot_{\tau(s)}\right\|(x|\frac{y}{g_0(s)}) + g_0'(s)\frac{y}{g_0(s)}\cdot z,$$

which concludes the proof. □

Before to characterize the derivative of the map $t \in \mathbb{R}_+ \mapsto u_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ at $t = 0$, let us detail $\mathrm{N}_{\overline{\mathrm{B}(0,1)}\cap(\mathbb{R}\mathrm{n}(s))^\perp}(y)$ the normal cone to $\overline{\mathrm{B}(0,1)} \cap (\mathbb{R}\mathrm{n}(s))^\perp$ at $y$, for almost all $s \in \Gamma_{\mathrm{S}}$.

**Proposition 6.2.6.** *Let* $s \in \Gamma_{\mathrm{S}}$. *Then for all* $y \in \overline{\mathrm{B}(0,1)} \cap (\mathbb{R}\mathrm{n}(s))^\perp$,

$$\mathrm{N}_{\overline{\mathrm{B}(0,1)}\cap(\mathbb{R}\mathrm{n}(s))^\perp}(y) = \begin{cases} \mathbb{R}\mathrm{n}(s) & \text{if } y \in \mathrm{B}(0,1)\cap(\mathbb{R}\mathrm{n}(s))^\perp, \\ \mathbb{R}\mathrm{n}(s) + \mathbb{R}_+ y & \text{if } y \in \partial\mathrm{B}(0,1)\cap(\mathbb{R}\mathrm{n}(s))^\perp. \end{cases}$$

*Proof.* Let $s \in \Gamma_{\mathrm{S}}$ and $y \in \overline{\mathrm{B}(0,1)} \cap (\mathbb{R}\mathrm{n}(s))^\perp$.

(i) First, let $y$ be in $\mathrm{B}(0,1) \cap (\mathbb{R}\mathrm{n}(s))^\perp$. If $v \in \mathbb{R}\mathrm{n}(s)$, then

$$v\cdot(y - z) = 0, \qquad \forall z \in \overline{\mathrm{B}(0,1)} \cap (\mathbb{R}\mathrm{n}(s))^\perp,$$

thus $v \in \mathrm{N}_{\overline{\mathrm{B}(0,1)}\cap(\mathbb{R}\mathrm{n}(s))^\perp}(y)$. Since this is true for any $v \in \mathbb{R}\mathrm{n}(s)$ one deduces that

$$\mathbb{R}\mathrm{n}(s) \subset \mathrm{N}_{\overline{\mathrm{B}(0,1)}\cap(\mathbb{R}\mathrm{n}(s))^\perp}(y).$$

Let us consider $v \in \mathrm{N}_{\overline{\mathrm{B}(0,1)}\cap(\mathbb{R}\mathrm{n}(s))^\perp}(y)$, then

$$v\cdot(z - y) \leq 0, \qquad \forall z \in \overline{\mathrm{B}(0,1)} \cap (\mathbb{R}\mathrm{n}(s))^\perp.$$

Moreover it exists $\varepsilon > 0$ such that $\mathrm{B}(y,\varepsilon) \cap (\mathbb{R}\mathrm{n}(s))^\perp \subset \mathrm{B}(0,1) \cap (\mathbb{R}\mathrm{n}(s))^\perp$. Therefore by considering $z = y + \varepsilon \frac{w}{2\|w\|}$, for all $w \in (\mathbb{R}\mathrm{n}(s))^{\perp}$ one deduces that

$$v \cdot w = 0, \qquad \forall w \in (\mathbb{R}\mathrm{n}(s))^{\perp}.$$

Thus $v \in ((\mathbb{R}\mathrm{n}(s))^{\perp})^{\perp} = \mathbb{R}\mathrm{n}(s)$. This is true for any $v \in \mathrm{N}_{\overline{\mathrm{B}(0,1)} \cap (\mathbb{R}\mathrm{n}(s))^{\perp}}(y)$, therefore

$$\mathrm{N}_{\overline{\mathrm{B}(0,1)} \cap (\mathbb{R}\mathrm{n}(s))^{\perp}}(y) \subset \mathbb{R}\mathrm{n}(s).$$

(ii) Let $y \in \partial \mathrm{B}(0,1) \cap (\mathbb{R}\mathrm{n}(s))^{\perp}$. If $v \in \mathbb{R}\mathrm{n}(s) + \mathbb{R}_+ y$ then

$$v \cdot (z-y) = v_{\tau(s)} \cdot (z-y) \leq \left\|v_{\tau(s)}\right\| \|z\| - v_{\tau(s)} \cdot y \leq \left\|v_{\tau(s)}\right\| - \left\|v_{\tau(s)}\right\| = 0,$$

for all $z \in \overline{\mathrm{B}(0,1)} \cap (\mathbb{R}\mathrm{n}(s))^{\perp}$. Thus it follows that

$$\mathbb{R}\mathrm{n}(s) + \mathbb{R}_+ y \subset \mathrm{N}_{\overline{\mathrm{B}(0,1)} \cap (\mathbb{R}\mathrm{n}(s))^{\perp}}(y).$$

Let us consider $v \in \mathrm{N}_{\overline{\mathrm{B}(0,1)} \cap (\mathbb{R}\mathrm{n}(s))^{\perp}}(y)$, then by considering $z = \frac{1}{2}\left(\frac{v_{\tau(s)}}{\left\|v_{\tau(s)}\right\|}\|y\| + y\right) \in \overline{\mathrm{B}(0,1)} \cap (\mathbb{R}\mathrm{n}(s))^{\perp}$ one deduces that

$$0 \geq v \cdot (z-y) = v_{\tau(s)} \cdot \frac{1}{2}\left(\frac{v_{\tau(s)}}{\left\|v_{\tau(s)}\right\|}\|y\| - y\right) = \frac{1}{2}\left(\left\|v_{\tau(s)}\right\| \|y\| - v_{\tau(s)} \cdot y\right) \geq 0,$$

thus $\left\|v_{\tau(s)}\right\| \|y\| = v_{\tau(s)} \cdot y$, hence $v_{\tau(s)} \in \mathbb{R}_+ y$. Since this is true for any $v \in \mathrm{N}_{\overline{\mathrm{B}(0,1)} \cap (\mathbb{R}\mathrm{n}(s))^{\perp}}(y)$, one deduces

$$\mathrm{N}_{\overline{\mathrm{B}(0,1)} \cap (\mathbb{R}\mathrm{n}(s))^{\perp}}(y) \subset \mathbb{R}\mathrm{n}(s) + \mathbb{R}_+ y.$$

The proof is complete. □

We can now prove the main result of this subsection.

**Theorem 6.2.7.** *Let $u_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ be the unique solution to the parameterized general Tresca problem* ($\mathrm{GTP}_t$)*, for all $t \geq 0$. Assume that:*

1. *the map $t \in \mathbb{R}_+ \mapsto h_t \in \mathrm{L}^2(\Gamma_{\mathrm{S}})$ is differentiable at $t=0$, with its derivative denoted by $h_0' \in \mathrm{L}^2(\Gamma_{\mathrm{S}})$;*
2. *for almost all $s \in \Gamma_{\mathrm{S}}$, the map $t \in \mathbb{R}_+ \mapsto g_t(s) \in \mathbb{R}_+$ is differentiable at $t=0$, with its derivative denoted by $g_0'(s)$, and also $g_0' \in \mathrm{L}^2(\Gamma_{\mathrm{S}})$;*
3. *the map $s \in \Gamma_{\mathrm{S}_{\mathrm{R}}^{u_0,g_0}} \mapsto \frac{g_0(s)}{\|u_{0_\tau}(s)\|} \in \mathbb{R}_+^*$ is in $\mathrm{L}^4(\Gamma_{\mathrm{S}_{\mathrm{R}}^{u_0,g_0}})$ (see below for the set $\Gamma_{\mathrm{S}_{\mathrm{R}}^{u_0,g_0}}$);*
4. *the parameterized Tresca friction functional $\Phi$ defined in* (6.8) *is twice epi-differentiable at $u_0$ for $F_0 - u_0 \in \partial\Phi(0,\cdot)(u_0)$, with*

$$\mathrm{D}_e^2\Phi(u_0|F_0-u_0)(v) = \int_{\Gamma_{\mathrm{S}}} \mathrm{D}_e^2 H(s)(u_0(s)|\sigma_\tau(F_0-u_0)(s))(v(s))\mathrm{d}s,$$

*for all $v \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$, where $F_0 \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ is the unique solution to the parameterized general Dirichlet-Neumann problem* (6.9) *for the parameter $t=0$.*

*Then the map $t \in \mathbb{R}_+ \mapsto u_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ is differentiable at $t=0$, and its derivative denoted by $u_0' \in \mathcal{K}_{u_0,\frac{\sigma_\tau(F_0-u_0)}{g_0}} \subset \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ is the unique solution to the variational inequality*

$$\begin{aligned}\langle u_0', v-u_0'\rangle_{\mathrm{A}} \geq \int_\Omega f_0'\cdot(v-u_0') + \int_{\Gamma_{\mathrm{S}}} h_0'\left(v_{\mathrm{n}}-u'_{0\mathrm{n}}\right) + \int_{\Gamma_{\mathrm{S}_{\mathrm{S}}^{u_0,g_0}}} g_0'\frac{\sigma_\tau(u_0)}{g_0}\cdot\left(v_\tau - u'_{0_\tau}\right)\\ - \int_\Omega \left(\mathrm{A}\left[\nabla u_0\right]\mathrm{B}_0'^{\top} + \mathrm{A}\left[\nabla u_0 \mathrm{B}_0'\right] + z_0'\mathrm{A}\nabla u_0\right) : \nabla(v-u_0')\\ + \int_{\Gamma_{\mathrm{S}_{\mathrm{R}}^{u_0,g_0}}} \left(-g_0'\frac{u_{0_\tau}}{\|u_{0_\tau}\|} - \frac{g_0}{\|u_{0_\tau}\|}\left(u'_{0_\tau} - \left(u'_{0_\tau}\cdot\frac{u_{0_\tau}}{\|u_{0_\tau}\|}\right)\frac{u_{0_\tau}}{\|u_{0_\tau}\|}\right)\right)\cdot\left(v_\tau-u'_{0_\tau}\right), \quad (6.13)\end{aligned}$$

*for all $v \in \mathcal{K}_{u_0,\frac{\sigma_\tau(F_0-u_0)}{g_0}}$, where $\mathcal{K}_{u_0,\frac{\sigma_\tau(F_0-u_0)}{g_0}}$ is the nonempty closed convex subset of $\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ defined by*

$$\mathcal{K}_{u_0,\frac{\sigma_\tau(F_0-u_0)}{g_0}} := \left\{ v \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d) \mid v_\tau = 0 \text{ a.e. on } \Gamma_{\mathrm{S}_{\mathrm{D}}^{u_0,g_0}} \text{ and } v_\tau \in \mathbb{R}_-\frac{\sigma_\tau(u_0)}{g_0} \text{ a.e. on } \Gamma_{\mathrm{S}_{\mathrm{S}}^{u_0,g_0}} \right\},$$

*and where $\Gamma_{\mathrm{S}}$ is decomposed, up to a null set, as $\Gamma_{\mathrm{S}_{\mathrm{R}}^{u_0,g_0}} \cup \Gamma_{\mathrm{S}_{\mathrm{D}}^{u_0,g_0}} \cup \Gamma_{\mathrm{S}_{\mathrm{S}}^{u_0,g_0}}$ with*

$$\begin{aligned}\Gamma_{\mathrm{S}_{\mathrm{R}}^{u_0,g_0}} &:= \left\{ s\in\Gamma_{\mathrm{S}} \mid u_{0_\tau}(s)\neq 0\right\},\\ \Gamma_{\mathrm{S}_{\mathrm{D}}^{u_0,g_0}} &:= \left\{ s\in\Gamma_{\mathrm{S}} \mid u_{0_\tau}(s)= 0 \text{ and } \tfrac{\sigma_\tau(u_0)(s)}{g_0(s)} \in \mathrm{B}(0,1)\cap(\mathbb{R}\mathrm{n}(s))^\perp\right\},\\ \Gamma_{\mathrm{S}_{\mathrm{S}}^{u_0,g_0}} &:= \left\{ s\in\Gamma_{\mathrm{S}} \mid u_{0_\tau}(s)= 0 \text{ and } \tfrac{\sigma_\tau(u_0)(s)}{g_0(s)} \in \partial\mathrm{B}(0,1)\cap(\mathbb{R}\mathrm{n}(s))^\perp\right\}.\end{aligned}$$

*Proof.* From Hypothesis (H4), (H5), (H7) and that the map $t\in\mathbb{R}_+ \mapsto h_t \in \mathrm{L}^2(\Gamma_{\mathrm{S}})$ is differentiable at $t=0$, one can prove similarly to Lemma 6.1.1, that the map $t\in\mathbb{R}_+\mapsto E_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ is differentiable at $t=0$ and its derivative, denoted by $E_0' \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$, is the unique solution to the variational equality

$$\langle E_0', v\rangle_{\mathrm{A}} = \int_\Omega f_0'\cdot v + \int_{\Gamma_{\mathrm{S}}} h_0' v_{\mathrm{n}} - \int_\Omega \left(\mathrm{A}\left[\nabla u_0\right]\mathrm{B}_0'^{\top} + \mathrm{A}\left[\nabla u_0\mathrm{B}_0'\right] + z_0'\mathrm{A}\nabla u_0\right) : \nabla v,$$

for all $v\in\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$. Moreover from Hypothesis 2, 4 and Proposition 6.2.5, it follows that

$$\begin{aligned}\mathrm{D}_e^2\Phi(u_0|F_0-u_0)(v) = \int_{\Gamma_{\mathrm{S}_{\mathrm{R}}^{u_0,g_0}}} \left(\frac{g_0}{2\|u_{0_\tau}\|}\left(\|v_\tau\|^2 - \left|v_\tau\cdot\frac{u_{0_\tau}}{\|u_{0_\tau}\|}\right|^2\right) + g_0'\frac{u_{0_\tau}}{\|u_{0_\tau}\|}\cdot v\right) +\\ \int_{\Gamma_{\mathrm{S}}\backslash\Gamma_{\mathrm{S}_{\mathrm{R}}^{u_0,g_0}}} \iota_{\mathrm{N}_{\overline{\mathrm{B}(0,1)}\cap(\mathbb{R}\mathrm{n}(s))^\perp}\left(\frac{\sigma_\tau(F_0-u_0)(s)}{g_0(s)}\right)}(v(s))\mathrm{d}s + \int_{\Gamma_{\mathrm{S}}\backslash\Gamma_{\mathrm{S}_{\mathrm{R}}^{u_0,g_0}}} g_0'\frac{\sigma_\tau(F_0-u_0)}{g_0}\cdot v,\end{aligned}$$

for all $v \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$. This is also

$$\mathrm{D}^2_e\Phi(u_0|F_0-u_0)(v) = \Psi(v)+\int_{\Gamma_{\mathrm{S}_{\mathrm{R}}^{u_0,g_0}}} g_0' \frac{u_{0_\tau}}{\|u_{0_\tau}\|}\cdot v_\tau + \iota_{\mathcal{K}_{u_0,\frac{\sigma_\tau(F_0-u_0)}{g_0}}}(v)+\int_{\Gamma_{\mathrm{S}}\backslash\Gamma_{\mathrm{S}_{\mathrm{R}}^{u_0,g_0}}} g_0' \frac{\sigma_\tau(F_0-u_0)}{g_0}\cdot v_\tau,$$

for all $v \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$, where $\Psi$ is defined by

$$\begin{array}{rccl} \Psi : & \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d) & \longrightarrow & \mathbb{R} \\ & v & \longmapsto & \Psi(v) := \displaystyle\int_{\Gamma_{\mathrm{S}_{\mathrm{R}}^{u_0,g_0}}} \frac{g_0}{2\,\|u_{0_\tau}\|}\left(\|v_\tau\|^2 - \left|v_\tau \cdot \frac{u_{0_\tau}}{\|u_{0_\tau}\|}\right|^2\right), \end{array}$$

which is well defined from Hypothesis 3 and from the continuous embedding $\mathrm{H}^1(\Omega,\mathbb{R}^d)\hookrightarrow\mathrm{L}^4(\Gamma,\mathbb{R}^d)$ (see Proposition 2.1.7), and where $\mathcal{K}_{u_0,\frac{\sigma_\tau(F_0-u_0)}{g_0}}$ is the nonempty closed convex subset of $\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ defined by

$$\mathcal{K}_{u_0,\frac{\sigma_\tau(F_0-u_0)}{g_0}} := \left\{ v \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d) \mid v(s) \in \mathrm{N}_{\overline{\mathrm{B}(0,1)}\cap(\mathbb{R}\mathrm{n}(s))^\perp}\left(\frac{\sigma_\tau(F_0-u_0)(s)}{g_0(s)}\right) \right. \\ \left. \text{for almost all } s \in \Gamma_{\mathrm{S}}\backslash\Gamma_{\mathrm{S}_{\mathrm{R}}^{u_0,g_0}} \right\}.$$

Moreover, since $F_0 \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ is the solution to the parameterized general Dirichlet-Neumann problem (6.9) for the parameter $t=0$, then $\sigma_\tau(F_0)=0$ *a.e.* on $\Gamma_{\mathrm{S}}$, thus it follows that

$$\mathcal{K}_{u_0,\frac{\sigma_\tau(F_0-u_0)}{g_0}} = \left\{ v \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d) \mid v_\tau = 0 \text{ } a.e. \text{ on } \Gamma_{\mathrm{S}_{\mathrm{D}}^{u_0,g_0}} \text{ and } v_\tau \in \mathbb{R}_- \frac{\sigma_\tau(u_0)}{g_0} \text{ } a.e. \text{ on } \Gamma_{\mathrm{S}_{\mathrm{S}}^{u_0,g_0}} \right\}.$$

Since $\frac{g_0}{||u_{0_\tau}||} > 0$ *a.e.* on $\Gamma_{\mathrm{S}_{\mathrm{R}}^{u_0,g_0}}$ and Hypothesis 3, then from Lemma 4.2.17 one deduces that $\Psi$ is convex and Fréchet differentiable on $\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ endowed with the scalar product $\langle\cdot,\cdot\rangle_{\mathrm{A}}$, thus $\mathrm{D}^2_e\Phi(u_0|F_0-u_0)$ is a proper lower semi-continuous convex function on $\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$. Therefore the assumptions of Proposition 3.2.15 are satisfied and we deduce that the map $t \in \mathbb{R}_+ \mapsto u_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ is differentiable at $t=0$, and its derivative $u_0' \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ satisfies

$$u_0' = \mathrm{prox}_{\mathrm{D}^2_e\Phi(u_0|F_0-u_0)}(E_0'),$$

which, from the definition of the proximal operator leads to

$$E_0' - u_0' \in \partial\mathrm{D}^2_e\Phi(u_0|F_0-u_0)(u_0'),$$

which means that

$$\langle E_0'-u_0', v-u_0'\rangle_{\mathrm{A}} \leq \mathrm{D}^2_e\Phi(u_0|F_0-u_0)(v) - \mathrm{D}^2_e\Phi(u_0|F_0-u_0)(u_0'),$$

for all $v \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$. Hence we get

$$\langle E_0' - u_0', v - u_0' \rangle_{\mathrm{A}} \leq \Psi(v) - \Psi(u_0') + \iota_{\mathcal{K}_{u_0, \frac{\sigma_\tau(F_0 - u_0)}{g_0}}}(v) - \iota_{\mathcal{K}_{u_0, \frac{\sigma_\tau(F_0 - u_0)}{g_0}}}(u_0') + \int_{\Gamma_{\mathrm{S}_{\mathrm{R}}^{u_0, g_0}}} g_0' \frac{u_{0_\tau}}{\|u_{0_\tau}\|} \cdot \left(v_\tau - u_{0_\tau}'\right) + \int_{\Gamma_{\mathrm{S}} \backslash \Gamma_{\mathrm{S}_{\mathrm{R}}^{u_0, g_0}}} g_0' \frac{\sigma_\tau(F_0 - u_0)}{g_0} \cdot \left(v_\tau - u_{0_\tau}'\right),$$

for all $v \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$. Thus $u_0' \in \mathcal{K}_{u_0, \frac{\sigma_\tau(F_0 - u_0)}{g_0}}$ and since, for all $v \in \mathcal{K}_{u_0, \frac{\sigma_\tau(F_0 - u_0)}{g_0}}$, $v_\tau = 0$ *a.e.* on $\Gamma_{\mathrm{S}_{\mathrm{D}}^{u_0, g_0}}$, one deduces that

$$\langle u_0', v - u_0' \rangle_{\mathrm{A}} + \Psi(v) - \Psi(u_0') \geq \int_\Omega f_0' \cdot (v - u_0') + \int_{\Gamma_{\mathrm{S}}} h_0' \left(v_{\mathrm{n}} - u_{0\mathrm{n}}'\right) + \int_{\Gamma_{\mathrm{S}_{\mathrm{S}}^{u_0, g_0}}} g_0' \frac{\sigma_\tau(u_0)}{g_0} \cdot \left(v_\tau - u_{0_\tau}'\right) - \int_\Omega \left(\mathrm{A}\left[\nabla u_0\right] \mathrm{B}_0'^{\top} + \mathrm{A}\left[\nabla u_0 \mathrm{B}_0'\right] + z_0' \mathrm{A} \nabla u_0\right) : \nabla(v - u_0') - \int_{\Gamma_{\mathrm{S}_{\mathrm{R}}^{u_0, g_0}}} g_0' \frac{u_{0_\tau}}{\|u_{0_\tau}\|} \cdot \left(v_\tau - u_{0_\tau}'\right),$$

for all $v \in \mathcal{K}_{u_0, \frac{\sigma_\tau(F_0 - u_0)}{g_0}}$. Moreover $\nabla\Psi(u_0') \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ is the unique solution to the Dirichlet-Neumann problem

$$\begin{cases} -\mathrm{div}\left(\mathrm{Ae}(\nabla\Psi(u_0'))\right) = 0 & \text{in } \Omega, \\ \nabla\Psi(u_0') = 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \mathrm{Ae}(\nabla\Psi(u_0'))\mathrm{n} = 0 & \text{on } \Gamma_{\mathrm{S}_{\mathrm{D}}^{u_0, g_0}} \cup \Gamma_{\mathrm{S}_{\mathrm{S}}^{u_0, g_0}}, \\ \sigma_{\mathrm{n}}(\nabla\Psi(u_0')) = 0 & \text{on } \Gamma_{\mathrm{S}_{\mathrm{R}}^{u_0, g_0}}, \\ \sigma_\tau(\nabla\Psi(u_0')) = \frac{g_0}{\|u_{0_\tau}\|}\left(u_{0_\tau}' - \left(u_{0_\tau}' \cdot \frac{u_{0_\tau}}{\|u_{0_\tau}\|}\right)\frac{u_{0_\tau}}{\|u_{0_\tau}\|}\right) & \text{on } \Gamma_{\mathrm{S}_{\mathrm{R}}^{u_0, g_0}}. \end{cases}$$

Thus one gets from Proposition 3.1.7,

$$\langle \nabla\Psi(u_0'), v - u_0' \rangle_{\mathrm{A}} \geq -\langle u_0', v - u_0' \rangle_{\mathrm{A}} + \int_\Omega f_0' \cdot (v - u_0') + \int_{\Gamma_{\mathrm{S}}} h_0' \left(v_{\mathrm{n}} - u_{0\mathrm{n}}'\right) + \int_{\Gamma_{\mathrm{S}_{\mathrm{S}}^{u_0, g_0}}} g_0' \frac{\sigma_\tau(u_0)}{g_0} \cdot \left(v_\tau - u_{0_\tau}'\right) - \int_\Omega \left(\mathrm{A}\left[\nabla u_0\right] \mathrm{B}_0'^{\top} + \mathrm{A}\left[\nabla u_0 \mathrm{B}_0'\right] + z_0' \mathrm{A} \nabla u_0\right) : \nabla(v - u_0') - \int_{\Gamma_{\mathrm{S}_{\mathrm{R}}^{u_0, g_0}}} g_0' \frac{u_{0_\tau}}{\|u_{0_\tau}\|} \cdot \left(v_\tau - u_{0_\tau}'\right),$$

for all $v \in \mathcal{K}_{u_0, \frac{\sigma_\tau(F_0 - u_0)}{g_0}}$. Finally using the variational formulation satisfied by $\nabla\Psi(u_0') \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$, it follows that

$$\langle u_0', v - u_0' \rangle_{\mathrm{A}} \geq \int_\Omega f_0' \cdot (v - u_0') + \int_{\Gamma_{\mathrm{S}}} h_0' \left(v_{\mathrm{n}} - u_{0\mathrm{n}}'\right) + \int_{\Gamma_{\mathrm{S}_{\mathrm{S}}^{u_0, g_0}}} g_0' \frac{\sigma_\tau(u_0)}{g_0} \cdot \left(v_\tau - u_{0_\tau}'\right) - \int_\Omega \left(\mathrm{A}\left[\nabla u_0\right] \mathrm{B}_0'^{\top} + \mathrm{A}\left[\nabla u_0 \mathrm{B}_0'\right] + z_0' \mathrm{A} \nabla u_0\right) : \nabla(v - u_0') + \int_{\Gamma_{\mathrm{S}_{\mathrm{R}}^{u_0, g_0}}} \left(-g_0' \frac{u_{0_\tau}}{\|u_{0_\tau}\|} - \frac{g_0}{\|u_{0_\tau}\|}\left(u_{0_\tau}' - \left(u_{0_\tau}' \cdot \frac{u_{0_\tau}}{\|u_{0_\tau}\|}\right)\frac{u_{0_\tau}}{\|u_{0_\tau}\|}\right)\right) \cdot \left(v_\tau - u_{0_\tau}'\right),$$

for all $v \in \mathcal{K}_{u_0, \frac{\sigma_\tau(F_0-u_0)}{g_0}}$. The proof is complete. □

With some additional assumptions one can characterize $u_0' \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ as weak solution to a tangential Signorini problem.

**Corollary 6.2.8.** *Consider the framework of Theorem 6.2.7, and assume that* $\mathrm{div}(\mathrm{A}[\nabla u_0]\mathrm{B}_0'^\top + \mathrm{A}[\nabla u_0 \mathrm{B}_0'] + z_0'\mathrm{A}\nabla u_0) \in \mathrm{L}^2(\Omega, \mathbb{R}^d)$ *and* $(\mathrm{A}[\nabla u_0]\mathrm{B}_0'^\top + \mathrm{A}[\nabla u_0 \mathrm{B}_0'] + z_0'\mathrm{A}\nabla u_0)\mathrm{n} \in \mathrm{L}^2(\Gamma, \mathbb{R}^d)$. *Then* $u_0' \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ *is the unique weak solution to the tangential Signorini problem*

$$\left\{\begin{array}{rcll} -\mathrm{div}(\mathrm{Ae}(u_0')) &=& \tilde{f} & \textit{in } \Omega, \\ u_0' &=& 0 & \textit{on } \Gamma_{\mathrm{D}}, \\ \sigma_{\mathrm{n}}(u_0') &=& h_0' + \xi_{\mathrm{n}} & \textit{on } \Gamma_{\mathrm{S}}, \\ \sigma_\tau(u_0') + \frac{g_0}{\|u_{0\tau}\|}\left(u_{0_\tau}' - \left(u_{0_\tau}' \cdot \frac{u_{0\tau}}{\|u_{0\tau}\|}\right)\frac{u_{0\tau}}{\|u_{0\tau}\|}\right) &=& -g_0'\frac{u_{0\tau}}{\|u_{0\tau}\|} + \xi_\tau & \textit{on } \Gamma_{\mathrm{S}_{\mathrm{R}}^{u_0,g_0}}, \\ u_{0_\tau}' &=& 0 & \textit{on } \Gamma_{\mathrm{S}_{\mathrm{D}}^{u_0,g_0}}, \\ u_{0_\tau}' \in \mathbb{R}_-\frac{\sigma_\tau(u_0)}{g_0}, \left(\sigma_\tau(u_0') - g_0'\frac{\sigma_\tau(u_0)}{g_0} - \xi_\tau\right)\cdot\frac{\sigma_\tau(u_0)}{g_0} \leq 0 \textit{ and} && & \\ u_{0_\tau}' \cdot \left(\sigma_\tau(u_0') - g_0'\frac{\sigma_\tau(u_0)}{g_0} - \xi_\tau\right) &=& 0 & \textit{on } \Gamma_{\mathrm{S}_{\mathrm{S}}^{u_0,g_0}}, \end{array}\right.$$

*where*

$$\tilde{f} := f_0' + \mathrm{div}\left(\mathrm{A}[\nabla u_0]\mathrm{B}_0'^\top + \mathrm{A}[\nabla u_0 \mathrm{B}_0'] + z_0'\mathrm{A}\nabla u_0\right) \in \mathrm{L}^2(\Omega, \mathbb{R}^d),$$

*and*

$$\xi := -\left(\mathrm{A}[\nabla u_0]\mathrm{B}_0'^\top + \mathrm{A}[\nabla u_0 \mathrm{B}_0'] + z_0'\mathrm{A}\nabla u_0\right)\mathrm{n} \in \mathrm{L}^2(\Gamma, \mathbb{R}^d).$$

*Proof.* From the assumption $\mathrm{div}(\mathrm{A}[\nabla u_0]\mathrm{B}_0'^\top + \mathrm{A}[\nabla u_0 \mathrm{B}_0'] + z_0'\mathrm{A}\nabla u_0) \in \mathrm{L}^2(\Omega, \mathbb{R}^d)$, one can apply the divergence formula in Inequality (6.13), and since $(\mathrm{A}[\nabla u_0]\mathrm{B}_0'^\top + \mathrm{A}[\nabla u_0 \mathrm{B}_0'] + z_0'\mathrm{A}\nabla u_0)\mathrm{n} \in \mathrm{L}^2(\Gamma, \mathbb{R}^d)$, this concludes the proof from Subsection 4.2.3. □

**Remark 6.2.9.** Note that the assumptions in Corollary 6.2.8 are, for instance, satisfied if $u_0 \in \mathrm{H}^2(\Omega, \mathbb{R}^d)$, $\mathrm{B}_0' \in \mathrm{W}^{1,\infty}(\Omega, \mathbb{R}^d)$ and $z_0' \in \mathrm{W}^{1,\infty}(\Omega)$, or if $\mathrm{B}_0' = 0$ and $z_0' = 0$ *a.e.* on $\Omega$.

### 6.2.2 With perturbation of the unit normal vector to the boundary in the two-dimensional case

In this subsection, the framework is considered in the two-dimensional case $d = 2$. Thus if $\tau$ is the orthonormal vector to n, i.e. $\tau \cdot \mathrm{n} = 0$ on $\Gamma$, then $\mathrm{M}_t^{-1^\top}\tau \cdot \mathrm{M}_t\mathrm{n} = 0$ on $\Gamma$. Let us consider the parameterized general Tresca friction problem given by

$$\left\{\begin{array}{rcll} -\mathrm{div}\left(z_t\mathrm{A}[\nabla u_t \mathrm{B}_t]\mathrm{B}_t^\top\right) &=& f_t & \text{in } \Omega, \\ u_t &=& 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \left(z_t\mathrm{A}[\nabla u_t \mathrm{B}_t]\mathrm{B}_t^\top\right)\mathrm{n} \cdot \frac{\mathrm{M}_t\mathrm{n}}{\|\mathrm{M}_t\mathrm{n}\|^2} &=& h_t & \text{on } \Gamma_{\mathrm{S}}, \\ \left|\left(z_t\mathrm{A}[\nabla u_t \mathrm{B}_t]\mathrm{B}_t^\top\right)\mathrm{n} \cdot \frac{\mathrm{M}_t^{-1^\top}\tau}{||\mathrm{M}_t^{-1^\top}\tau||^2}\right| \leq g_t \text{ and} && & \\ \left(\left(z_t\mathrm{A}[\nabla u_t \mathrm{B}_t]\mathrm{B}_t^\top\right)\mathrm{n} \cdot \frac{\mathrm{M}_t^{-1^\top}\tau}{||\mathrm{M}_t^{-1^\top}\tau||^2}\right)\mathrm{M}_t^{-1^\top}\tau \cdot u_t + g_t\left|u_t \cdot \mathrm{M}_t^{-1^\top}\tau\right| &=& 0 & \text{on } \Gamma_{\mathrm{S}}, \end{array}\right. \qquad (\mathrm{GTP2d}_t)$$

for all $t \geq 0$. From Subsection 4.2.4, for all $t \geq 0$, there exists a unique weak solution $u_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ to (GTP2d$_t$) which satisfies

$$\int_\Omega z_t \mathrm{A}\left[\nabla u_t \mathrm{B}_t\right] \mathrm{B}_t^\top : \nabla(v - u_t) + \int_{\Gamma_\mathrm{S}} g_t \left| v \cdot {\mathrm{M}_t^{-1}}^\top \tau \right| - \int_{\Gamma_\mathrm{S}} g_t \left| u_t \cdot {\mathrm{M}_t^{-1}}^\top \tau \right| \geq \int_\Omega f_t \cdot (v - u_t) + \int_{\Gamma_\mathrm{S}} h_t \mathrm{M}_t \mathrm{n} \cdot (v - u_t), \qquad \forall v \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d), \quad (6.14)$$

and it can be expressed as

$$u_t = \mathrm{prox}_{\phi_t}(F_t),$$

where $F_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ is the unique solution to the parameterized general Dirichlet-Neumann problem

$$\begin{cases} -\mathrm{div}\left(z_t \mathrm{A}\left[(\nabla F_t)\mathrm{B}_t\right] \mathrm{B}_t^\top\right) = f_t & \text{in } \Omega, \\ F_t = 0 & \text{on } \Gamma_\mathrm{D}, \\ \left(z_t \mathrm{A}\left[(\nabla F_t)\mathrm{B}_t\right] \mathrm{B}_t^\top\right) \mathrm{n} = h_t \mathrm{M}_t \mathrm{n} & \text{on } \Gamma_\mathrm{S}, \end{cases} \quad (6.15)$$

where $\mathrm{prox}_{\phi_t}$ stands for the proximal operator associated with the Tresca friction functional $\phi_t$ defined by

$$\begin{array}{rccl} \phi_t : & \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d) & \longrightarrow & \mathbb{R} \\ & v & \longmapsto & \phi_t(v) := \displaystyle\int_{\Gamma_\mathrm{S}} g_t \left| v \cdot {\mathrm{M}_t^{-1}}^\top \tau \right|, \end{array}$$

considered on the perturbed Hilbert space $(\mathrm{H}^1(\Omega), \langle \cdot, \cdot \rangle_{z_t, \mathrm{A}, \mathrm{B}_t})$. Nevertheless, in order to avoid the computation of the twice epi-differentiability of the $t$-dependent tangential norm $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d \mapsto |x \cdot {\mathrm{M}_t^{-1}}^\top \tau| \in \mathbb{R}_+$ (see Remark 6.2.13), we follow the same strategy as Section 6.1. Since $\mathrm{M}_t \in \mathrm{C}^{1,\infty}(\overline{\Omega}, \mathrm{GL}_d(\mathbb{R}))$ then $\overline{u}_t := \mathrm{M}_t^{-1} u_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ satisfies

$$\int_\Omega z_t \mathrm{A}\left[\nabla(\mathrm{M}_t \overline{u}_t)\mathrm{B}_t\right] \mathrm{B}_t^\top : \nabla(\mathrm{M}_t (v - \overline{u}_t)) + \int_{\Gamma_\mathrm{S}} g_t |v \cdot \tau| - \int_{\Gamma_\mathrm{S}} g_t |\overline{u}_t \cdot \tau| \geq \int_\Omega {\mathrm{M}_t}^\top f_t \cdot (v - \overline{u}_t) + \int_{\Gamma_\mathrm{S}} h_t {\mathrm{M}_t}^\top \mathrm{M}_t \mathrm{n} \cdot (v - \overline{u}_t), \qquad \forall v \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d), \quad (6.16)$$

and is characterized as

$$\overline{u}_t = \mathrm{prox}_{\Phi(t,\cdot)}(E_t),$$

where $E_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ stands for the unique solution to the parameterized variational equality

$$\langle E_t, v \rangle_\mathrm{A} = \int_\Omega \mathrm{M}_t^\top f_t \cdot v + \int_{\Gamma_\mathrm{S}} h_t {\mathrm{M}_t}^\top \mathrm{M}_t \mathrm{n} \cdot v - \int_\Omega z_t \mathrm{A}\left[\nabla(\mathrm{M}_t \overline{u}_t)\mathrm{B}_t\right] \mathrm{B}_t^\top : \nabla(\mathrm{M}_t v) + \int_\Omega \mathrm{A} \nabla \overline{u}_t : \nabla v,$$

for all $v \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$, and where $\mathrm{prox}_{\Phi(t,\cdot)}$ stands for the proximal operator associated with $\Phi(t, \cdot)$ (already defined in (6.8) with, in our case, $v_\tau = (v \cdot \tau)\tau$ for all $v \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$) considered on the Hilbert space $(\mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d), \langle \cdot, \cdot \rangle_\mathrm{A})$ whose scalar product is $t$-independent. Note that $\overline{u}_0$ (resp. $E_0$) coincides with the solution $u_0$ (resp. $F_0$) to the parameterized general Tresca friction problem (GTP2d$_t$) (resp. to the parameterized general Dirichlet-Neumann problem (6.15)) for the parameter $t = 0$. Since we have already investigated the twice epi-differentiability of $\Phi$ on the Hilbert space $(\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d),\langle\cdot,\cdot\rangle_{\mathrm{A}})$ in Subsection 6.2.1, we can directly prove that the map $t\in\mathbb{R}_+\mapsto\overline{u}_t\in\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ is differentiable at $t=0$.

**Theorem 6.2.10.** *Let $\overline{u}_t\in\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ be the unique solution to the parameterized variational inequality* (6.16)*, for all $t\geq 0$. Assume that:*

1. *the map $t\in\mathbb{R}_+\mapsto h_t\in\mathrm{L}^2(\Gamma_{\mathrm{S}})$ is differentiable at $t=0$, with its derivative denoted by $h_0'\in\mathrm{L}^2(\Gamma_{\mathrm{S}})$;*
2. *for almost all $s\in\Gamma_{\mathrm{S}}$, the map $t\in\mathbb{R}_+\mapsto g_t(s)\in\mathbb{R}_+$ is differentiable at $t=0$, with its derivative denoted by $g_0'(s)$, and also $g_0'\in\mathrm{L}^2(\Gamma_{\mathrm{S}})$;*
3. *the parameterized Tresca friction functional $\Phi$ is twice epi-differentiable at $u_0$ for $F_0-u_0\in\partial\Phi(0,\cdot)(u_0)$, with*

$$\mathrm{D}^2_e\Phi(u_0|F_0-u_0)(v)=\int_{\Gamma_{\mathrm{S}}}\mathrm{D}^2_eH(s)(u_0(s)|\sigma_\tau(F_0-u_0)(s))(v(s))\mathrm{d}s,$$

*for all $v\in\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$, where $F_0\in\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ is the unique solution to the parameterized general Dirichlet-Neumann problem* (6.15) *for the parameter $t=0$, and $\mathrm{D}^2_eH(s)$ is defined in Proposition 6.2.5 for almost all $s\in\Gamma_{\mathrm{S}}$.*

*Then the map $t\in\mathbb{R}_+\mapsto\overline{u}_t\in\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ is differentiable at $t=0$, and its derivative denoted by $\overline{u}_0'\in\mathcal{K}_{u_0,\frac{\sigma_\tau(F_0-u_0)}{g_0}}\subset\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ is the unique solution to the variational inequality*

$$\begin{aligned}\langle\overline{u}_0',v-\overline{u}_0'\rangle_{\mathrm{A}}\geq&\int_\Omega\left({\mathrm{M}_0'}^\top f_0+f_0'\right)\cdot(v-\overline{u}_0')+\int_{\Gamma_{\mathrm{S}}}\left(h_0'\mathrm{I}+h_0\left({\mathrm{M}_0'}^\top+\mathrm{M}_0'\right)\right)\mathrm{n}\cdot(v-\overline{u}_0')\\-\int_\Omega\mathrm{A}\nabla u_0:\nabla(\mathrm{M}_0'(v-\overline{u}_0'))&-\int_\Omega\left(\mathrm{A}\left[\nabla u_0\right]{\mathrm{B}_0'}^\top+\mathrm{A}\left[\nabla u_0\mathrm{B}_0'\right]+\mathrm{A}\nabla(\mathrm{M}_0'u_0)+z_0'\mathrm{A}\nabla u_0\right):\nabla(v-\overline{u}_0')\\&+\int_{\Gamma_{\mathrm{S}^{u_0,g_0}_{\mathrm{S}-}}\cup\Gamma_{\mathrm{S}^{u_0,g_0}_{\mathrm{S}+}}}g_0'\frac{\sigma_\tau(u_0)}{g_0}\cdot\left(v_\tau-\overline{u}_{0_\tau}'\right)-\int_{\Gamma_{\mathrm{S}^{u_0,g_0}_{\mathrm{R}}}}g_0'\frac{u_{0_\tau}}{\|u_{0_\tau}\|}\cdot\left(v_\tau-\overline{u}_{0_\tau}'\right),\end{aligned}$$

*for all $v\in\mathcal{K}_{u_0,\frac{\sigma_\tau(F_0-u_0)}{g_0}}$, where*

$$\mathcal{K}_{u_0,\frac{\sigma_\tau(F_0-u_0)}{g_0}}=\Big\{v\in\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)\mid v\cdot\tau=0\text{ a.e. on }\Gamma_{\mathrm{S}^{u_0,g_0}_{\mathrm{D}}},v\cdot\tau\leq 0\text{ a.e. on }\Gamma_{\mathrm{S}^{u_0,g_0}_{\mathrm{S}-}}\\ \text{and }v\cdot\tau\geq 0\text{ a.e. on }\Gamma_{\mathrm{S}^{u_0,g_0}_{\mathrm{S}+}}\Big\},$$

*is a nonempty closed convex subset of $\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$, and where $\Gamma_{\mathrm{S}}$ is decomposed (up to a null set) as $\Gamma_{\mathrm{S}^{u_0,g_0}_{\mathrm{R}}}\cup\Gamma_{\mathrm{S}^{u_0,g_0}_{\mathrm{D}}}\cup\Gamma_{\mathrm{S}^{u_0,g_0}_{\mathrm{S}-}}\cup\Gamma_{\mathrm{S}^{u_0,g_0}_{\mathrm{S}+}}$ with*

$$\begin{aligned}\Gamma_{\mathrm{S}^{u_0,g_0}_{\mathrm{R}}}&:=\left\{s\in\Gamma_{\mathrm{S}}\mid u_0(s)\cdot\tau(s)\neq 0\right\},\\\Gamma_{\mathrm{S}^{u_0,g_0}_{\mathrm{D}}}&:=\left\{s\in\Gamma_{\mathrm{S}}\mid u_0(s)\cdot\tau(s)=0\text{ and }\sigma_\tau(u_0)(s)\cdot\tau(s)\in(-g_0(s),g_0(s))\right\},\\\Gamma_{\mathrm{S}^{u_0,g_0}_{\mathrm{S}-}}&:=\left\{s\in\Gamma_{\mathrm{S}}\mid u_0(s)\cdot\tau(s)=0\text{ and }\sigma_\tau(u_0)(s)\cdot\tau(s)=g_0(s)\right\},\\\Gamma_{\mathrm{S}^{u_0,g_0}_{\mathrm{S}+}}&:=\left\{s\in\Gamma_{\mathrm{S}}\mid u_0(s)\cdot\tau(s)=0\text{ and }\sigma_\tau(u_0)(s)\cdot\tau(s)=-g_0(s)\right\}.\end{aligned}$$

*Proof.* From Hypothesis (H4), (H5), (H6), (H7) and that the map $t \in \mathbb{R}_+ \mapsto h_t \in \mathrm{L}^2(\Gamma_\mathrm{S})$ is differentiable at $t = 0$, one can prove similarly to Lemma 6.1.1, that the map $t \in \mathbb{R}_+ \mapsto E_t \in \mathrm{H}^1_\mathrm{D}(\Omega, \mathbb{R}^d)$ is differentiable at $t = 0$ and its derivative, denoted by $E'_0 \in \mathrm{H}^1_\mathrm{D}(\Omega, \mathbb{R}^d)$, is the unique solution to the variational equality

$$\langle E'_0, v\rangle_\mathrm{A} = \int_\Omega \left(\mathrm{M}_0'^\top f_0 + f'_0\right)\cdot v + \int_{\Gamma_\mathrm{S}} \left(h'_0\mathrm{I} + h_0\left(\mathrm{M}_0'^\top + \mathrm{M}'_0\right)\right)\mathrm{n}\cdot v - \int_\Omega \mathrm{A}\nabla u_0 : \nabla(\mathrm{M}'_0 v)$$
$$- \int_\Omega \left(\mathrm{A}\left[\nabla u_0\right]\mathrm{B}_0'^\top + \mathrm{A}\left[\nabla u_0 \mathrm{B}'_0\right] + \mathrm{A}\nabla(\mathrm{M}'_0 u_0) + z'_0\mathrm{A}\nabla u_0\right) : \nabla v,$$

for all $v \in \mathrm{H}^1_\mathrm{D}(\Omega, \mathbb{R}^d)$. Moreover since $d = 2$, one deduces that

$$\int_{\Gamma_{\mathrm{S}^{u_0,g_0}_\mathrm{R}}} \left(\frac{g_0}{2\left\|u_{0_\tau}\right\|}\left(\left\|v_\tau\right\|^2 - \left|v_\tau\cdot\frac{u_{0_\tau}}{\left\|u_{0_\tau}\right\|}\right|^2\right)\right) = 0,$$

for all $v \in \mathrm{H}^1_\mathrm{D}(\Omega, \mathbb{R}^d)$, thus from Hypothesis 2 and 3, it follows that

$$\mathrm{D}^2_e\Phi(u_0|F_0 - u_0)(v) = \int_{\Gamma_{\mathrm{S}^{u_0,g_0}_\mathrm{R}}} g'_0\frac{u_{0_\tau}}{\left\|u_{0_\tau}\right\|}\cdot v_\tau + \int_{\Gamma_\mathrm{S}\backslash\Gamma_{\mathrm{S}^{u_0,g_0}_\mathrm{R}}} g'_0\frac{\sigma_\tau(F_0 - u_0)}{g_0}\cdot v_\tau + \iota_{\mathcal{K}_{u_0,\frac{\sigma_\tau(F_0-u_0)}{g_0}}}(v),$$

for all $v \in \mathrm{H}^1_\mathrm{D}(\Omega, \mathbb{R}^d)$, where $\mathcal{K}_{u_0,\frac{\sigma_\tau(F_0-u_0)}{g_0}}$ defined in Theorem 6.2.7 is also given, for $d = 2$, by

$$\mathcal{K}_{u_0,\frac{\sigma_\tau(F_0-u_0)}{g_0}} = \Big\{v \in \mathrm{H}^1_\mathrm{D}(\Omega, \mathbb{R}^d) \mid v\cdot\tau = 0 \text{ } a.e. \text{ on } \Gamma_{\mathrm{S}^{u_0,g_0}_\mathrm{D}}, v\cdot\tau \leq 0 \text{ } a.e. \text{ on } \Gamma_{\mathrm{S}^{u_0,g_0}_{\mathrm{S}-}}$$
$$\text{and } v\cdot\tau \geq 0 \text{ } a.e. \text{ on } \Gamma_{\mathrm{S}^{u_0,g_0}_{\mathrm{S}+}}\Big\}.$$

The rest of the proof is similar to Theorem 6.2.7. □

Since $u_t = \mathrm{M}_t\overline{u}_t \in \mathrm{H}^1_\mathrm{D}(\Omega, \mathbb{R}^d)$, one can prove in the same way as Theorem 6.1.4 the following result. Note that we have assumed at the beginning of Section 6.2, that almost every point of $\Gamma_\mathrm{S}$ is in $\mathrm{int}_\Gamma(\Gamma_\mathrm{S})$, thus $u_0 \in \mathrm{H}^1_\mathrm{D}(\Omega, \mathbb{R}^d)$ is a (strong) solution to the parameterized general Tresca friction problem (GTP2d$_t$) for the parameter $t = 0$ (see Subsection 4.2.4), and therefore $\mathrm{A}\mathrm{e}(u_0)\mathrm{n} \in \mathrm{L}^2(\Gamma, \mathbb{R}^d)$.

**Theorem 6.2.11.** *Consider the framework of Theorem 6.2.10. Let $u_t \in \mathrm{H}^1_\mathrm{D}(\Omega, \mathbb{R}^d)$, be the unique solution to the parameterized general Tresca friction problem* (GTP2d$_t$). *Then the map $t \in \mathbb{R}_+ \mapsto u_t \in \mathrm{H}^1_\mathrm{D}(\Omega, \mathbb{R}^d)$ is differentiable at $t = 0$, and its derivative denoted by $u'_0 \in \mathcal{K}_{u_0,\frac{\sigma_\tau(F_0-u_0)}{g_0}} + \mathrm{M}'_0 u_0$ is* *the unique solution to the variational inequality*

$$\begin{aligned}\langle u_0', v-u_0'\rangle_{\mathrm{A}} \geq \int_\Omega f_0'\cdot(v-u_0') + \int_{\Gamma_{\mathrm{S}}}\left(h_0'\mathrm{I}+h_0\left({\mathrm{M}_0'}^\top+\mathrm{M}_0'\right)\right)\mathrm{n}\cdot(v-u_0') \\ -\int_{\Gamma_{\mathrm{S}}}\mathrm{A}\nabla u_0\mathrm{n}\cdot\mathrm{M}_0'(v-u_0') - \int_\Omega\left(\mathrm{A}\left[\nabla u_0\right]{\mathrm{B}_0'}^\top+\mathrm{A}\left[\nabla u_0\mathrm{B}_0'\right]+z_0'\mathrm{A}\nabla u_0\right):\nabla(v-u_0') \\ +\int_{\Gamma_{\mathrm{S}_{\mathrm{S}-}^{u_0,g_0}}\cup\Gamma_{\mathrm{S}_{\mathrm{S}+}^{u_0,g_0}}} g_0'\frac{\sigma_\tau(u_0)}{g_0}\cdot\left(v_\tau-u_{0_\tau}'\right) - \int_{\Gamma_{\mathrm{S}_{\mathrm{R}}^{u_0,g_0}}} g_0'\frac{u_{0_\tau}}{\|u_{0_\tau}\|}\cdot\left(v_\tau-u_{0_\tau}'\right),\end{aligned}$$

*for all* $v\in\mathcal{K}_{u_0,\frac{\sigma_\tau(F_0-u_0)}{g_0}}+\mathrm{M}_0'u_0$, *where*

$$\begin{aligned}\mathcal{K}_{u_0,\frac{\sigma_\tau(F_0-u_0)}{g_0}}+\mathrm{M}_0'u_0 := \\ \Big\{v\in\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)\mid v\cdot\tau=\mathrm{M}_0'u_0\cdot\tau \text{ a.e. on } \Gamma_{\mathrm{S}_{\mathrm{D}}^{u_0,g_0}},(v-\mathrm{M}_0'u_0)\cdot\tau\leq 0 \text{ a.e. on } \Gamma_{\mathrm{S}_{\mathrm{S}-}^{u_0,g_0}} \\ \text{and } (v-\mathrm{M}_0'u_0)\cdot\tau\geq 0 \text{ a.e. on } \Gamma_{\mathrm{S}_{\mathrm{S}+}^{u_0,g_0}}\Big\}.\end{aligned}$$

As we did in Corollary 6.2.8, one can characterize $u_0'\in\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ as weak solution to a tangential Signorini problem.

**Corollary 6.2.12.** *Consider the framework of Theorem 6.2.11. Assume that* $\operatorname{div}(\mathrm{A}\left[\nabla u_0\right]{\mathrm{B}_0'}^\top+\mathrm{A}\left[\nabla u_0\mathrm{B}_0'\right]+z_0'\mathrm{A}\nabla u_0)\in\mathrm{L}^2(\Omega,\mathbb{R}^d)$ *and* $(\mathrm{A}\left[\nabla u_0\right]{\mathrm{B}_0'}^\top+\mathrm{A}\left[\nabla u_0\mathrm{B}_0'\right]+z_0'\mathrm{A}\nabla u_0)\mathrm{n}\in\mathrm{L}^2(\Gamma,\mathbb{R}^d)$. *Then* $u_0'\in\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ *is the unique weak solution to the tangential Signorini problem*

$$\left\{\begin{array}{rcll} -\operatorname{div}(\mathrm{Ae}(u_0')) &=& \tilde{f} & \text{in } \Omega,\\ u_0' &=& 0 & \text{on } \Gamma_{\mathrm{D}},\\ \sigma_{\mathrm{n}}(u_0') &=& \xi_{\mathrm{n}} & \text{on } \Gamma_{\mathrm{S}},\\ \sigma_\tau(u_0')\cdot\tau &=& -g_0'\frac{u_0\cdot\tau}{|u_0\cdot\tau|}+\xi\cdot\tau & \text{on } \Gamma_{\mathrm{S}_{\mathrm{R}}^{u_0,g_0}},\\ u_0'\cdot\tau &=& \mathrm{M}_0'u_0\cdot\tau & \text{on } \Gamma_{\mathrm{S}_{\mathrm{D}}^{u_0,g_0}},\\ (u_0'-\mathrm{M}_0'u_0)\cdot\tau\leq 0,\ \sigma_\tau(u_0')\cdot\tau\leq g_0'+\xi\cdot\tau \text{ and} && & \\ (u_0'-\mathrm{M}_0'u_0)\cdot\tau\left(\sigma_\tau(u_0')\cdot\tau-g_0'-\xi\cdot\tau\right) &=& 0 & \text{on } \Gamma_{\mathrm{S}_{\mathrm{S}-}^{u_0,g_0}},\\ (u_0'-\mathrm{M}_0'u_0)\cdot\tau\geq 0,\ \sigma_\tau(u_0')\cdot\tau\geq -g_0'+\xi\cdot\tau \text{ and} && & \\ (u_0'-\mathrm{M}_0'u_0)\cdot\tau\left(\sigma_\tau(u_0')\cdot\tau+g_0'-\xi\cdot\tau\right) &=& 0 & \text{on } \Gamma_{\mathrm{S}_{\mathrm{S}+}^{u_0,g_0}}. \end{array}\right.$$

*where* $\tilde{f}:=f_0'+\operatorname{div}\mathrm{A}\left[\nabla u_0\right]{\mathrm{B}_0'}^\top+\mathrm{A}\left[\nabla u_0\mathrm{B}_0'\right]+z_0'\mathrm{A}\nabla u_0)\in\mathrm{L}^2(\Omega,\mathbb{R}^d)$, *and*

$$\xi:=h_0'\mathrm{n}+h_0\left({\mathrm{M}_0'}^\top+\mathrm{M}_0'\right)\mathrm{n}-{\mathrm{M}_0'}^\top(\mathrm{A}\nabla u_0\mathrm{n})-\left(\mathrm{A}\left[\nabla u_0\right]{\mathrm{B}_0'}^\top+\mathrm{A}\left[\nabla u_0\mathrm{B}_0'\right]+z_0'\mathrm{A}\nabla u_0\right)\mathrm{n}\in\mathrm{L}^2(\Gamma_{\mathrm{S}},\mathbb{R}^d).$$

**Remark 6.2.13.** Note that, in the three-dimensional case, the parameterized Tresca friction functional is given by

$$\begin{array}{rccl}\Phi: & \mathbb{R}_+ \times \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d) & \longrightarrow & \mathbb{R} \\ & (t,v) & \longmapsto & \Phi(t,v) := \displaystyle\int_{\Gamma_{\mathrm{S}}} g_t \left\| v_{(\mathrm{M}_t\mathrm{n})^\perp} \right\|, \end{array}$$

where $v_{(\mathrm{M}_t\mathrm{n})^\perp} := v - (v \cdot \frac{\mathrm{M}_t\mathrm{n}}{||\mathrm{M}_t\mathrm{n}||^2})\mathrm{M}_t\mathrm{n}$. Unlike the two-dimensional case, to the best of our knowledge, we cannot make disappear the $t$-dependance of the norm. Therefore, for almost all $s \in \Gamma_{\mathrm{S}}$, we have to deal with the twice epi-differentiability of the $t$-dependent map $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d \mapsto ||x_{(\mathrm{M}_t\mathrm{n})^\perp}|| \in \mathbb{R}_+$, which is an a highly nontrivial work and an interesting topic for future research.

Chapter 7

# Numerical simulations for the sensitivity analysis of the scalar Tresca friction problem

---

In the two previous chapters of this part, we have investigated the sensitivity analysis of a general (scalar) Signorini/Tresca friction problem. Roughly speaking, the main theorems obtained claim that, the first-order approximation of the solution $u_t$ to the problem are given by $u_0 + tu_0'$ for small values of $t \geq 0$, where $u_0'$ is the solution to a boundary value problem involving (scalar) Signorini unilateral conditions or tangential Signorini unilateral conditions.

In this chapter, we focus on the scalar Tresca friction problem studied in Subsection 5.2.2, and we illustrate Theorem 5.2.12 with some numerical simulations. Thus, the same notations are preserved and, for an explicit two-dimensional example described in Section 7.1, we compare in $\mathrm{H}^1$-norm the solution $u_t$ to the parameterized Tresca friction problem (STP2$_t$) with its first-order approximation $u_0 + tu_0'$ for small values of $t \geq 0$, where $u_0'$ is the solution to the Signorini problem (5.18).

Numerical simulations are performed using Freefem++ software (see [44]) and iterative switching algorithms (see [4] or Appendix B for details). The results are presented in Section 7.2. In a nutshell, let us recall that those iterative switching algorithms operate by checking at each iteration if the boundary conditions are satisfied, and if they are not, by imposing them and restarting the computation. The convergence proof of these algorithms is not established yet but their performance are experimentally validated. Let us emphasize that our aim in this section is not to study these algorithms rigorously but only to illustrate the main result of Section 5.2.2 with a simple and easily implementable method to solve the Signorini problem and the Tresca friction problem. Let us mention that there exist other algorithms, like for instance Nitsche methods (see, e.g., [21, 22, 23, 64]) which will be used in Chapter 9, hybrid methods (see, e.g., [12]), mixed methods (see, e.g., [41]) and more, that are not used here, but could be more efficient for future investigations.

## 7.1 Mathematical framework

In this section we describe the example used for numerical simulations. This example is inspired from the one introduced in the paper [3]. Let $d = 2$ and $\Omega$ be the unit disk of $\mathbb{R}^2$, and let $f \in \mathrm{L}^2(\Omega)$

be the function defined by

$$\begin{array}{rccl} f: & \Omega & \longrightarrow & \mathbb{R} \\ & (x,y) & \longmapsto & \dfrac{1}{2}\left(x^2+y^2-5\right)\xi(x) - 2x\xi'(x) - \dfrac{1}{2}(x^2+y^2-1)\xi''(x), \end{array}$$

where $\xi$ is given by

$$\begin{array}{rccl} \xi: & [-1,1] & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \xi(x) := \begin{cases} -1 & \text{if} \quad -1 \leq x \leq -\frac{1}{2}, \\ \sin(\pi x) & \text{if} \quad -\frac{1}{2} \leq x \leq \frac{1}{2}, \\ 1 & \text{if} \quad \frac{1}{2} \leq x \leq 1. \end{cases} \end{array}$$

Let us introduce, for all $t \geq 0$, the function $f_t \in \mathrm{L}^2(\Omega)$ defined by

$$\begin{array}{rccl} f_t: & \Omega & \longrightarrow & \mathbb{R} \\ & (x,y) & \longmapsto & f_t(x,y) := \exp(t) f(x,y), \end{array}$$

the function $g_t \in \mathrm{L}^2(\Gamma)$ defined by

$$\begin{array}{rccl} g_t: & \Gamma & \longrightarrow & \mathbb{R} \\ & (x,y) & \longmapsto & g_t(x,y) := 1+t. \end{array}$$

This choice of functions is justified by the fact that we are able to determinate explicitly the solution $u_0$ to the parameterized Tresca friction problem (STP2$_t$) for $t=0$, which is given by

$$u_0(x,y) = \frac{1}{2}\left(x^2+y^2-1\right)\xi(x),$$

for all $(x,y) \in \Omega$. The knowledge of the solution $u_0$ reduces errors due to the approximations for the numerical computations of $u_0'$ and $u_0 + tu_0'$. Indeed, since $\partial_{\mathrm{n}} u = \xi$ and $u_0 = 0$ *a.e.* on $\Gamma$, we can directly express the decomposition

$$\Gamma = \Gamma_{\mathrm{N}}^{u_0,g_0} \cup \Gamma_{\mathrm{D}}^{u_0,g_0} \cup \Gamma_{\mathrm{S-}}^{u_0,g_0} \cup \Gamma_{\mathrm{S+}}^{u_0,g_0},$$

which is given by

$$\begin{aligned} \Gamma_{\mathrm{N}}^{u_0,g_0} &= \emptyset, \\ \Gamma_{\mathrm{D}}^{u_0,g_0} &= \left\{(x,y) \in \Gamma,\ -\tfrac{1}{2} < x < \tfrac{1}{2}\right\}, \\ \Gamma_{\mathrm{S-}}^{u_0,g_0} &= \left\{(x,y) \in \Gamma,\ x \geq \tfrac{1}{2}\right\}, \\ \Gamma_{\mathrm{S+}}^{u_0,g_0} &= \left\{(x,y) \in \Gamma \mid x \leq -\tfrac{1}{2}\right\}, \end{aligned}$$

(see Figure 7.1). Moreover, since $f_0' = f$ in $\mathrm{L}^2(\Omega)$ and $g_0' = 1$ in $\mathrm{L}^2(\Gamma)$, we are now in a position to compute numerically $u_0'$ and $u_t$, and then to compare $u_t$ with its first-order approximation $u_0 + tu_0'$ in $\mathrm{H}^1$-norm for several small values of $t \geq 0$.

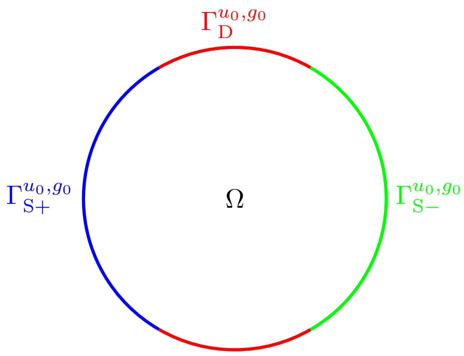


Figure 7.1 – Unit disk $\Omega$ and its boundary $\Gamma = \Gamma_{\mathrm{N}}^{u_0,g_0} \cup \Gamma_{\mathrm{D}}^{u_0,g_0} \cup \Gamma_{\mathrm{S-}}^{u_0,g_0} \cup \Gamma_{\mathrm{S+}}^{u_0,g_0}$.

## 7.2 Numerical results

Here we present the numerical results obtained for the two-dimensional example described in Section 7.1. Numerical simulations have been made using P2 finite element method and with a discretization of the boundary of 190 points. We concatenate in Table 7.1 some values of $\|u_t - u_0 - tu_0'\|_{\mathrm{H}^1(\Omega)}$ for several small values of $t \geq 0$. Figure 7.2 gives the representation in logarithmic scale of the maps $t \in \mathbb{R}_+ \mapsto \|u_t - u_0 - tu_0'\|_{\mathrm{H}^1(\Omega)} \in \mathbb{R}_+$ and $t \in \mathbb{R}_+ \mapsto t^2 \in \mathbb{R}_+$. Finally, Figure 7.3 is the illustration of $u_t$ and its first-order approximation $u_0 + tu_0'$ for $t = 0.1$.

Roughly speaking, we observe from Figure 7.2 that

$$\left\| \frac{u_t - u_0}{t} - u_0' \right\|_{\mathrm{H}^1(\Omega)} = O(t),$$

where $O$ stands for the standard Bachmann–Landau notation, which is in accordance with Theorem 5.2.12.

| Paramètre t | 0.60 | 0.40 | 0.20 | 0.1 | 0.075 | 0.05 | 0.025 | 0.01 |
|---|---|---|---|---|---|---|---|---|
| $\|u_t - u_0 - tu_0'\|_{\mathrm{H}^1(\Omega)}$ | 0.5254 | 0.2135 | 0.0484 | 0.0114 | 0.0065 | 0.0033 | 0.0021 | 0.0021 |

Table 7.1 – $\mathrm{H}^1$-norm of the difference between $u_t$ and its first-order approximation $u_0 + tu_0'$ for several small values of $t$.

**Remark 7.2.1.** Note that the representation of $\|u_t - u_0 - tu_0'\|_{\mathrm{H}^1(\Omega)}$ with respect to $t$ in logarithmic scale got a threshold for $t \approx 0.03$. This is a classical phenomenon due to the numerical approximations we made and the numerical algorithms we used.

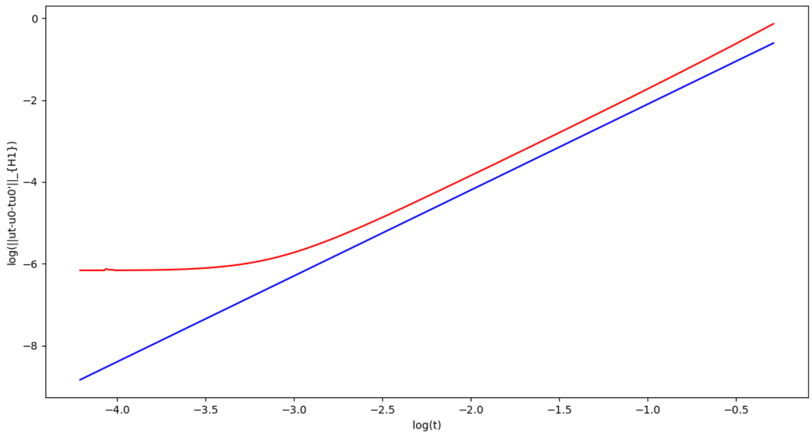


Figure 7.2 – The representation in logarithmic scale of the map $t \in \mathbb{R}_+ \mapsto \|u_t - u_0 - tu_0'\|_{\mathrm{H}^1(\Omega)} \in \mathbb{R}_+$ (red) and of the map $t \in \mathbb{R}_+ \mapsto t^2 \in \mathbb{R}_+$ (blue).

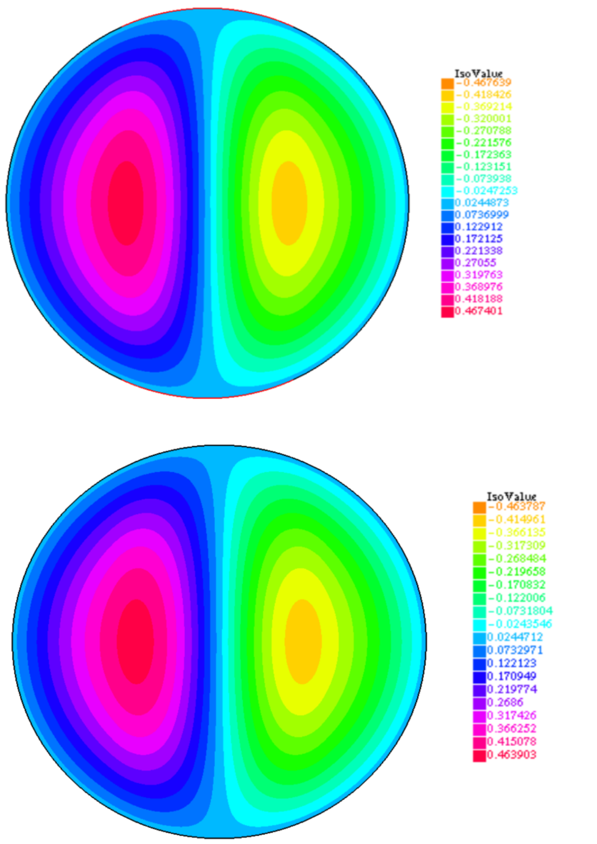


Figure 7.3 – The first figure is the representation of $u_t$ and the second its first-order approximation $u_0 + \ tu_0'$ for $t = 0.1$.

Part III

# Applications

In this last part, we present several applications of the sensitivity analysis investigated in Part II. More precisely, in Chapter 8, we investigate a shape optimization problem involving the scalar Tresca friction law. Thus, the results obtained in Subsection 5.2.3 are used in order to characterize the material and shape directional derivatives as solutions to a scalar Signorini problem. Then, using the material/shape directional derivative, we explicitly characterize the shape gradient of the corresponding energy functional and we exhibit a descent direction in order to perform numerical simulations. In Chapter 9, we investigate a shape optimization problem involving a Signorini problem in the linear elastic model. Thus, the results from Section 6.1 are used in order to prove that the material and shape directional derivatives are solutions to Signorini problems, and we characterize the shape gradient of the corresponding energy functional. Then some numerical simulations are performed to illustrate our result. We mention that the shape optimization problems in Chapter 8 and Chapter 9 are investigated using the Hadamard's boundary variation method (see, e.g., [6, 39]). Finally in Chapter 10, an optimal control problem is studied, which involves the Tresca friction law in the linear elastic model, and we search for a control that minimizes the cost functional defined in (10.2).

The characterizations of the material and shape directional derivative as the solutions to a (scalar) Signorini problems, with also a suitable characterization of the shape gradient of the corresponding energy functional are the main novelties in shape optimization, while the suitable characterization of the gradient of the cost functional defined in (10.2) is the main novelty in optimal control. To the best of my knowledge, those results are new and have never been obtained in the literature.

CHAPTER 8

# SHAPE OPTIMIZATION FOR VARIATIONAL INEQUALITIES: THE TRESCA FRICTION PROBLEM IN THE SCALAR MODEL

Let $d \in \mathbb{N}^*$ be a positive integer and let $f \in \mathrm{H}^1(\mathbb{R}^d)$ and $g \in \mathrm{H}^2(\mathbb{R}^d)$ be such that $g > 0$ *a.e.* on $\mathbb{R}^d$. In this chapter we consider the shape optimization problem given by

$$\underset{\substack{\Omega \in \mathcal{U} \\ |\Omega| = \lambda}}{\text{minimize}} \ \mathcal{J}(\Omega), \tag{8.1}$$

where

$$\mathcal{U} := \{\Omega \subset \mathbb{R}^d \mid \Omega \text{ nonempty connected bounded open subset of } \mathbb{R}^d \text{ with Lipschitz boundary}\},$$

with the volume constraint $|\Omega| = \lambda > 0$, where $\mathcal{J} : \mathcal{U} \to \mathbb{R}$ is the *Tresca energy functional* defined by

$$\mathcal{J}(\Omega) := \frac{1}{2} \int_\Omega \left( \|\nabla u_\Omega\|^2 + |u_\Omega|^2 \right) + \int_\Gamma g|u_\Omega| - \int_\Omega f u_\Omega,$$

where $\Gamma := \partial\Omega$ is the boundary of $\Omega$ and where $u_\Omega \in \mathrm{H}^1(\Omega)$ stands for the unique solution to the scalar Tresca friction problem given by

$$\left\{ \begin{aligned} -\Delta u + u &= f && \text{in } \Omega, \\ |\partial_{\mathrm{n}} u| \leq g \text{ and } u\partial_{\mathrm{n}} u + g|u| &= 0 && \text{on } \Gamma, \end{aligned} \right. \tag{STP$_\Omega$}$$

for all $\Omega \in \mathcal{U}$, where n is the outward-pointing unit normal vector to $\Gamma$. Note that we focus here on minimizing the energy functional (as in [36, 45, 81]) which corresponds to maximize the compliance (see [8]). From Subsection 4.1.3, note that $\mathcal{J}$ can also be expressed as

$$\mathcal{J}(\Omega) = -\frac{1}{2} \int_\Omega \left( \|\nabla u_\Omega\|^2 + |u_\Omega|^2 \right),$$

for all $\Omega \in \mathcal{U}$.

In the whole section let us fix $\Omega_0 \in \mathcal{U}$. We denote by id : $\mathbb{R}^d \to \mathbb{R}^d$ the identity operator. Our aim here is to prove that, under appropriate assumptions, the functional $\mathcal{J}$ is *shape differentiable* at $\Omega_0$, in the sense that the map

$$\begin{array}{rcl} \mathcal{C}^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d) & \longrightarrow & \mathbb{R} \\ \theta & \longmapsto & \mathcal{J}(\mathrm{id}+\theta)(\Omega_0)), \end{array}$$

where $\mathcal{C}^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d) := \mathcal{C}^1(\mathbb{R}^d,\mathbb{R}^d) \cap \mathrm{W}^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d)$, is Gateaux differentiable at 0, and to give an expression of the Gateaux differential, denoted by $\mathcal{J}'(\Omega_0)$, which is called the *shape gradient* of $\mathcal{J}$ at $\Omega_0$. To this aim we have to perform the sensitivity analysis of the scalar Tresca friction problem ($\mathrm{STP}_\Omega$) with respect to the shape, and then we characterize *the material and shape directional derivatives*. For better organization, this chapter will be done in the following three separate sections below. In Section 8.1 we perturb the scalar Tresca friction problem ($\mathrm{TP}_{\Omega_0}$) with respect to the shape, and under appropriate assumptions, we characterize the material directional derivative as solution to a variational inequality (see Theorem 8.1.1). Moreover, with additional assumptions we characterize the material and shape directional derivatives as being weak solutions to scalar Signorini problems (see Corollaries 8.1.3 and 8.1.5). In Section 8.2 we prove that, under appropriate assumptions, the functional $\mathcal{J}$ is shape differentiable at $\Omega_0$ and we provide an expression of the shape gradient $\mathcal{J}'(\Omega_0)$ (see Theorem 8.2.1 and Corollary 8.2.2). Finally in Section 8.3 numerical simulations are performed to solve the shape optimization problem (8.1) on a two-dimensional example.

## 8.1 Setting of the shape perturbation and derivatives

Consider $\theta \in \mathcal{C}^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d)$ and, for all $t \geq 0$ sufficiently small such that id$+t\theta$ is a $\mathcal{C}^1$-diffeomorphism of $\mathbb{R}^d$, consider the shape perturbed scalar Tresca friction problem given by

$$\left\{ \begin{aligned} -\Delta u_t + u_t &= f \quad \text{in } \Omega_t, \\ |\partial_{\mathrm{n}} u_t| \leq g \text{ and } u_t\partial_{\mathrm{n}} u_t + g|u_t| &= 0 \quad \text{on } \Gamma_t, \end{aligned} \right. \tag{PSTP$_t$}$$

where $\Omega_t := (\mathrm{id}+t\theta)(\Omega_0) \in \mathcal{U}$ and $\Gamma_t := \partial\Omega_t = (\mathrm{id}+t\theta)(\Gamma_0)$. From Subsection 4.1.3, there exists a unique solution $u_t \in \mathrm{H}^1(\Omega_t)$ to ($\mathrm{PSTP}_t$) which satisfies

$$\int_{\Omega_t} \nabla u_t \cdot \nabla(v-u_t) + \int_{\Omega_t} u_t(v-u_t) + \int_{\Gamma_t} g|v| - \int_{\Gamma_t} g|u_t| \geq \int_{\Omega_t} f(v-u_t), \qquad \forall v \in \mathrm{H}^1(\Omega_t).$$

Following the usual strategy in shape optimization literature (see, e.g., [8, 46]) and using the change of variables $\mathrm{id}+t\theta$, we prove that $\overline{u}_t := u_t \circ (\mathrm{id}+t\theta) \in \mathrm{H}^1(\Omega_0)$ satisfies

$$\begin{aligned} \int_{\Omega_0} \mathrm{M}_t\nabla\overline{u}_t \cdot \nabla(v-\overline{u}_t) + \int_{\Omega_0} \overline{u}_t(v-\overline{u}_t)\mathrm{J}_t + \int_{\Gamma_0} g_t\mathrm{J}_{\mathrm{T}_t}|v| - \int_{\Gamma_0} g_t\mathrm{J}_{\mathrm{T}_t}|\overline{u}_t| \\ \geq \int_{\Omega_0} f_t\mathrm{J}_t(v-\overline{u}_t), \qquad \forall v \in \mathrm{H}^1(\Omega_0), \end{aligned}$$

where $f_t := f \circ (\mathrm{id}+t\theta) \in \mathrm{H}^1(\mathbb{R}^d)$, $g_t := g \circ (\mathrm{id}+t\theta) \in \mathrm{H}^2(\mathbb{R}^d)$, $\mathrm{J}_t := \det(\mathrm{I}+t\nabla\theta) \in \mathrm{L}^\infty(\mathbb{R}^d)$ is the Jacobian determinant, $\mathrm{M}_t := \det(\mathrm{I}+t\nabla\theta)(\mathrm{I}+t\nabla\theta)^{-1}(\mathrm{I}+t\nabla\theta^\top)^{-1} \in \mathrm{L}^\infty(\mathbb{R}^d,\mathbb{R}^{d\times d})$, $\mathrm{J}_{\mathrm{T}_t} :=$

$\det(\mathrm{I}+t\nabla\theta)\|(\mathrm{I}+t\nabla\theta^\top)^{-1}\mathrm{n}\| \in \mathcal{C}^0(\Gamma_0)$ is the tangential Jacobian, and where $\nabla\theta$ stands for the standard Jacobian matrix of $\theta$. Therefore, we deduce from Subsection 4.1.3 that $\overline{u}_t \in \mathrm{H}^1(\Omega_0)$ is the unique solution to the perturbed general scalar Tresca friction problem

$$\begin{cases} -\mathrm{div}\left(\mathrm{M}_t\nabla\overline{u}_t\right) + \overline{u}_t\mathrm{J}_t = f_t\mathrm{J}_t & \text{in } \Omega_0, \\ |\mathrm{M}_t\nabla\overline{u}_t\cdot\mathrm{n}| \leq g_t\mathrm{J}_{\mathrm{T}_t} \text{ and } \overline{u}_t\mathrm{M}_t\nabla\overline{u}_t\cdot\mathrm{n} + g_t\mathrm{J}_{\mathrm{T}_t}|\overline{u}_t| = 0 & \text{on } \Gamma_0, \end{cases} \tag{PGSTP$_t$}$$

and thus can be characterized as

$$\overline{u}_t = \mathrm{prox}_{\Phi(t,\cdot)}(F_t),$$

where $F_t \in \mathrm{H}^1(\Omega_0)$ stands for the unique solution to the perturbed general scalar Neumann problem given by

$$\begin{cases} -\mathrm{div}(\mathrm{M}_t\nabla F_t) + F_t\mathrm{J}_t = f_t\mathrm{J}_t & \text{in } \Omega, \\ \mathrm{M}_t\nabla F_t\cdot\mathrm{n} = 0 & \text{on } \Gamma, \end{cases}$$

and where $\mathrm{prox}_{\Phi(t,\cdot)} : \mathrm{H}^1(\Omega_0) \to \mathrm{H}^1(\Omega_0)$ is the proximal operator associated with the parameterized Tresca friction functional defined by

$$\begin{array}{rrcl} \Phi : & \mathbb{R}_+ \times \mathrm{H}^1(\Omega_0) & \longrightarrow & \mathbb{R} \\ & (t,v) & \longmapsto & \Phi(t,v) := \displaystyle\int_{\Gamma_0} g_t\mathrm{J}_{\mathrm{T}_t}|v|, \end{array}$$

considered on the perturbed Hilbert space $(\mathrm{H}^1(\Omega), \langle\cdot,\cdot\rangle_{\mathrm{M}_t,\mathrm{J}_t})$ (see the notations in Section 4.1). Let us recall that (see [46]):

- (i) the map $t \in \mathbb{R}_+ \mapsto \mathrm{J}_t \in \mathrm{L}^\infty(\mathbb{R}^d)$ is differentiable at $t = 0$ with derivative given by $\mathrm{div}(\theta)$;
- (ii) the map $t \in \mathbb{R}_+ \mapsto f_t\mathrm{J}_t \in \mathrm{L}^2(\mathbb{R}^d)$ is differentiable at $t = 0$ with derivative given by $f\mathrm{div}(\theta) + \nabla f\cdot\theta$;
- (iii) the map $t \in \mathbb{R}_+ \mapsto \mathrm{M}_t \in \mathrm{L}^\infty(\mathbb{R}^d, \mathbb{R}^{d\times d})$ is differentiable at $t = 0$ with derivative given by $\mathrm{M}_0' := -\nabla\theta - \nabla\theta^\top + \mathrm{div}(\theta)\mathrm{I}$;
- (iv) the map $t \in \mathbb{R}_+ \mapsto g_t\mathrm{J}_{\mathrm{T}_t} \in \mathrm{L}^2(\Gamma_0)$ is differentiable at $t = 0$ with derivative given by $\nabla g\cdot\theta + g\mathrm{div}_\tau(\theta)$;

where $\mathrm{div}_\tau$ is the tangential divergence (see Proposition 2.1.4). In particular, if we assume that for almost all $s \in \Gamma_0$, $g$ has a directional derivative at $s$ in any direction, then the map $t \in \mathbb{R}_+ \mapsto g_t\mathrm{J}_{\mathrm{T}_t}(s) \in \mathbb{R}_+$ is differentiable at $t = 0$. Therefore one can apply Theorem 5.2.14 to characterize the material derivative.

**Theorem 8.1.1** (Material directional derivative)**.** *Let $\overline{u}_t \in \mathrm{H}^1(\Omega_0)$ be the unique solution to the perturbed general scalar Tresca friction problem* (PGSTP$_t$)*. Assume that*

- (i) *for almost all $s \in \Gamma_0$, $g$ has a directional derivative at $s$ in any direction;*
- (ii) *$\Phi$ is twice epi-differentiable at $u_0$ for $F_0 - u_0 \in \partial\Phi(0,\cdot)(u_0)$ with*

$$\mathrm{D}_e^2\Phi(u_0|F_0 - u_0)(w) = \int_{\Gamma_0} \mathrm{D}_e^2 G(s)(u_0(s)|\partial_{\mathrm{n}}(F_0 - u_0)(s))(v(s))\,\mathrm{d}s,$$

*for all* $v \in \mathrm{H}^1(\Omega)$.

*Then the map* $t \in \mathbb{R}_+ \mapsto \overline{u}_t \in \mathrm{H}^1(\Omega_0)$ *is differentiable at* $t = 0$*, and its derivative (that is the material directional derivative), denoted by* $\overline{u}_0' \in \mathcal{K}_{u_0, \frac{\partial_{\mathrm{n}}(F_0 - u_0)}{g}} \subset \mathrm{H}^1(\Omega_0)$*, is the unique solution to the variational inequality*

$$\begin{aligned}\langle \overline{u}_0', v - \overline{u}_0' \rangle_{\mathrm{H}^1(\Omega_0)} \geq & \int_{\Omega_0} \theta \cdot \nabla u_0 \, (v - \overline{u}_0') - \int_{\Omega_0} \mathrm{div}\,(\Delta u_0 \theta)\,(v - \overline{u}_0') \\ & - \int_{\Omega_0} \left( \left( -\nabla\theta - \nabla\theta^\top + \mathrm{div}(\theta)\mathrm{I} \right) \nabla u_0 \right) \cdot \nabla(v - \overline{u}_0') \\ & + \int_{\Gamma_0} \left( \frac{\nabla g}{g} \cdot \theta + \mathrm{div}_\tau(\theta) \right) \partial_{\mathrm{n}} u_0 \, (v - \overline{u}_0'), \qquad \forall v \in \mathcal{K}_{u_0, \frac{\partial_{\mathrm{n}}(F_0 - u_0)}{g}}, \end{aligned} \tag{8.2}$$

*where* $\mathcal{K}_{u_0, \frac{\partial_{\mathrm{n}}(F_0 - u_0)}{g}}$ *is the nonempty closed convex subset of* $\mathrm{H}^1(\Omega_0)$ *defined by*

$$\mathcal{K}_{u_0, \frac{\partial_{\mathrm{n}}(F_0 - u_0)}{g}} := \left\{ v \in \mathrm{H}^1(\Omega_0) \mid v \leq 0 \text{ a.e. on } \Gamma_{\mathrm{S}-}^{u_0,g},\ v \geq 0 \text{ a.e. on } \Gamma_{\mathrm{S}+}^{u_0,g},\ v = 0 \text{ a.e. on } \Gamma_{\mathrm{D}}^{u_0,g} \right\},$$

*where* $\Gamma_0$ *is decomposed, up to a null set, as* $\Gamma_{\mathrm{N}}^{u_0,g} \cup \Gamma_{\mathrm{D}}^{u_0,g} \cup \Gamma_{\mathrm{S}-}^{u_0,g} \cup \Gamma_{\mathrm{S}+}^{u_0,g}$*, where*

$$\begin{aligned} \Gamma_{\mathrm{N}}^{u_0,g} &:= \{ s \in \Gamma_0 \mid u_0(s) \neq 0 \}, \\ \Gamma_{\mathrm{D}}^{u_0,g} &:= \{ s \in \Gamma_0 \mid u_0(s) = 0 \text{ and } \partial_{\mathrm{n}} u_0(s) \in (-g(s), g(s)) \}, \\ \Gamma_{\mathrm{S}-}^{u_0,g} &:= \{ s \in \Gamma_0 \mid u_0(s) = 0 \text{ and } \partial_{\mathrm{n}} u_0(s) = g(s) \}, \\ \Gamma_{\mathrm{S}+}^{u_0,g} &:= \{ s \in \Gamma_0 \mid u_0(s) = 0 \text{ and } \partial_{\mathrm{n}} u_0(s) = -g(s) \}. \end{aligned}$$

*Proof.* From Theorem 5.2.14 it follows that $\overline{u}_0' \in \mathcal{K}_{u_0, \frac{\partial_{\mathrm{n}}(F_0 - u_0)}{g}}$ is the unique solution to the variational inequality

$$\begin{aligned}\langle \overline{u}_0', v - \overline{u}_0' \rangle_{\mathrm{H}^1(\Omega_0)} \geq & \int_{\Omega_0} \mathrm{div}(f\theta)\,(v - \overline{u}_0') - \int_{\Omega_0} \mathrm{div}(\theta) u_0 \,(v - \overline{u}_0') \\ & - \int_{\Omega_0} \left( -\nabla\theta - \nabla\theta^\top + \mathrm{div}(\theta)\mathrm{I} \right) \nabla u_0 \cdot \nabla(v - \overline{u}_0') + \int_{\Gamma_0} \left( \nabla g \cdot \theta + g \mathrm{div}_\tau(\theta) \right) \frac{\partial_{\mathrm{n}} u_0}{g} \,(v - \overline{u}_0'), \end{aligned}$$

for all $v \in \mathcal{K}_{u_0, \frac{\partial_{\mathrm{n}}(F_0 - u_0)}{g}}$. Using the equality $-\Delta u_0 + u_0 = f$ in $\mathrm{H}^1(\Omega_0)$, we obtain the result. □

**Remark 8.1.2.** Consider the framework of Theorem 8.1.1 which is dependent on $\theta \in \mathcal{C}^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ and let us denote by $\overline{u}_0'(\theta) := \overline{u}_0'$. One can easily see that

$$\overline{u}_0'(\alpha_1\theta_1 + \alpha_2\theta_2) = \alpha_1 \overline{u}_0'(\theta_1) + \alpha_2 \overline{u}_0'(\theta_2).$$

for any $\theta_1$, $\theta_2 \in \mathcal{C}^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ and for any nonnegative real numbers $\alpha_1 \geq 0$, $\alpha_2 \geq 0$. However, this is not true for negative real numbers and justify why, in the present work, we call $\overline{u}_0'$ as material *directional* derivative (instead of simply material derivative as usually in the literature). This nonlinearity is standard in shape optimization for variational inequalities (see, e.g., [47] or [78, Section 4]).

The presentation of Theorem 8.1.1 can be improved under additional regularity assumptions.

**Corollary 8.1.3.** *Consider the framework of Theorem 8.1.1 with the additional assumptions that* $u_0 \in \mathrm{H}^3(\Omega_0)$ *and* $\theta \in \mathcal{C}^{2,\infty}(\mathbb{R}^d,\mathbb{R}^d) := \mathcal{C}^2(\mathbb{R}^d,\mathbb{R}^d) \cap \mathrm{W}^{2,\infty}(\mathbb{R}^d,\mathbb{R}^d)$. *Then the material derivative* $\overline{u}_0' \in \mathrm{H}^1(\Omega_0)$ *is the unique weak solution to the scalar Signorini problem given by*

$$\begin{cases} -\Delta\overline{u}_0' + \overline{u}_0' = -\Delta\left(\theta\cdot\nabla u_0\right) + \theta\cdot\nabla u_0 & \text{in } \Omega_0,\\ \overline{u}_0' = 0 & \text{on } \Gamma_{\mathrm{D}}^{u_0,g},\\ \partial_{\mathrm{n}}\overline{u}_0' = h^m(\theta) & \text{on } \Gamma_{\mathrm{N}}^{u_0,g},\\ \overline{u}_0' \leq 0,\ \partial_{\mathrm{n}}\overline{u}_0' \leq h^m(\theta) \text{ and } \overline{u}_0'\left(\partial_{\mathrm{n}}\overline{u}_0' - h^m(\theta)\right) = 0 & \text{on } \Gamma_{\mathrm{S-}}^{u_0,g},\\ \overline{u}_0' \geq 0,\ \partial_{\mathrm{n}}\overline{u}_0' \geq h^m(\theta) \text{ and } \overline{u}_0'\left(\partial_{\mathrm{n}}\overline{u}_0' - h^m(\theta)\right) = 0 & \text{on } \Gamma_{\mathrm{S+}}^{u_0,g}, \end{cases}$$

*where* $h^m(\theta) := (\frac{\nabla g}{g}\cdot\theta - \nabla\theta\mathrm{n}\cdot\mathrm{n})\partial_{\mathrm{n}}u_0 + (\nabla\theta + \nabla\theta^\top)\nabla u_0\cdot\mathrm{n} \in \mathrm{L}^2(\Gamma_0)$.

*Proof.* Since $u_0 \in \mathrm{H}^2(\Omega_0)$ and $\theta \in \mathcal{C}^{2,\infty}(\mathbb{R}^d,\mathbb{R}^d)$, we deduce that $\mathrm{div}((-\nabla\theta - \nabla\theta^\top + \mathrm{div}(\theta)\mathrm{I})\nabla u_0) \in \mathrm{L}^2(\Omega_0)$ and that $(-\nabla\theta - \nabla\theta^\top + \mathrm{div}(\theta)\mathrm{I})\nabla u_0\cdot\mathrm{n} \in \mathrm{L}^2(\Gamma_0)$. Thus, in the same way as Corollary 5.2.15, using the divergence formula in Inequality (8.2), we get that

$$\begin{aligned} \langle\overline{u}_0', v - \overline{u}_0'\rangle_{\mathrm{H}^1(\Omega_0)} \geq & \int_{\Omega_0}\theta\cdot\nabla u_0\,(v-\overline{u}_0') - \int_{\Omega_0}\mathrm{div}\left(\Delta u_0\theta\right)(v-\overline{u}_0')\\ &+ \int_{\Omega_0}\mathrm{div}\left(\left(-\nabla\theta - \nabla\theta^\top + \mathrm{div}(\theta)\mathrm{I}\right)\nabla u_0\right)(v-\overline{u}_0')\\ &+ \int_{\Gamma_0}\left(\left(\nabla\theta + \nabla\theta^\top\right)\nabla u_0\cdot\mathrm{n} + \left(\frac{\nabla g}{g}\cdot\theta - \nabla\theta\mathrm{n}\cdot\mathrm{n}\right)\partial_{\mathrm{n}}u_0\right)(v-\overline{u}_0'), \end{aligned} \tag{8.3}$$

for all $v \in \mathcal{K}_{u_0,\frac{\partial_{\mathrm{n}}(F_0-u_0)}{g}}$. Furthermore, one has $\Delta(\theta\cdot\nabla u_0) \in \mathrm{L}^2(\Omega_0)$ from $u_0 \in \mathrm{H}^3(\Omega_0)$, thus, using Proposition 2.1.3, it follows that

$$\langle\overline{u}_0', v - \overline{u}_0'\rangle_{\mathrm{H}^1(\Omega_0)} \geq \int_{\Omega_0} -\Delta\left(\theta\cdot\nabla u_0\right)(v-\overline{u}_0') + \int_{\Omega_0}\theta\cdot\nabla u_0\,(v-\overline{u}_0') + \int_{\Gamma_0}h^m(\theta)\,(v-\overline{u}_0'),$$

for all $v \in \mathcal{K}_{u_0,\frac{\partial_{\mathrm{n}}(F_0-u_0)}{g}}$ which concludes the proof from Subsection 4.1.2. □

**Remark 8.1.4.** As mentioned in Remark 5.2.16, if $\Gamma_0$ is sufficiently regular, then $u_0 \in \mathrm{H}^2(\Omega_0)$, and this is the best regularity result that can be obtained. We refer to [14, Chapter 1, Theorem I.10 p.43] and [14, Chapter 1, Remark I.26 p.47] for details. It does not mean that $u_0 \notin \mathrm{H}^3(\Omega_0)$ in general. It just means that, in this reference, there is a counterexample in which $u_0 \notin \mathrm{H}^3(\Omega_0)$ even if $\Gamma_0$ is very smooth. Note that, from Corollary 5.2.15, one can get, under the weaker assumption $u_0 \in \mathrm{H}^2(\Omega_0)$, that the material directional derivative $\overline{u}_0'$ is the weak solution to a scalar Signorini problem, and more precisely, the solution to the variational inequality (8.3) which is, from Subsection 4.1.2, the weak formulation of a scalar Signorini problem with the source term given by $-\mathrm{div}((\Delta u_0)\,\theta - \mathrm{div}(\theta)\nabla u_0 + (\nabla\theta + \nabla\theta^\top)\nabla u_0) + \theta\cdot\nabla u_0 \in \mathrm{L}^2(\Omega_0)$.

Thanks to Corollary 8.1.3, we are now in a position to characterize the shape directional derivative.

**Corollary 8.1.5** (Shape directional derivative)**.** *Consider the framework of Corollary 8.1.3 with the additional assumption that* $\Gamma_0$ *is of class* $\mathcal{C}^3$. *Then the shape directional derivative, defined by* $u_0' :=$

$\overline{u}_0' - \nabla u_0 \cdot \theta \in \mathrm{H}^1(\Omega_0)$, *is the unique weak solution to the scalar Signorini problem given by*

$$\begin{cases} -\Delta u_0' + u_0' = 0 & \text{in } \Omega_0, \\ u_0' = -\theta \cdot \nabla u_0 & \text{on } \Gamma_{\mathrm{D}}^{u_0,g}, \\ \partial_{\mathrm{n}} u_0' = h^s(\theta) & \text{on } \Gamma_{\mathrm{N}}^{u_0,g}, \\ u_0' \leq -\theta\cdot\nabla u_0,\ \partial_{\mathrm{n}} u_0' \leq h^s(\theta) \text{ and } (u_0' + \theta\cdot\nabla u_0)(\partial_{\mathrm{n}} u_0' - h^s(\theta)) = 0 & \text{on } \Gamma_{\mathrm{S-}}^{u_0,g}, \\ u_0' \geq -\theta\cdot\nabla u_0,\ \partial_{\mathrm{n}} u_0' \geq h^s(\theta) \text{ and } (u_0' + \theta\cdot\nabla u_0)(\partial_{\mathrm{n}} u_0' - h^s(\theta)) = 0 & \text{on } \Gamma_{\mathrm{S+}}^{u_0,g}, \end{cases}$$

*where* $h^s(\theta) := \theta\cdot \mathrm{n}(\partial_{\mathrm{n}}(\partial_{\mathrm{n}} u_0) - \frac{\partial^2 u_0}{\partial \mathrm{n}^2}) + \nabla_\tau u_0 \cdot \nabla_\tau(\theta\cdot\mathrm{n}) - g\nabla(\frac{\partial_{\mathrm{n}} u_0}{g})\cdot\theta \in \mathrm{L}^2(\Gamma_0)$.

*Proof.* From the weak variational formulation of $\overline{u}_0'$ given in Corollary 8.1.3 and using the divergence formula, one can easily obtain that

$$\langle u_0', v - \theta\cdot\nabla u_0 - u_0'\rangle_{\mathrm{H}^1(\Omega_0)} \geq \int_{\Gamma_0} (h^m(\theta) - \nabla(\theta\cdot\nabla u_0)\cdot\mathrm{n})(v - \theta\cdot\nabla u_0 - u_0'),$$

for all $v \in \mathcal{K}_{u_0, \frac{\partial_{\mathrm{n}}(F_0 - u_0)}{g}}$, which can be rewritten as

$$\langle u_0', w - u_0'\rangle_{\mathrm{H}^1(\Omega_0)} \geq \int_{\Gamma_0} (h^m(\theta) - \nabla(\theta\cdot\nabla u_0)\cdot\mathrm{n})(w - u_0'),$$

for all $w \in \mathcal{K}_{u_0, \frac{\partial_{\mathrm{n}}(F_0 - u_0)}{g}} - \theta\cdot\nabla u_0$. Since $\Gamma_0$ is of class $\mathcal{C}^3$ and $u_0 \in \mathrm{H}^3(\Omega_0)$, the normal derivative of $u_0$ can be extended into a function defined in $\Omega_0$ such that $\partial_{\mathrm{n}} u_0 \in \mathrm{H}^2(\Omega_0)$. Thus, it holds that $v\partial_{\mathrm{n}} u_0 \in \mathrm{W}^{2,1}(\Omega_0)$ for all $v \in \mathcal{C}^\infty(\overline{\Omega_0})$, and one can use Propositions 2.1.3 and 2.1.4 to obtain that

$$\begin{aligned}\int_{\Gamma_0} &(h^m(\theta) - \nabla(\theta\cdot\nabla u_0)\cdot\mathrm{n})\, v \\ &= \int_{\Gamma_0} \theta\cdot\mathrm{n}\,(-\nabla u_0\cdot\nabla v - u_0 v + f v + H v \partial_{\mathrm{n}} u_0 + \partial_{\mathrm{n}}(v\partial_{\mathrm{n}} u_0)) - \int_{\Gamma_0} g v \nabla\left(\frac{\partial_{\mathrm{n}} u_0}{g}\right)\cdot\theta,\end{aligned}$$

for all $v \in \mathcal{C}^\infty(\overline{\Omega_0})$. Then, by using Proposition 2.1.6, one deduces that

$$\begin{aligned}\int_{\Gamma_0} &(h^m(\theta) - \nabla(\theta\cdot\nabla u_0)\cdot\mathrm{n})\, v \\ &= \int_{\Gamma_0} \left(\theta\cdot\mathrm{n}\left(\partial_{\mathrm{n}}(\partial_{\mathrm{n}} u_0) - \frac{\partial^2 u_0}{\partial \mathrm{n}^2}\right) + \nabla_\tau u_0\cdot\nabla_\tau(\theta\cdot\mathrm{n}) - g\nabla\left(\frac{\partial_{\mathrm{n}} u_0}{g}\right)\cdot\theta\right) v,\end{aligned}$$

for all $v \in \mathcal{C}^\infty(\overline{\Omega_0})$, and also for all $v \in \mathrm{H}^1(\Omega_0)$ by density. Thus it follows that

$$\begin{aligned}&\langle u_0', w - u_0'\rangle_{\mathrm{H}^1(\Omega_0)} \\ &\geq \int_{\Gamma_0} \left(\theta\cdot\mathrm{n}\left(\partial_{\mathrm{n}}(\partial_{\mathrm{n}} u_0) - \frac{\partial^2 u_0}{\partial \mathrm{n}^2}\right) + \nabla_\tau u_0\cdot\nabla_\tau(\theta\cdot\mathrm{n}) - g\nabla\left(\frac{\partial_{\mathrm{n}} u_0}{g}\right)\cdot\theta\right)(w - u_0'),\end{aligned}$$

for all $w \in \mathcal{K}_{u_0, \frac{\partial_{\mathrm{n}}(F_0 - u_0)}{g}} - \theta\cdot\nabla u_0$, which concludes the proof from Subsection 4.1.2. □

## 8.2 Shape gradient of the Tresca energy functional

Thanks to the characterization of the material directional derivative obtained in Theorem 8.1.1, we can now prove the shape differentiability of the Tresca energy functional.

**Theorem 8.2.1.** *Consider the framework of Theorem 8.1.1. Then the Tresca energy functional $\mathcal{J}$ admits a shape gradient at $\Omega_0$ in any direction $\theta \in \mathcal{C}^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ given by*

$$\mathcal{J}'(\Omega_0)(\theta) = \frac{1}{2}\int_{\Omega_0} \mathrm{div}(\theta) \left\|\nabla u_0\right\|^2 - \int_{\Omega_0} \nabla u_0 \cdot (\nabla\theta\nabla u_0 + \Delta u_0 \theta) + \int_{\Gamma_0} \left(\theta \cdot \mathrm{n}\left(\frac{|u_0|^2}{2} - f u_0\right) - \left(\frac{\nabla g}{g}\cdot\theta + \mathrm{div}_\tau(\theta)\right) u_0 \partial_{\mathrm{n}} u_0\right). \quad (8.4)$$

*Proof.* By following the usual strategy developed in the shape optimization literature (see, e.g., [8, 46]) to compute the shape gradient of $\mathcal{J}$ at $\Omega_0$ in a direction $\theta \in \mathcal{C}^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$, one gets

$$\mathcal{J}'(\Omega_0)(\theta) = -\frac{1}{2}\int_{\Omega_0}\left(\left\|\nabla u_0\right\|^2 + |u_0|^2\right)\mathrm{div}(\theta) + \int_{\Omega_0} \nabla u_0 \cdot \nabla\theta\nabla u_0 - \langle \overline{u}_0', u_0 \rangle_{\mathrm{H}^1(\Omega_0)}.$$

On the other hand, since $\overline{u}_0' \pm u_0 \in \mathcal{K}_{u_0, \frac{\partial_{\mathrm{n}}(F_0 - u_0)}{g}}$, we deduce from the weak variational formulation of $\overline{u}_0'$ that

$$\langle \overline{u}_0', u_0 \rangle_{\mathrm{H}^1(\Omega_0)} = \int_{\Omega_0} u_0\theta \cdot \nabla u_0 - \int_{\Omega_0} \mathrm{div}\left(\Delta u_0 \theta\right) u_0 - \int_{\Omega_0} \left(\left(-\nabla\theta - \nabla\theta^\top + \mathrm{div}(\theta)\mathrm{I}\right)\nabla u_0\right)\cdot\nabla u_0 + \int_{\Gamma_0}\left(\frac{\nabla g}{g}\cdot\theta + \mathrm{div}_\tau(\theta)\right) u_0 \partial_{\mathrm{n}} u_0.$$

The proof is complete thanks to the divergence formula. □

As we did in Corollary 8.1.3 for the material directional derivative, the presentation of Theorem 8.2.1 can be improved under additional assumptions.

**Corollary 8.2.2.** *Consider the framework of Theorem 8.2.1 with the additional assumptions that $d \in \{1, 2, 3, 4, 5\}$, $\Gamma_0$ is of class $\mathcal{C}^3$ and $u_0 \in \mathrm{H}^3(\Omega_0)$. Then the shape gradient of the Tresca energy functional $\mathcal{J}$ at $\Omega_0$ in any direction $\theta \in \mathcal{C}^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ is given by*

$$\mathcal{J}'(\Omega_0)(\theta) = \int_{\Gamma_0} \theta\cdot\mathrm{n}\left(\frac{\left\|\nabla u_0\right\|^2 + |u_0|^2}{2} - f u_0 + H g\,|u_0| - \partial_{\mathrm{n}}\left(u_0\partial_{\mathrm{n}} u_0\right) + g u_0 \nabla\left(\frac{\partial_{\mathrm{n}} u_0}{g}\right)\cdot\mathrm{n}\right),$$

*where $H$ is the mean curvature of $\Gamma_0$.*

*Proof.* Let $\theta \in \mathcal{C}^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$. Since $u_0 \in \mathrm{H}^2(\Omega_0) \subset \mathrm{H}^3(\Omega_0)$, it holds that

$$\int_{\Omega_0} \mathrm{div}(\theta)\left\|\nabla u_0\right\|^2 = -\int_{\Omega_0}\theta\cdot\nabla\left(\left\|\nabla u_0\right\|^2\right) + \int_{\Gamma_0}\theta\cdot\mathrm{n}\left\|\nabla u_0\right\|^2,$$

and

$$\int_{\Omega_0}\Delta u_0\theta\cdot\nabla u_0 = -\int_{\Omega_0}\nabla u_0\cdot\nabla(\theta\cdot\nabla u_0) + \int_{\Gamma_0}\partial_{\mathrm{n}} u_0\theta\cdot\nabla u_0.$$

One deduces from (8.4) that

$$\mathcal{J}'(\Omega_0)(\theta) = \int_{\Gamma_0} \theta \cdot \mathrm{n} \left( \frac{\|\nabla u_0\|^2 + |u_0|^2}{2} - f u_0 \right) \\ - \int_{\Gamma_0} \left( \partial_{\mathrm{n}} u_0 \theta \cdot \nabla u_0 + \left( \frac{\nabla g}{g} \cdot \theta + \mathrm{div}_\tau(\theta) \right) u_0 \partial_{\mathrm{n}} u_0 \right). \quad (8.5)$$

Moreover, since $\Gamma_0$ is of class $\mathcal{C}^3$ and $u_0 \in \mathrm{H}^3(\Omega_0)$, the normal derivative of $u_0$ can be extended into a function defined in $\Omega_0$ such that $\partial_{\mathrm{n}} u_0 \in \mathrm{H}^2(\Omega_0)$. Therefore, using Proposition 2.1.4 with $v = u_0 \partial_{\mathrm{n}} u_0 \in \mathrm{W}^{2,1}(\Omega_0)$, one gets

$$\mathcal{J}'(\Omega_0)(\theta) = \int_{\Gamma_0} \theta \cdot \mathrm{n} \left( \frac{\|\nabla u_0\|^2 + |u_0|^2}{2} - f u_0 - H u_0 \partial_{\mathrm{n}} u_0 - \partial_{\mathrm{n}} \left( u_0 \partial_{\mathrm{n}} u_0 \right) \right) + \int_{\Gamma_0} g u_0 \nabla \left( \frac{\partial_{\mathrm{n}} u_0}{g} \right) \cdot \theta.$$

From the scalar Tresca friction law, one has $H u_0 \partial_{\mathrm{n}} u_0 = -H g |u_0|$ *a.e.* on $\Gamma_0$. Now let us focus on the last term. Since $u_0 = 0$ on $\Gamma_{\mathrm{D}}^{u_0,g} \cup \Gamma_{\mathrm{S}-}^{u_0,g} \cup \Gamma_{\mathrm{S}+}^{u_0,g}$, we have

$$\int_{\Gamma_0} g u_0 \nabla \left( \frac{\partial_{\mathrm{n}} u_0}{g} \right) \cdot \theta = \int_{\Gamma_{\mathrm{N}}^{u_0,g}} g u_0 \nabla \left( \frac{\partial_{\mathrm{n}} u_0}{g} \right) \cdot \theta.$$

Let us introduce two disjoint subsets of $\Gamma_0$ given by

$$\Gamma_{\mathrm{N}+}^{u_0,g} := \{ s \in \Gamma_0 \mid u_0(s) > 0 \} \qquad \text{and} \qquad \Gamma_{\mathrm{N}-}^{u_0,g} := \{ s \in \Gamma_0 \mid u_0(s) < 0 \}.$$

Hence it follows that $\Gamma_{\mathrm{N}}^{u_0,g} = \Gamma_{\mathrm{N}+}^{u_0,g} \cup \Gamma_{\mathrm{N}-}^{u_0,g}$, with $\partial_{\mathrm{n}} u_0 = -g$ *a.e.* on $\Gamma_{\mathrm{N}+}^{u_0,g}$, and $\partial_{\mathrm{n}} u_0 = g$ *a.e.* on $\Gamma_{\mathrm{N}-}^{u_0,g}$. Moreover, since $u_0 \in \mathrm{H}^3(\Omega)$ and $d \in \{1,2,3,4,5\}$, we get from Sobolev embeddings (see, e.g., [1, Chapter 4, p.79]) that $u_0$ is continuous over $\Gamma_0$, thus $\Gamma_{\mathrm{N}+}^{u_0,g}$ and $\Gamma_{\mathrm{N}-}^{u_0,g}$ are open subsets of $\Gamma_0$. Hence $\nabla_\tau(\frac{\partial_{\mathrm{n}} u_0}{g}) = 0$ *a.e.* on $\Gamma_{\mathrm{N}+}^{u_0,g} \cup \Gamma_{\mathrm{N}-}^{u_0,g}$, and one deduces that

$$\int_{\Gamma_{\mathrm{N}}^{u_0,g}} g u_0 \nabla \left( \frac{\partial_{\mathrm{n}} u_0}{g} \right) \cdot \theta = \int_{\Gamma_{\mathrm{N}}^{u_0,g}} \theta \cdot \mathrm{n} \left( g u_0 \nabla \left( \frac{\partial_{\mathrm{n}} u_0}{g} \right) \cdot \mathrm{n} \right),$$

which concludes the proof. □

**Remark 8.2.3.** Under the weaker condition $u_0 \in \mathrm{H}^2(\Omega_0)$ (satisfied if $\Gamma_0$ is sufficiently regular, see Remark 8.1.4), one can follow the proof of Corollary 8.2.2 and obtain that the shape gradient of $\mathcal{J}$ is given by Equality (8.5).

**Remark 8.2.4.** Consider the framework of Theorem 8.2.1. We have seen in Remark 8.1.2 that the expression of the material directional derivative $\overline{u}_0'$ is not linear with respect to $\theta$. However one can observe that the scalar product $\langle \overline{u}_0', u_0 \rangle_{\mathrm{H}^1(\Omega_0)}$, that appears in the proof of Theorem 8.2.1, is linear. This leads to an expression of the shape gradient $\mathcal{J}'(\Omega_0)(\theta)$ in Theorem 8.2.1 that is linear with respect to $\theta$. Hence we deduce that the Tresca energy functional $\mathcal{J}$ is shape differentiable at $\Omega_0$. Furthermore note that the shape gradient $\mathcal{J}'(\Omega_0)(\theta)$ depends only on $u_0$ (and not on $u_0'$)

and therefore does not require the introduction of an appropriate adjoint problem to be computed explicitly. The linear explicit expression of $\mathcal{J}'(\Omega_0)(\theta)$ with respect to the direction $\theta$ will allow us in the next Section 8.3 to exhibit a descent direction for numerical simulations in order to solve the shape optimization problem (8.1) on a two-dimensional example. It is worth noting that all previous comments are specific to the Tresca energy functional $\mathcal{J}$.

**Remark 8.2.5.** Let us recall that the standard Neumann energy functional is

$$\mathcal{J}_{\mathrm{N}}(\Omega) := \frac{1}{2}\int_{\Omega}\left(\|\nabla w_{\mathrm{N},\Omega}\|^2 + |w_{\mathrm{N},\Omega}|^2\right) + \int_{\Gamma} g w_{\mathrm{N},\Omega} - \int_{\Omega} f w_{\mathrm{N},\Omega},$$

for all $\Omega \in \mathcal{U}$, where $w_{\mathrm{N},\Omega} \in \mathrm{H}^1(\Omega)$ is the unique solution to the standard Neumann problem

$$\begin{cases} -\Delta w_{\mathrm{N},\Omega} + w_{\mathrm{N},\Omega} = f & \text{in } \Omega, \\ \partial_{\mathrm{n}} w_{\mathrm{N},\Omega} = -g & \text{on } \Gamma. \end{cases} \tag{SNP$_\Omega$}$$

One can prove (see, e.g., [8, 46]) that the shape gradient of the Neumann energy functional $\mathcal{J}_{\mathrm{N}}$ at $\Omega_0 \in \mathcal{U}$ in any direction $\theta \in \mathcal{C}^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ is given by

$$\mathcal{J}_{\mathrm{N}}'(\Omega_0)(\theta) = \int_{\Gamma_0} \theta\cdot\mathrm{n}\left(\frac{\|\nabla w_{\mathrm{N},\Omega_0}\|^2 + |w_{\mathrm{N},\Omega_0}|^2}{2} - f w_{\mathrm{N},\Omega_0} + H g w_{\mathrm{N},\Omega_0} + \partial_{\mathrm{n}}\left(g w_{\mathrm{N},\Omega_0}\right)\right).$$

Thus the shape gradient of the Tresca energy functional $\mathcal{J}$ obtained in Corollary 8.2.2 is close to the one of $\mathcal{J}_{\mathrm{N}}$ with the additional term

$$\int_{\Gamma_0} g u_0 \nabla\left(\frac{\partial_{\mathrm{n}} u_0}{g}\right)\cdot\theta.$$

Note that, if $\partial_{\mathrm{n}} u_0 = -g$ *a.e.* on $\Gamma_0$, then they coincide.

**Remark 8.2.6.** Let us recall that the standard Dirichlet energy functional is

$$\mathcal{J}_{\mathrm{D}}(\Omega) := \frac{1}{2}\int_{\Omega}\left(\|\nabla w_{\mathrm{D},\Omega}\|^2 + |w_{\mathrm{D},\Omega}|^2\right) - \int_{\Omega} f w_{\mathrm{D},\Omega},$$

for all $\Omega \in \mathcal{U}$, where $w_{\mathrm{D},\Omega} \in \mathrm{H}^1(\Omega)$ is the unique solution to the Dirichlet problem

$$\begin{cases} -\Delta w_{\mathrm{D},\Omega} + w_{\mathrm{D},\Omega} = f & \text{in } \Omega, \\ w_{\mathrm{D},\Omega} = 0 & \text{on } \Gamma. \end{cases} \tag{SDP$_\Omega$}$$

One can prove (see, e.g., [8, 46]) that the shape gradient of $\mathcal{J}_{\mathrm{D}}$ at $\Omega_0 \in \mathcal{U}$ in any direction $\theta \in \mathcal{C}^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ is given by

$$\mathcal{J}_{\mathrm{D}}'(\Omega_0)(\theta) = -\int_{\Gamma_0} \theta\cdot\mathrm{n}\left(\frac{\|\nabla w_{\mathrm{D},\Omega_0}\|^2 + |w_{\mathrm{D},\Omega_0}|^2}{2}\right).$$

Note that, if $u_0 = 0$ *a.e.* on $\Gamma_0$, then $\nabla_\tau u_0 = 0$ *a.e.* on $\Gamma_0$, thus $(\partial_{\mathrm{n}} u_0)^2 = ||\nabla u_0||^2$ *a.e.* on $\Gamma_0$ and thus the shape gradient of $\mathcal{J}$ obtained in Corollary 8.2.2 coincides with the one of $\mathcal{J}_{\mathrm{D}}$.

## 8.3 Numerical simulations

In this section we numerically solve an example of the shape optimization problem (8.1) in the two-dimensional case $d = 2$, by making use of our theoretical results obtained in Section 8.2. The numerical simulations have been performed using Freefem++ software [44] with P1-finite elements and standard affine mesh. We could use the expression of the shape gradient of $\mathcal{J}$ obtained in Theorem 8.2.1 but, for the purpose of simplifying the computations, we chose to use the expression provided in Corollary 8.2.2 under additional assumptions (such as $u_0 \in \mathrm{H}^3(\Omega_0)$ that we assumed to be true at each iteration). The $\mathcal{C}^3$ regularity of the shapes required in Corollary 8.2.2 is not satisfied since we use a classical affine mesh and thus the discretized domains have boundaries that are only Lipschitz. Nevertheless it could be possible to impose more regularity by using curved mesh for example. However the use of such numerical techniques falls outside the scope of this work in which the numerical simulations are intended to illustrate our theoretical results.

### 8.3.1 Numerical methodology

Consider an initial shape $\Omega_0 \in \mathcal{U}$ (see the beginning of Section 8 for the definition of $\mathcal{U}$). Note that Corollary 8.2.2 allows to exhibit a descent direction $\theta_0$ of the Tresca energy functional $\mathcal{J}$ at $\Omega_0$ as the unique solution to the Neumann problem

$$\begin{cases} -\Delta\theta_0 + \theta_0 = 0 & \text{in } \Omega_0, \\ \nabla\theta_0 \mathrm{n} = -\left(\frac{\|\nabla u_0\|^2 + |u_0|^2}{2} - f u_0 + Hg\,|u_0| - \partial_{\mathrm{n}}\left(u_0 \partial_{\mathrm{n}} u_0\right) + g u_0 \nabla\left(\frac{\partial_{\mathrm{n}} u_0}{g}\right) \cdot \mathrm{n}\right)\mathrm{n} & \text{on } \Gamma_0, \end{cases}$$

since it satisfies $\mathcal{J}'(\Omega_0)(\theta_0) = -\left\|\theta_0\right\|^2_{\mathrm{H}^1(\Omega_0,\mathbb{R}^d)} \leq 0$.

In order to numerically solve the shape optimization problem (8.1) on a given example, we also have to deal with the volume constraint $|\Omega| = \lambda > 0$. To this aim, the Uzawa algorithm (see, e.g., [8, Chapter 3 p.64] or [20]) is used. In a nutshell it consists in augmenting the Tresca energy functional $\mathcal{J}$ by adding an initial Lagrange multiplier $p_0 \in \mathbb{R}$ multiplied by the standard volume functional minus $\lambda$. From [8, Chapter 6, Section 6.5], we know that the shape gradient of the volume functional at $\Omega_0$ is given by

$$\theta \in \mathcal{C}^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d) \mapsto \int_{\Gamma_0} \theta \cdot \mathrm{n} \in \mathbb{R},$$

and thus one can easily obtain a descent direction $\theta_0(p_0)$ of the *augmented* Tresca energy functional at $\Omega_0$ by adding $p_0\mathrm{n}$ in the Neumann boundary condition of $\theta_0$. This descent direction leads to a new shape $\Omega_1 := (\mathrm{id} + \tau\theta_0(p_0))(\Omega_0)$, where $\tau > 0$ is a fixed parameter. Finally the Lagrange multiplier is updated as follows

$$p_1 := p_0 + \eta\left(|\Omega_1| - \lambda\right),$$

where $\eta > 0$ is a fixed parameter, and the algorithm restarts with $\Omega_1$ and $p_1$, and so on.

Let us mention that the scalar Tresca friction problem is numerically solved using an adaptation of iterative switching algorithm (see Appendix B for detailed explanations). We also precise that, for all $i \in \mathbb{N}^*$, the difference between the Tresca energy functional $\mathcal{J}$ at the iteration $20 \times i$ and at the iteration $20 \times (i-1)$ is computed. The smallness of this difference is used as a stopping criterion for the algorithm. Finally the curvature term $H$ is numerically computed by extending the normal $\mathrm{n}$ into a function $\tilde{\mathrm{n}}$ which is defined on the whole domain $\Omega_0$. Then the curvature is given by $H = \mathrm{div}(\tilde{\mathrm{n}}) - \nabla(\tilde{\mathrm{n}})\mathrm{n} \cdot \mathrm{n}$ (see, e.g., [46, Proposition 5.4.8 p.194]).

### 8.3.2 Example and numerical results

In this subsection, take $d = 2$ and $f \in \mathrm{H}^1(\mathbb{R}^2)$ given by

$$\begin{array}{rccl} f: & \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ & (x,y) & \longmapsto & f(x,y) = \dfrac{5 - x^2 - y^2 + xy}{4}\eta(x,y), \end{array}$$

and, for a given parameter $\beta > 0$, let $g_\beta \in \mathrm{H}^2(\mathbb{R}^2)$ be given by

$$\begin{array}{rccl} g_\beta: & \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ & (x,y) & \longmapsto & g(x,y) = \beta\left(1 + \dfrac{(\sin x)^2}{0.8}\right)\eta(x,y), \end{array}$$

where $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ is a cut-off function chosen appropriately so that $f$ and $g$ satisfy the assumptions of the present chapter. The volume constraint considered is $\lambda = \pi$ and the initial shape $\Omega_0 \subset \mathbb{R}^2$ is an ellipse centered at $(0,0) \in \mathbb{R}^2$, with semi-major axis $a = 1.3$ and semi-minor axis $b = 1/a$.

In what follows, we present the numerical results obtained for this two-dimensional example using the methodology described in Subsection 8.3.1, and for different values of $\beta$:

— Figure 8.1 shows on the left the shape which solves Problem (8.1) for $\beta = 0.49$, and on the right the one when the Tresca problem and its energy functional are replaced by Dirichlet ones (see Remark 8.2.6). We observe that both shapes are very close. Indeed, with $\beta \geq 0.49$, one can check numerically that the solution $w_{\mathrm{D},\Omega}$ to the Dirichlet problem ($\mathrm{SDP}_\Omega$) satisfies $|\partial_{\mathrm{n}} w_{\mathrm{D},\Omega}| < g_\beta$ on $\Gamma$, and thus is also the solution to the scalar Tresca friction problem ($\mathrm{STP}_\Omega$). One deduces from Remark 8.2.6 that the shape gradient of $\mathcal{J}$ and the one of $\mathcal{J}_\mathrm{D}$ coincide. Therefore, since the shape minimizing the Dirichlet energy functional $\mathcal{J}_\mathrm{D}$ under the volume constraint $\lambda = \pi$ is a critical shape of the *augmented* Dirichlet energy functional, it is also a critical shape of the *augmented* Tresca energy functional.

— Figure 8.2 shows the shapes which solve Problem (8.1) for $\beta = 0.46, 0.43, 0.37, 0.31$. The shapes are different from the one obtained on the left of Figure 8.1. In that context, note that the normal derivative of the solution $u$ to the scalar Tresca friction problem ($\mathrm{STP}_\Omega$) reaches the friction threshold $g_\beta$ on some parts of the boundary.

— Figure 8.3 shows on the left the shapes which solve Problem (8.1) for $\beta = 0.28, 0.1, 0.01$. Here the normal derivative of the solution $u$ to the scalar Tresca friction problem ($\mathrm{STP}_\Omega$) reaches the friction threshold $g_\beta$ on the entire boundary. Moreover we can notice that these shapes are very close to the ones (presented on the right of Figure 8.3) that minimize $\mathcal{J}_\mathrm{N}$ with $g = g_\beta$ (see Remark 8.2.5) under the same volume constraint $\lambda = \pi$. Indeed, for these values of $\beta$, one can check numerically that the solution $w_{\mathrm{N},\Omega}$ to the Neumann problem ($\mathrm{SNP}_\Omega$) with $g = g_\beta$

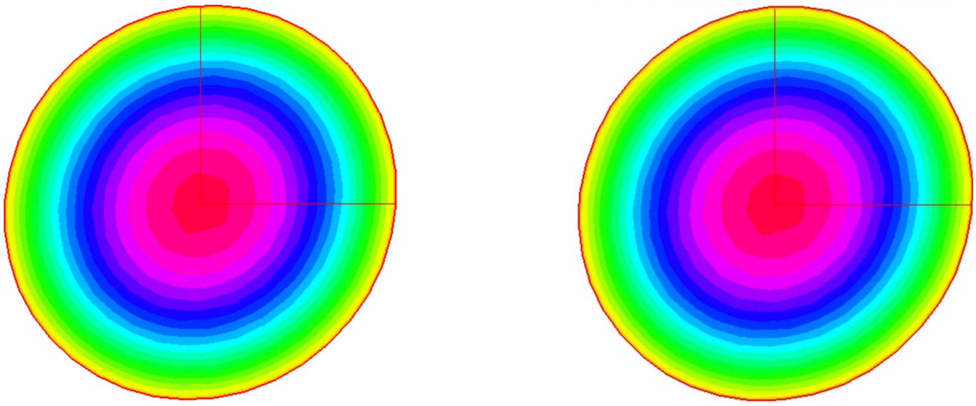

Figure 8.1 – Shapes minimizing $\mathcal{J}$ (left) and $\mathcal{J}_{\mathrm{D}}$ (right), under the volume constraint $\lambda = \pi$, and with $\beta = 0.49$.

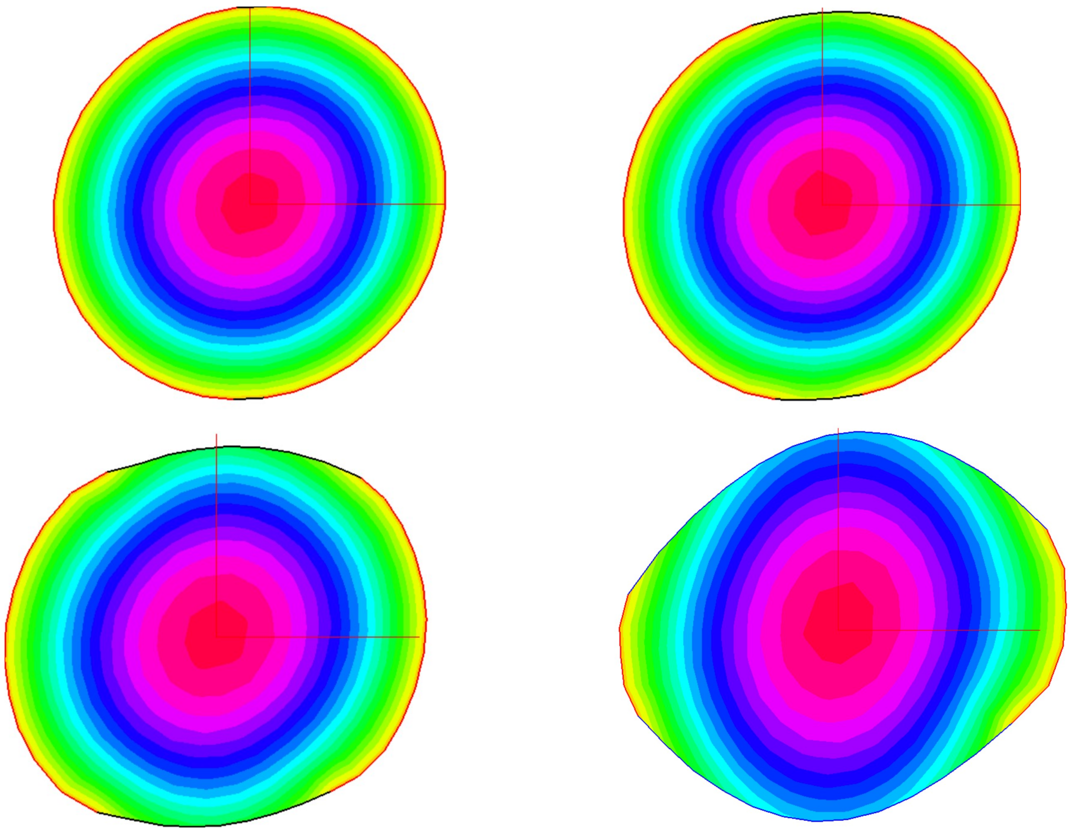

Figure 8.2 – Shapes minimizing $\mathcal{J}$ under the volume constraint $\lambda = \pi$. From top-left to bottom-right, $\beta = 0.46, 0.43, 0.37, 0.31$. The red boundary shows where $u = 0$ and the black/blue boundary shows where $|\partial_{\mathrm{n}} u| = g_\beta$.

satisfies $w_{\mathrm{N},\Omega} > 0$ on $\Gamma$, and thus is also the solution to the scalar Tresca friction problem ($\mathrm{STP}_\Omega$). One deduces from Remark 8.2.5 that the shape gradient of $\mathcal{J}$ and the one of $\mathcal{J}_{\mathrm{N}}$ coincide. Therefore, since the shape minimizing the Neumann energy functional $\mathcal{J}_{\mathrm{N}}$ under the volume constraint $\lambda = \pi$ is a critical shape of the *augmented* Neumann energy functional, it is also a critical shape of the *augmented* Tresca energy functional.

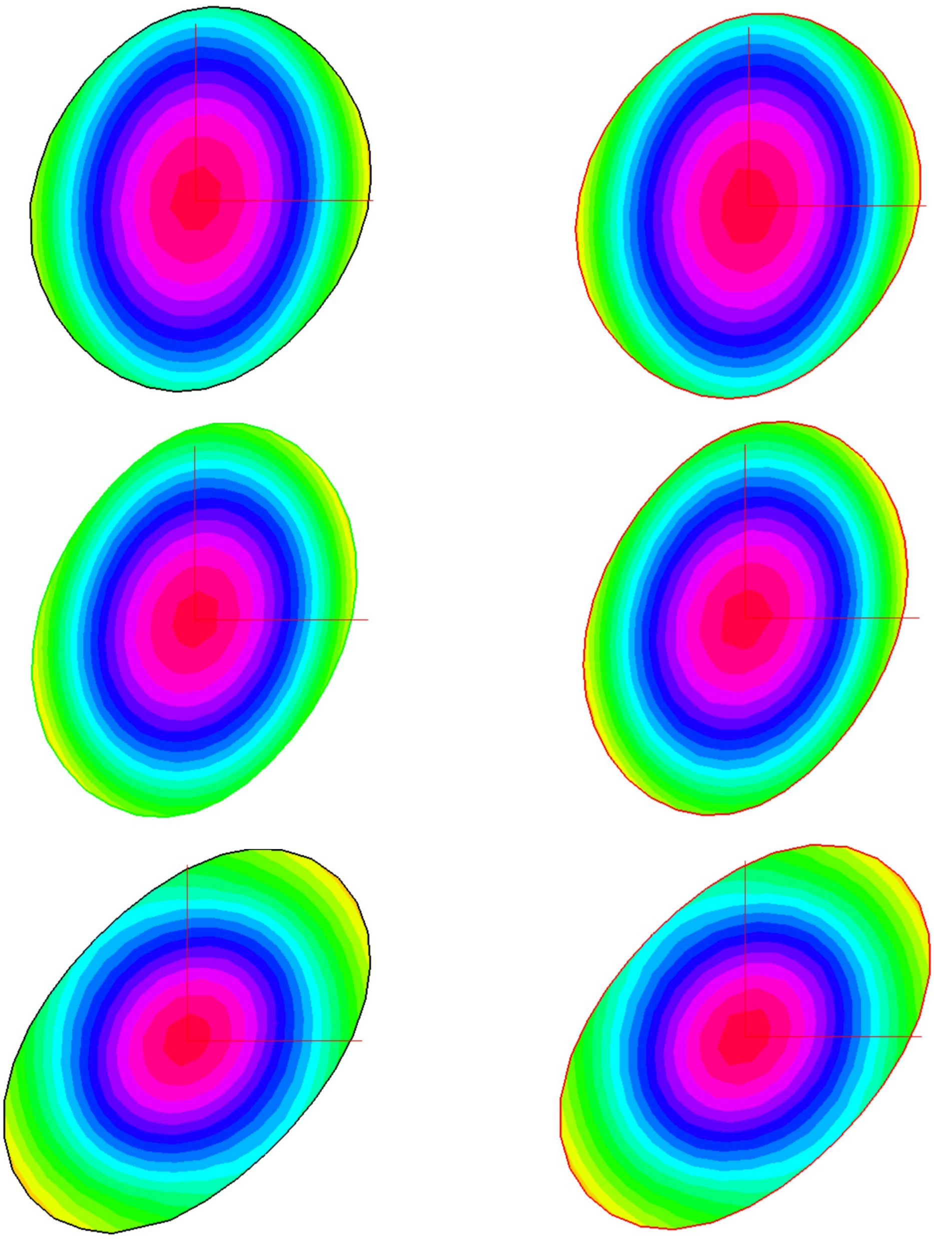

Figure 8.3 – Shapes minimizing $\mathcal{J}$ (left) and $\mathcal{J}_{\text{N}}$ (right), under the volume constraint $\lambda = \pi$. From top to bottom, $\beta = 0.28, 0.1, 0.01$.

For more details and an animated illustration, we would like to suggest to the reader to watch the video `https://youtu.be/_MufZx3zsew` presenting all numerical results we obtained for different values of $\beta$ from 0.7 to 0.01.

To conclude this chapter, we would like to bring to the attention of the reader that, in the above numerical simulations, it seems that there is a kind of transition from optimal shapes associated with the Neumann energy functional to optimal shapes associated with the Dirichlet energy functional. This transition is carried out by optimal shapes associated with the Tresca energy functional, continuously with respect to the friction threshold (precisely with respect to the parameter $\beta$). However, we do not have a proof of such a highly nontrivial result. This may constitute an interesting topic for future investigations.

Chapter 9

# Shape optimization for variational inequalities: the Signorini problem in the linear elastic model

Let $d \in \{2,3\}$, $f$ be a function in $\mathrm{H}^1(\mathbb{R}^d,\mathbb{R}^d)$, $\Omega_{\mathrm{ref}}$ be a nonempty connected bounded open subset of $\mathbb{R}^d$ with Lipschitz boundary $\Gamma_{\mathrm{ref}} := \partial\Omega_{\mathrm{ref}}$, such that $\Gamma_{\mathrm{ref}} = \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{S}_{\mathrm{ref}}}$, where $\Gamma_{\mathrm{D}}$ and $\Gamma_{\mathrm{S}_{\mathrm{ref}}}$ are two measurable pairwise disjoint subsets of $\Gamma_{\mathrm{ref}}$, and $\Gamma_{\mathrm{D}}$ has a positive measure.

In this chapter we consider the shape optimization problem given by

$$\underset{\substack{\Omega\in\mathcal{U}_{\mathrm{ref}}\\|\Omega|=|\Omega_{\mathrm{ref}}|}}{\operatorname{minimize}}\ \mathcal{J}(\Omega), \tag{9.1}$$

where

$$\mathcal{U}_{\mathrm{ref}} := \Big\{\Omega\subset\mathbb{R}^d \mid \Omega \text{ nonempty connected bounded open subset of } \mathbb{R}^d \\ \text{with Lipschitz boundary } \Gamma:=\partial\Omega \text{ such that } \Gamma_{\mathrm{D}}\subset\Gamma\Big\},$$

with the volume constraint $|\Omega| = |\Omega_{\mathrm{ref}}| > 0$, $\Omega$ is an elastic solid satisfying the linear elastic model (see Chapter 1), and where $\mathcal{J} : \mathcal{U}_{\mathrm{ref}} \to \mathbb{R}$ is the *Signorini energy functional* defined by

$$\mathcal{J}(\Omega) := \frac{1}{2}\int_\Omega \mathrm{Ae}\left(u_\Omega\right) : \mathrm{e}\left(u_\Omega\right) - \int_\Omega f\cdot u_\Omega, \tag{9.2}$$

where $u_\Omega \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ stands for the unique solution to the Signorini problem given by

$$\left\{\begin{aligned} -\mathrm{div}(\mathrm{Ae}(u)) &= f && \text{in } \Omega,\\ u &= 0 && \text{on } \Gamma_{\mathrm{D}},\\ \sigma_\tau(u) &= 0 && \text{on } \Gamma_{\mathrm{S}},\\ u_{\mathrm{n}}\leq 0, \sigma_{\mathrm{n}}(u)\leq 0 \text{ and } u_{\mathrm{n}}\sigma_{\mathrm{n}}(u) &= 0 && \text{on } \Gamma_{\mathrm{S}}, \end{aligned}\right. \tag{SP$_\Omega$}$$

where, for all $\Omega\in\mathcal{U}_{\mathrm{ref}}$, $\Gamma := \partial\Omega$, $\Gamma_{\mathrm{S}} := \Gamma\backslash\Gamma_{\mathrm{D}}$, n is the outward-pointing unit normal vector to $\Gamma$.

Recall that A is the stiffness tensor, e is the infinitesimal strain tensor, $\sigma_{\mathrm{n}}$ is the normal stress, $\sigma_\tau$ is the shear stress, and $f$ models volume forces (see Chapter1 for details). From Subsection 4.2.2, note that the Signorini energy functional $\mathcal{J}$, given by (9.2), can also be expressed as

$$\mathcal{J}(\Omega) = -\frac{1}{2}\int_\Omega \mathrm{Ae}\left(u_\Omega\right) : \mathrm{e}\left(u_\Omega\right),$$

for all $\Omega \in \mathcal{U}_{\mathrm{ref}}$.

In the whole section let us fix $\Omega_0 \in \mathcal{U}_{\mathrm{ref}}$. Similarly to Chapter 8, our aim here is to prove that, under appropriate assumptions, the functional $\mathcal{J}$ is *shape differentiable* at $\Omega_0$, in the sense that the map

$$\begin{array}{rcl} \mathcal{C}^{2,\infty}_{\mathrm{D}}(\mathbb{R}^d,\mathbb{R}^d) & \longrightarrow & \mathbb{R} \\ \theta & \longmapsto & \mathcal{J}((\mathrm{id}+\theta)(\Omega_0)), \end{array}$$

is Gateaux differentiable at 0, and to give an expression of the Gateaux differential, denoted by $\mathcal{J}'(\Omega_0)$, where

$$\mathcal{C}^{2,\infty}_{\mathrm{D}}(\mathbb{R}^d,\mathbb{R}^d) := \left\{\theta \in \mathcal{C}^2(\mathbb{R}^d,\mathbb{R}^d) \cap \mathrm{W}^{2,\infty}(\mathbb{R}^d,\mathbb{R}^d) \mid \theta = 0 \text{ on } \Gamma_{\mathrm{D}}\right\}.$$

This chapter is separated as follows. In Section 9.1 we perturb the Signorini problem with respect to the shape and we characterize the material derivative as solution to a variational inequality (see Theorem 9.1.2). Then, with additional regularity assumptions, we characterize the material and shape derivatives as being weak solutions to Signorini problems (see Corollaries 9.1.3 and 9.1.5). In Section 9.2 we prove under appropriate assumptions, that the Signorini functional $\mathcal{J}$ is shape differentiable at $\Omega_0$, and we provide an expression of its shape gradient (see Theorem 9.2.1 and Corollary 9.2.2). Finally in Section 9.3 numerical simulations are performed to solve the shape optimization problem (9.1) on a two-dimensional example.

In the sequel we denote, for all $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$, by $xy^\top$ the matrix whose the $i$-th line is given by the vector $x_i y$, for all $i \in [[1,d]]$, and by $\langle\cdot,\cdot\rangle_{\mathrm{A},\Omega_0}$ the scalar product given in (4.1) and defined on $\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)$. We also recall that we denote (when it is necessary to avoid any confusion) A[S] the stiffness tensor A applied to a matrix $\mathrm{S} \in \mathbb{R}^{d\times d}$.

## 9.1 Setting of the shape perturbation and derivatives

Consider $\theta \in \mathcal{C}^{2,\infty}_{\mathrm{D}}(\mathbb{R}^d,\mathbb{R}^d)$ and, for all $t \geq 0$ sufficiently small such that id$+t\theta$ is a $\mathcal{C}^2$-diffeomorphism of $\mathbb{R}^d$, consider the shape perturbed Signorini problem given by

$$\left\{\begin{array}{rcll} -\mathrm{div}(\mathrm{Ae}(u_t)) & = & f & \text{in } \Omega_t, \\ u_t & = & 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \sigma_{\tau_t}(u_t) & = & 0 & \text{on } \Gamma_{\mathrm{S}_t}, \\ u_{t,\mathrm{n}_t} \leq 0,\ \sigma_{\mathrm{n}_t}(u_t) \leq 0 \text{ and } u_{t,\mathrm{n}_t}\sigma_{\mathrm{n}_t}(u_t) & = & 0 & \text{on } \Gamma_{\mathrm{S}_t}, \end{array}\right. \tag{PSP$_t$}$$

where $\Omega_t := (\mathrm{id}+t\theta)(\Omega_0) \in \mathcal{U}_{\mathrm{ref}}$ and $\Gamma_t := (\mathrm{id}+t\theta)(\Gamma_0)$ and $\mathrm{n}_t$ is the outward-pointing unit normal vector to $\Gamma_t$. From Subsection 4.2.2, there exists a unique weak solution $u_t \in \mathrm{H}^1(\Omega_t,\mathbb{R}^d)$ to (PSP$_t$)

which satisfies

$$\int_{\Omega_t} \mathrm{Ae}(u_t) : \mathrm{e}(v-u_t) \geq \int_{\Omega_t} f\cdot(v-u_t), \qquad \forall v\in\mathcal{K}^1(\Omega_t,\mathbb{R}^d),$$

where

$$\mathcal{K}^1(\Omega_t,\mathbb{R}^d) := \left\{ v\in \mathrm{H}^1_{\mathrm{D}}(\Omega_t,\mathbb{R}^d) \mid v_{\mathrm{n}_t} \leq 0 \ a.e. \text{ on } \Gamma_{\mathrm{S}_t} \right\}.$$

Using the change of variables $\mathrm{id}+t\theta$ and the equality

$$\mathrm{n}_t \circ (\mathrm{id}+t\theta) = \frac{(\mathrm{I}+t\nabla\theta^\top)^{-1}\mathrm{n}}{\|(\mathrm{I}+t\nabla\theta^\top)^{-1}\mathrm{n}\|},$$

where $\mathrm{n} := \mathrm{n}_0$ (see, e.g., [78, Chapter 2, Proposition 2.48 p.79]), we prove that $\overline{u}_t := u_t\circ(\mathrm{id}+t\theta) \in \mathcal{K}^1_t(\Omega_0,\mathbb{R}^d) \subset \mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)$ satisfies

$$\int_{\Omega_0} \mathrm{J}_t\mathrm{A}\left[\nabla\overline{u}_t\,(\mathrm{I}+t\nabla\theta)^{-1}\right] : \nabla(v-\overline{u}_t)\,(\mathrm{I}+t\nabla\theta)^{-1} \geq \int_{\Omega_0} f_t\mathrm{J}_t\cdot(v-\overline{u}_t), \qquad \forall v\in\mathcal{K}^1_t(\Omega_0,\mathbb{R}^d),$$

where $\mathcal{K}^1_t(\Omega_0,\mathbb{R}^d) := \{v\in\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d) \mid v\cdot(\mathrm{I}+t\nabla\theta^\top)^{-1}\mathrm{n} \leq 0 \ a.e. \text{ on } \Gamma_{\mathrm{S}_0}\}$, $f_t := f\circ(\mathrm{id}+t\theta) \in \mathrm{H}^1(\mathbb{R}^d,\mathbb{R}^d)$ and $\mathrm{J}_t := \det(\mathrm{I}+t\nabla\theta) \in \mathrm{L}^\infty(\mathbb{R}^d,\mathbb{R})$ is the Jacobian determinant. Therefore, we deduce from Subsection 4.2.2 that $\overline{u}_t \in \mathcal{K}^1_t(\Omega_0,\mathbb{R}^d)$ is the unique weak solution to the perturbed general Signorini problem

$$\left\{\begin{array}{rcll} -\mathrm{div}\left(\mathrm{J}_t\mathrm{A}\left[\nabla\overline{u}_t\mathrm{B}_t\right]\mathrm{B}_t^\top\right) & = & f_t\mathrm{J}_t & \text{in } \Omega_0,\\ \overline{u}_t & = & 0 & \text{on } \Gamma_{\mathrm{D}},\\ \left(\left(\mathrm{J}_t\mathrm{A}\left[\nabla\overline{u}_t\mathrm{B}_t\right]\mathrm{B}_t^\top\right)\mathrm{n}\right)_{\left(\mathrm{M}_t^{-1^\top}\mathrm{n}\right)^\perp} & = & 0 & \text{on } \Gamma_{\mathrm{S}_0},\\ \overline{u}_t\cdot\mathrm{M}_t^{-1^\top}\mathrm{n}\leq 0,\ \left(\mathrm{J}_t\mathrm{A}\left[\nabla\overline{u}_t\mathrm{B}_t\right]\mathrm{B}_t^\top\right)\mathrm{n}\cdot\mathrm{M}_t^{-1^\top}\mathrm{n}\leq 0 \text{ and} & & & \\ \overline{u}_t\cdot\mathrm{M}_t^{-1^\top}\mathrm{n}\left(\left(\mathrm{J}_t\mathrm{A}\left[\nabla\overline{u}_t\mathrm{B}_t\right]\mathrm{B}_t^\top\right)\mathrm{n}\cdot\mathrm{M}_t^{-1^\top}\mathrm{n}\right) & = & 0 & \text{on } \Gamma_{\mathrm{S}_0}, \end{array}\right. \tag{PGSP$_t$}$$

where $\mathrm{M}_t := (\mathrm{I}+t\nabla\theta)$ and $\mathrm{B}_t := (\mathrm{I}+t\nabla\theta)^{-1}$, and thus can be characterized as

$$\overline{u}_t = \mathrm{prox}_{\iota_{\mathcal{K}^1_t(\Omega_0,\mathbb{R}^d)}}(F_t),$$

where $F_t\in\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)$ is the unique solution to the perturbed general Dirichlet-Neumann problem

$$\left\{\begin{array}{rcll} -\mathrm{div}\left(\mathrm{J}_t\mathrm{A}\left[(\nabla F_t)\mathrm{B}_t\right]\mathrm{B}_t^\top\right) & = & f_t\mathrm{J}_t & \text{in } \Omega_0,\\ F_t & = & 0 & \text{on } \Gamma_{\mathrm{D}},\\ \left(\mathrm{J}_t\mathrm{A}\left[(\nabla F_t)\mathrm{B}_t\right]\mathrm{B}_t^\top\right)\mathrm{n} & = & 0 & \text{on } \Gamma_{\mathrm{S}_0}. \end{array}\right.$$

and where $\mathrm{prox}_{\iota_{\mathcal{K}^1_t(\Omega_0,\mathbb{R}^d)}}$ stands for the proximal operator associated with the Signorini indicator function $\iota_{\mathcal{K}^1_t(\Omega_0,\mathbb{R}^d)}$ considered on the perturbed Hilbert space $(\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d),\langle\cdot,\cdot\rangle_{\mathrm{J}_t,\mathrm{A},\mathrm{B}_t})$ (see the notations in Section 4.2).

Let us recall that (see, e.g., [46]):

(i) the map $t\in\mathbb{R}_+\mapsto f_t\mathrm{J}_t\in\mathrm{L}^2(\mathbb{R}^d,\mathbb{R}^d)$ is differentiable at $t=0$ with derivative given by $f\mathrm{div}(\theta)+\nabla f\theta = \mathrm{div}(f\theta^\top)$;

(ii) the map $t\in\mathbb{R}_+\mapsto\mathrm{J}_t\in\mathrm{L}^\infty(\mathbb{R}^d)$ is differentiable at $t=0$ with derivative given by $\mathrm{div}(\theta)$;

(iii) the map $t \in \mathbb{R}_+ \mapsto \mathrm{I} + t\nabla\theta \in \mathrm{C}^{1,\infty}(\mathbb{R}^d, \mathbb{R}^{d\times d})$ is differentiable at $t = 0$ with derivative given by $\nabla\theta$;

(iv) the map $t \in \mathbb{R}_+ \mapsto (\mathrm{I} + t\nabla\theta)^{-1} \in \mathrm{L}^{\infty}(\mathbb{R}^d, \mathbb{R}^{d\times d})$ is differentiable at $t = 0$ with derivative given by $-\nabla\theta$.

From those differentiability results we can apply Theorem 6.1.3 to obtain the differentiability of the map $t \in \mathbb{R}_+ \mapsto \overline{\overline{u}}_t := \mathrm{M}_t^{-1}\overline{u}_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega_0, \mathbb{R}^d)$ at $t = 0$, and then to deduce, from Theorem 6.1.4, the material directional derivative.

**Theorem 9.1.1.** *Let $\overline{u}_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega_0, \mathbb{R}^d)$ be the unique weak solution to the perturbed general Signorini problem* (PGSP$_t$)*. Then the map $t \in \mathbb{R}_+ \mapsto \overline{\overline{u}}_t := \mathrm{M}_t^{-1}\overline{u}_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega_0, \mathbb{R}^d)$ is differentiable at $t = 0$ and its derivative, denoted by $\overline{\overline{u}}'_0 \in \mathrm{T}_{\mathrm{N}_{\mathcal{K}^1_0(\Omega_0,\mathbb{R}^d)}}(u_0) \cap (\mathbb{R}\,(F_0 - u_0))^{\perp}$, is the unique solution to the variational inequality*

$$\left\langle \overline{\overline{u}}'_0, v - \overline{\overline{u}}'_0 \right\rangle_{\mathrm{A},\Omega_0} \geq -\int_{\Omega_0} \operatorname{div}\left(\operatorname{div}\left(\mathrm{Ae}(u_0)\right)\theta^{\top}\right) \cdot \left(v - \overline{\overline{u}}'_0\right) - \left\langle \mathrm{Ae}(u_0)\mathrm{n}, \nabla\theta \left(v - \overline{\overline{u}}'_0\right)\right\rangle_{\mathrm{H}^{-1/2}(\Gamma_0,\mathbb{R}^d)\times\mathrm{H}^{1/2}(\Gamma_0,\mathbb{R}^d)} + \int_{\Omega_0} \left((\mathrm{Ae}(u_0))\,\nabla\theta^{\top} + \mathrm{A}\left(\nabla u_0 \nabla\theta\right) - \mathrm{Ae}\left(\nabla\theta u_0\right) - \operatorname{div}(\theta)\mathrm{Ae}(u_0)\right) : \nabla\left(v - \overline{\overline{u}}'_0\right), \quad (9.3)$$

*for all $v \in \mathrm{T}_{\mathrm{N}_{\mathcal{K}^1_0(\Omega_0,\mathbb{R}^d)}}(u_0) \cap (\mathbb{R}\,(F_0 - u_0))^{\perp}$, where*

$$\mathrm{T}_{\mathrm{N}_{\mathcal{K}^1_0(\Omega_0,\mathbb{R}^d)}}(u_0) \cap (\mathbb{R}\,(F_0 - u_0))^{\perp} = \left\{ v \in \mathrm{H}^1_{\mathrm{D}}(\Omega_0, \mathbb{R}^d) \mid v_{\mathrm{n}} \leq 0 \text{ q.e. on } \Gamma_{\mathrm{S}_0}^{u_{0\mathrm{n}}=0} \text{ and } \langle F_0 - u_0, v\rangle_{\mathrm{A},\Omega_0} = 0 \right\},$$

*and $\Gamma_{\mathrm{S}_0}^{u_{0\mathrm{n}}=0} := \{s \in \Gamma_{\mathrm{S}_0} \mid u_{0\mathrm{n}}(s) = 0\}$.*

*Proof.* From Theorem 6.1.3 it follows that $\overline{\overline{u}}'_0 \in \mathrm{T}_{\mathrm{N}_{\mathcal{K}^1_0(\Omega_0,\mathbb{R}^d)}}(u_0) \cap (\mathbb{R}\,(F_0 - u_0))^{\perp}$ is the unique solution to the variational inequality

$$\left\langle \overline{\overline{u}}'_0, v - \overline{\overline{u}}'_0 \right\rangle_{\mathrm{A},\Omega_0} \geq \int_{\Omega_0} \operatorname{div}\left(f\theta^{\top}\right) \cdot \left(v - \overline{\overline{u}}'_0\right) - \left\langle \mathrm{Ae}(u_0)\mathrm{n}, \nabla\theta \left(v - \overline{\overline{u}}'_0\right)\right\rangle_{\mathrm{H}^{-1/2}(\Gamma_0,\mathbb{R}^d)\times\mathrm{H}^{1/2}(\Gamma_0,\mathbb{R}^d)} + \int_{\Omega_0} \left((\mathrm{Ae}(u_0))\,\nabla\theta^{\top} + \mathrm{A}\left(\nabla u_0 \nabla\theta\right) - \mathrm{Ae}\left(\nabla\theta u_0\right) - \operatorname{div}(\theta)\mathrm{Ae}(u_0)\right) : \nabla\left(v - \overline{\overline{u}}'_0\right),$$

for all $v \in \mathrm{T}_{\mathrm{N}_{\mathcal{K}^1_0(\Omega_0,\mathbb{R}^d)}}(u_0) \cap (\mathbb{R}\,(F_0 - u_0))^{\perp}$. Using the equality $\operatorname{div}\left(\mathrm{Ae}(u_0)\right) = -f$ in $\mathrm{H}^1(\Omega_0, \mathbb{R}^d)$, we obtain the result. □

**Theorem 9.1.2** (Material directional derivative)**.** *Let $\overline{u}_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega_0, \mathbb{R}^d)$ be the unique weak solution to the perturbed general Signorini problem* (PGSP$_t$)*. Then the map $t \in \mathbb{R}_+ \mapsto \overline{u}_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega_0, \mathbb{R}^d)$ is differentiable at $t = 0$ and its derivative (that is, the material directional derivative), denoted* *by* $\overline{u}_0' \in \mathrm{T}_{\mathrm{N}_{\mathcal{K}_0^1(\Omega_0,\mathbb{R}^d)}}(u_0) \cap (\mathbb{R}\,(F_0-u_0))^{\perp} + \nabla\theta u_0$, *is the unique solution to the variational inequality*

$$\begin{aligned}\langle \overline{u}_0', v-\overline{u}_0'\rangle_{\mathrm{A},\Omega_0} \geq &-\int_{\Omega_0} \mathrm{div}\left(\mathrm{div}\left(\mathrm{Ae}(u_0)\right)\theta^{\top}\right)\cdot(v-\overline{u}_0')\\ &- \langle \mathrm{Ae}(u_0)\mathrm{n}, \nabla\theta\,(v-\overline{u}_0')\rangle_{\mathrm{H}^{-1/2}(\Gamma_0,\mathbb{R}^d)\times \mathrm{H}^{1/2}(\Gamma_0,\mathbb{R}^d)}\\ &+\int_{\Omega_0}\left((\mathrm{Ae}(u_0))\,\nabla\theta^{\top} + \mathrm{A}\,(\nabla u_0\nabla\theta) - \mathrm{div}(\theta)\mathrm{Ae}(u_0)\right):\nabla(v-\overline{u}_0'), \quad (9.4)\end{aligned}$$

*for all* $v \in \mathrm{T}_{\mathrm{N}_{\mathcal{K}_0^1(\Omega_0,\mathbb{R}^d)}}(u_0) \cap (\mathbb{R}\,(F_0-u_0))^{\perp} + \nabla\theta u_0$, *where*

$$\begin{aligned}&\mathrm{T}_{\mathrm{N}_{\mathcal{K}_0^1(\Omega_0,\mathbb{R}^d)}}(u_0) \cap (\mathbb{R}\,(F_0-u_0))^{\perp} + \nabla\theta u_0 =\\ &\qquad\left\{ v\in \mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d) \mid (v-\nabla\theta u_0)_{\mathrm{n}} \leq 0 \text{ q.e. on } \Gamma_{\mathrm{S}_0}^{u_{0\mathrm{n}}=0} \text{ and } \langle F_0-u_0, v-\nabla\theta u_0\rangle_{\mathrm{A},\Omega_0}=0\right\}.\end{aligned}$$

In [19, 57, 78], the authors get the same result using the conical differentiability of the projection operator since the projection operator on $\mathcal{K}_0^1(\Omega_0,\mathbb{R}^d)$ is conically differentiable at $u_0$ for $F_0 -\ u_0$, and its conical derivative is given by $\mathrm{proj}_{\mathrm{T}_{\mathrm{N}_{\mathcal{K}_0^1(\Omega_0,\mathbb{R}^d)}}(u_0)\cap(\mathbb{R}(F_0-u_0))^{\perp}}(E_0')$. Nevertheless, to the best of our knowledge, no one notices that it was possible to improve this result under additional assumptions, in order to characterize the material derivative as weak solution to a Signorini problem.

**Corollary 9.1.3.** *Consider the framework of Theorem 9.1.2 with the additional assumptions that the decomposition* $\Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{S}_0}$ *of* $\Gamma_0$ *is consistent,* $u_0\in \mathrm{H}^3(\Omega_0,\mathbb{R}^d)$ *and* $\Gamma_{\mathrm{S}_0}^{\mathrm{u}_{0\mathrm{n}}=0} = \overline{\mathrm{int}_{\Gamma_{\mathrm{S}_0}}(\Gamma_{\mathrm{S}_0}^{\mathrm{u}_{0\mathrm{n}}=0})}$. *Then the material directional derivative* $\overline{u}_0' \in \mathrm{T}_{\mathrm{N}_{\mathcal{K}_0^1(\Omega_0,\mathbb{R}^d)}}(u_0) \cap (\mathbb{R}\,(F_0-u_0))^{\perp} + \nabla\theta u_0$ *is the unique weak solution to the Signorini problem*

$$\begin{cases} -\mathrm{div}(\mathrm{Ae}(\overline{u}_0')) = -\mathrm{div}(\mathrm{Ae}(\nabla u_0\theta)) & \text{in } \Omega_0,\\ \overline{u}_0' = 0 & \text{on } \Gamma_{\mathrm{D}},\\ \sigma_\tau(\overline{u}_0') = h^m(\theta)_\tau & \text{on } \Gamma_{\mathrm{S}_0},\\ \sigma_{\mathrm{n}}(\overline{u}_0') = h^m(\theta)_{\mathrm{n}} & \text{on } \Gamma_{\mathrm{S}_{0\mathrm{N}}}^{u_{0\mathrm{n}}},\\ (\overline{u}_0'-\nabla\theta u_0)_{\mathrm{n}} = 0 & \text{on } \Gamma_{\mathrm{S}_{0\mathrm{D}}}^{u_{0\mathrm{n}}},\\ (\overline{u}_0'-\nabla\theta u_0)_{\mathrm{n}} \leq 0, \sigma_{\mathrm{n}}(\overline{u}_0') \leq h^m(\theta)_{\mathrm{n}} \text{ and } (\overline{u}_0'-\nabla\theta u_0)_{\mathrm{n}}\,(\sigma_{\mathrm{n}}(\overline{u}_0')-h^m(\theta)_{\mathrm{n}})) = 0 & \text{on } \Gamma_{\mathrm{S}_{0-}}^{u_{0\mathrm{n}}}, \end{cases}$$

*where* $h^m(\theta) := ((\mathrm{Ae}(u_0))\nabla\theta^{\top} + \mathrm{A}(\nabla u_0\nabla\theta) - \sigma_{\mathrm{n}}(u_0)(\mathrm{div}(\theta)\mathrm{I}+\nabla\theta^{\top}))\mathrm{n} \in \mathrm{L}^2(\Gamma_{\mathrm{S}_0},\mathbb{R}^d)$, *and* $\Gamma_{\mathrm{S}_0}$ *is decomposed, up to a null set, as* $\Gamma_{\mathrm{S}_{0\mathrm{N}}}^{u_{0\mathrm{n}}}\cup\Gamma_{\mathrm{S}_{0\mathrm{D}}}^{u_{0\mathrm{n}}}\cup\Gamma_{\mathrm{S}_{0-}}^{u_{0\mathrm{n}}}$, *where*

$$\begin{aligned}\Gamma_{\mathrm{S}_{0\mathrm{N}}}^{u_{0\mathrm{n}}} &:= \left\{s\in\Gamma_{\mathrm{S}_0} \mid u_{0\mathrm{n}}(s)\neq 0\right\},\\ \Gamma_{\mathrm{S}_{0\mathrm{D}}}^{u_{0\mathrm{n}}} &:= \left\{s\in\Gamma_{\mathrm{S}_0} \mid u_{0\mathrm{n}}(s)= 0 \text{ and } \sigma_{\mathrm{n}}(u_0)(s)<0\right\},\\ \Gamma_{\mathrm{S}_{0-}}^{u_{0\mathrm{n}}} &:= \left\{s\in\Gamma_{\mathrm{S}_0} \mid u_{0\mathrm{n}}(s)= 0 \text{ and } \sigma_{\mathrm{n}}(u_0)(s)=0\right\}.\end{aligned}$$

*Proof.* Since $u_0\in\mathrm{H}^2(\Omega_0,\mathbb{R}^d)$ and $\theta\in\mathcal{C}^{2,\infty}_{\mathrm{D}}(\mathbb{R}^d,\mathbb{R}^d)$, one deduces that

$$\mathrm{div}\left((\mathrm{Ae}(u_0))\,\nabla\theta^{\top} + \mathrm{A}\,(\nabla u_0\nabla\theta) - \mathrm{div}(\theta)\mathrm{Ae}(u_0)\right) \in \mathrm{L}^2(\Omega_0,\mathbb{R}^d),$$

and that

$$\left((\mathrm{Ae}(u_0))\,\nabla\theta^{\top} + \mathrm{A}\,(\nabla u_0\nabla\theta) - \mathrm{div}(\theta)\mathrm{Ae}(u_0)\right)\mathrm{n} \in \mathrm{L}^2(\Gamma_0,\mathbb{R}^d).$$

Moreover, since the decomposition $\Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{S}_0}$ of $\Gamma_0$ is consistent and $\mathrm{Ae}(u_0)\mathrm{n} \in \mathrm{L}^2(\Gamma_0, \mathbb{R}^d)$, then $u_0$ is a (strong) solution to the Signorini problem ($\mathrm{PSP}_t$) for the parameter $t = 0$ (see Proposition 4.2.9). Thus $\sigma_\tau(u_0) = 0$ *a.e.* on $\Gamma_{\mathrm{S}_0}$, and using the divergence in Inequality (9.4), we get that

$$\begin{aligned}\langle \overline{u}_0', v - \overline{u}_0' \rangle_{\mathrm{A},\Omega_0} \geq & \int_{\Gamma_{\mathrm{S}_0}} h^m \cdot (v - \overline{u}_0') \\ & - \int_{\Omega_0} \mathrm{div}\left(\mathrm{div}(\mathrm{Ae}(u_0))\theta^\top + (\mathrm{Ae}(u_0))\nabla\theta^\top + \mathrm{A}\left(\nabla u_0 \nabla\theta\right) - \mathrm{div}(\theta)\mathrm{Ae}(u_0)\right) \cdot (v - \overline{u}_0'), \quad (9.5)\end{aligned}$$

for all $v \in \mathrm{T}_{\mathrm{N}_{\mathcal{K}_0^1(\Omega_0,\mathbb{R}^d)}}(u_0) \cap (\mathbb{R}\left(F_0 - u_0\right))^\perp + \nabla\theta u_0$. Furthermore, one has $\mathrm{div}\left(\mathrm{Ae}\left(\nabla u_0 \theta\right)\right) \in \mathrm{L}^2(\Omega_0, \mathbb{R}^d)$ from $u_0 \in \mathrm{H}^3(\Omega_0, \mathbb{R}^d)$. Thus, using the equality

$$\mathrm{div}\left(\mathrm{Ae}\left(\nabla u_0 \theta\right)\right) = \mathrm{div}\left(\mathrm{div}(\mathrm{Ae}(u_0))\theta^\top + (\mathrm{Ae}(u_0))\nabla\theta^\top + \mathrm{A}\left(\nabla u_0 \nabla\theta\right) - \mathrm{div}(\theta)\mathrm{Ae}(u_0)\right),$$

in $\mathrm{L}^2(\Omega_0, \mathbb{R}^d)$, we can conclude similarly to the proof of Corollary 6.1.5, and also from Subsection 4.2.2. □

**Remark 9.1.4.** Note that, from Corollary 6.1.5, one can get under the weaker assumption $u_0 \in \mathrm{H}^2(\Omega_0, \mathbb{R}^d)$, that the material directional derivative $\overline{u}_0'$ is the weak solution to a Signorini problem, and more precisely, the solution to the variational inequality (9.5) which is, from Subsection 4.2.2, the weak formulation of a Signorini problem with the source term given by

$$-\mathrm{div}\left(\mathrm{div}(\mathrm{Ae}(u_0))\theta^\top + (\mathrm{Ae}(u_0))\nabla\theta^\top + \mathrm{A}\left(\nabla u_0 \nabla\theta\right) - \mathrm{div}(\theta)\mathrm{Ae}(u_0)\right) \in \mathrm{L}^2(\Omega_0, \mathbb{R}^d).$$

It is important to note that, to the best of our knowledge, there is no regularity result for the solution to the Signorini problem in the linear elastic model, with respect to the data. To obtain this regularity result in our case is a highly nontrivial work. However, we can mention the works [71, 72] which deal with regularity results for variational inequalities concerning the Stokes equations.

Thanks to Corollary 9.1.3, we are now in a position to characterize the shape directional derivative.

**Corollary 9.1.5** (Shape directional derivative)**.** *Consider the framework of Corollary 9.1.3 with the additional assumption that* $\Gamma_0$ *is of class* $\mathcal{C}^3$*. Then the shape directional derivative, defined by* $u_0' := \overline{u}_0' - \nabla u_0 \theta \in \mathrm{T}_{\mathrm{N}_{\mathcal{K}_0^1(\Omega_0,\mathbb{R}^d)}}(u_0) \cap (\mathbb{R}\left(F_0 - u_0\right))^\perp + \nabla\theta u_0 - \nabla u_0 \theta$ *is the unique weak solution to the Signorini problem*

$$\left\{\begin{aligned} -\mathrm{div}(\mathrm{Ae}(u_0')) &= 0 && \text{in } \Omega_0, \\ u_0' &= 0 && \text{on } \Gamma_{\mathrm{D}}, \\ \sigma_\tau(u_0') &= h^s(\theta)_\tau && \text{on } \Gamma_{\mathrm{S}_0}, \\ \sigma_{\mathrm{n}}(u_0') &= h^s(\theta)_{\mathrm{n}} && \text{on } \Gamma_{\mathrm{S}_{0\mathrm{N}}}^{u_{0\mathrm{n}}}, \\ \left(u_0' - W(\theta)\right)_{\mathrm{n}} &= 0 && \text{on } \Gamma_{\mathrm{S}_{0\mathrm{D}}}^{u_{0\mathrm{n}}}, \\ \left(u_0' - W(\theta)\right)_{\mathrm{n}} \leq 0, \sigma_{\mathrm{n}}(u_0') \leq h^s(\theta)_{\mathrm{n}} \text{ and } \left(u_0' - W(\theta)\right)_{\mathrm{n}}\left(\sigma_{\mathrm{n}}(u_0') - h^s(\theta)_{\mathrm{n}}\right) &= 0 && \text{on } \Gamma_{\mathrm{S}_0-}^{u_{0\mathrm{n}}}, \end{aligned}\right.$$

*where* $W(\theta) := (\nabla\theta u_0) - (\nabla u_0 \theta) \in \mathrm{H}^{1/2}(\Gamma_0, \mathbb{R}^d)$,

$$h^s(\theta) := \theta \cdot \mathrm{n} \left( \partial_{\mathrm{n}} \left( \mathrm{Ae}(u_0)\mathrm{n} \right) - \partial_{\mathrm{n}} \left( \mathrm{Ae}(u_0) \right) \mathrm{n} \right) + \mathrm{Ae}(u_0) \nabla_\tau \left( \theta \cdot \mathrm{n} \right) \\ - \nabla(\mathrm{Ae}(u_0)\mathrm{n})\theta - \sigma_{\mathrm{n}}(u_0) \left( \mathrm{div}_\tau(\theta)\mathrm{I} + \nabla\theta^\top \right) \mathrm{n} \in \mathrm{L}^2(\Gamma_{\mathrm{S}_0}, \mathbb{R}^d),$$

*and where* $\mathrm{div}_\tau(\theta) := \mathrm{div}(\theta) - (\nabla\theta \mathrm{n} \cdot \mathrm{n}) \in \mathrm{L}^\infty(\Gamma_0)$ *is the tangential divergence of* $\theta$, $\partial_{\mathrm{n}} \left( \mathrm{Ae}(u_0)\mathrm{n} \right) := \nabla(\mathrm{Ae}(u_0)\mathrm{n})\mathrm{n}$ *stands for the normal derivative of* $\mathrm{Ae}(u_0)\mathrm{n}$, *and* $\partial_{\mathrm{n}} \left( \mathrm{Ae}(u_0) \right)$ *is the matrix whose the i-th line is given by the vector* $\partial_{\mathrm{n}} \left( \mathrm{Ae}(u_0)_i \right) := \nabla(\mathrm{Ae}(u_0)_i)\mathrm{n}$, *for all* $i \in [[1, d]]$.

*Proof.* Since $u_0' := \overline{u}_0' - \nabla u_0 \theta$, one deduces from the weak formulation of $\overline{u}_0'$ and using the divergence formula that,

$$\langle u_0', v - u_0' \rangle_{\mathrm{A},\Omega_0} \geq \\ \int_{\Omega_0} \left( \mathrm{div} \left( \mathrm{Ae}(u_0) \right) \theta^\top + \left( \mathrm{Ae}(u_0) \right) \nabla\theta^\top + \mathrm{A} \left( \nabla u_0 \nabla\theta \right) - \mathrm{Ae} \left( \nabla u_0 \theta \right) \right) : \nabla(v - u_0') \\ - \int_{\Omega_0} \mathrm{div}(\theta) \mathrm{Ae}(u_0) : \mathrm{e} \left( v - u_0' \right) - \int_{\Gamma_{\mathrm{S}_0}} \mathrm{Ae}(u_0)\mathrm{n} \cdot \nabla\theta \left( v - u_0' \right) + \int_{\Gamma_{\mathrm{S}_0}} \left( \theta \cdot \mathrm{n} \right) f \cdot \left( v - u_0' \right),$$

for all $v \in \mathrm{T}_{\mathrm{N}_{\mathcal{K}_0^1(\Omega_0,\mathbb{R}^d)}}(u_0) \cap \left( \mathbb{R} \left( F_0 - u_0 \right) \right)^\perp + \nabla\theta u_0 - \nabla u_0 \theta$. Moreover, one has

$$\int_{\Omega_0} \mathrm{div} \left( \mathrm{Ae}(u_0) \right) \theta^\top : \nabla\varphi = \int_{\Omega_0} \mathrm{div} \left( \mathrm{Ae}(u_0) \right) \cdot \left( \nabla\varphi \right) \theta = - \int_{\Omega_0} \mathrm{Ae}(u_0) : \nabla(\left( \nabla\varphi \right) \theta) + \int_{\Gamma_0} \mathrm{Ae}(u_0)\mathrm{n} \cdot \left( \nabla\varphi \right) \theta,$$

and also

$$- \int_{\Omega_0} \mathrm{div}(\theta) \mathrm{Ae}(u_0) : \mathrm{e}(\varphi) = \int_{\Omega_0} \theta \cdot \nabla(\mathrm{Ae}(u_0) : \mathrm{e}(\varphi)) - \int_{\Gamma_0} \theta \cdot \mathrm{n} \left( \mathrm{Ae}(u_0) : \mathrm{e}(\varphi) \right),$$

for all $\varphi \in \mathcal{C}^\infty(\overline{\Omega_0}, \mathbb{R}^d)$. Therefore, using the equality

$$\left( \left( \mathrm{Ae}(u_0) \right) \nabla\theta^\top + \mathrm{A} \left( \nabla u_0 \nabla\theta \right) - \mathrm{Ae} \left( \nabla u_0 \theta \right) \right) : \nabla\varphi + \theta \cdot \nabla(\mathrm{Ae}(u_0) : \mathrm{e}(\varphi)) - \mathrm{Ae}(u_0) : \nabla(\left( \nabla\varphi \right) \theta) = 0,$$

*a.e.* on $\Omega_0$, one deduces using the divergence formula that

$$\int_{\Omega_0} \left( \mathrm{div} \left( \mathrm{Ae}(u_0) \right) \theta^\top + \left( \mathrm{Ae}(u_0) \right) \nabla\theta^\top + \mathrm{A} \left( \nabla u_0 \nabla\theta \right) - \mathrm{Ae} \left( \nabla u_0 \theta \right) \right) : \nabla\varphi \\ - \int_{\Omega_0} \mathrm{div}(\theta) \mathrm{Ae}(u_0) : \mathrm{e} \left( \varphi \right) - \int_{\Gamma_0} \nabla\theta^\top \left( \mathrm{Ae}(u_0)\mathrm{n} \right) \cdot \varphi + \int_{\Gamma_0} \left( \theta \cdot \mathrm{n} \right) f \cdot \varphi \\ = \int_{\Gamma_0} \theta \cdot \mathrm{n} \left( -\mathrm{Ae}(u_0) : \mathrm{e}(\varphi) + f \cdot \varphi \right) + \nabla\varphi^\top \mathrm{Ae}(u_0)\mathrm{n} \cdot \theta - \nabla\theta^\top \mathrm{Ae}(u_0)\mathrm{n} \cdot \varphi,$$

for all $\varphi \in \mathcal{C}^\infty(\overline{\Omega_0}, \mathbb{R}^d)$. Furthermore, since $\Gamma_0$ is of class $\mathcal{C}^3$ and $u_0 \in \mathrm{H}^3(\Omega_0, \mathbb{R}^d)$, $\mathrm{Ae}(u_0)\mathrm{n}$ can be extended into a function defined in $\Omega_0$ such that $\mathrm{Ae}(u_0)\mathrm{n} \in \mathrm{H}^2(\Omega_0, \mathbb{R}^d)$. Thus it holds that $\mathrm{Ae}(u_0)\mathrm{n} \cdot$

$\varphi \in \mathrm{W}^{2,1}(\Omega_0, \mathbb{R}^d)$, and one can use Proposition 2.1.4 to get that

$$\int_{\Gamma_0} \theta \cdot \mathrm{n} \left(-\mathrm{Ae}(u_0) : \mathrm{e}(\varphi) + f \cdot \varphi\right) + \nabla\varphi^\top \mathrm{Ae}(u_0)\mathrm{n} \cdot \theta - \nabla\theta^\top \mathrm{Ae}(u_0)\mathrm{n} \cdot \varphi$$
$$= \int_{\Gamma_0} \theta \cdot \mathrm{n} \left(-\mathrm{Ae}(u_0) : \mathrm{e}(\varphi) + f \cdot \varphi + \partial_{\mathrm{n}} \left(\mathrm{Ae}(u_0)\mathrm{n} \cdot \varphi\right) + H\mathrm{Ae}(u_0)\mathrm{n} \cdot \varphi\right)$$
$$- \int_{\Gamma_0} \big(\nabla(\mathrm{Ae}(u_0)\mathrm{n})\theta + \nabla\theta^\top \mathrm{Ae}(u_0)\mathrm{n} + \mathrm{div}_\tau(\theta)\mathrm{Ae}(u_0)\mathrm{n}\big) \cdot \varphi,$$

for all $\varphi \in \mathcal{C}^\infty(\overline{\Omega_0}, \mathbb{R}^d)$. Moreover, since $-\mathrm{div}(\mathrm{Ae}(u_0)) = f \in \mathrm{H}^1(\Omega_0, \mathbb{R}^d)$, one deduces from Proposition 2.1.5 that

$$\int_{\Gamma_0} \theta \cdot \mathrm{n} \left(f + H\mathrm{Ae}(u_0)\mathrm{n}\right) \cdot \varphi = \int_{\Gamma_0} \mathrm{Ae}(u_0) : \nabla_\tau \left(\varphi \left(\theta \cdot \mathrm{n}\right)\right) - \left(\theta \cdot \mathrm{n}\right) \partial_{\mathrm{n}} \left(\mathrm{Ae}(u_0)\right) \mathrm{n} \cdot \varphi,$$

for all $\varphi \in \mathcal{C}^\infty(\overline{\Omega_0}, \mathbb{R}^d)$. Therefore, using the following equalities

$$\mathrm{Ae}(u_0) : \nabla_\tau \left(\varphi \left(\theta \cdot \mathrm{n}\right)\right) = \theta \cdot \mathrm{n} \left(\mathrm{Ae}(u_0) : \nabla_\tau \varphi\right) + (\mathrm{Ae}(u_0))\nabla_\tau \left(\theta \cdot \mathrm{n}\right) \cdot \varphi, \ \textit{a.e.} \text{ on } \Gamma_0,$$

and

$$\mathrm{Ae}(u_0) : \nabla_\tau \varphi = \mathrm{Ae}(u_0) : e(\varphi) - \nabla\varphi^\top (\mathrm{Ae}(u_0)\mathrm{n}) \cdot \mathrm{n} \ \textit{a.e} \text{ on } \Gamma_0,$$

one gets

$$\int_{\Gamma_0} \theta \cdot \mathrm{n} \left(-\mathrm{Ae}(u_0) : \mathrm{e}(\varphi) + f \cdot \varphi + \partial_{\mathrm{n}} \left(\mathrm{Ae}(u_0)\mathrm{n} \cdot \varphi\right) + H\mathrm{Ae}(u_0)\mathrm{n} \cdot \varphi\right)$$
$$- \int_{\Gamma_0} \big(\nabla(\mathrm{Ae}(u_0)\mathrm{n})\theta + \nabla\theta^\top \mathrm{Ae}(u_0)\mathrm{n} + \mathrm{div}_\tau(\theta)\mathrm{Ae}(u_0)\mathrm{n}\big) \cdot \varphi$$
$$= \int_{\Gamma_0} \left(\theta \cdot \mathrm{n} \left(\partial_{\mathrm{n}} \left(\mathrm{Ae}(u_0)\mathrm{n}\right) - \partial_{\mathrm{n}} \left(\mathrm{Ae}(u_0)\right) \mathrm{n}\right) + \mathrm{Ae}(u_0)\nabla_\tau \left(\theta \cdot \mathrm{n}\right)\right) \cdot \varphi$$
$$- \int_{\Gamma_0} \big(\nabla(\mathrm{Ae}(u_0)\mathrm{n})\theta + \big(\mathrm{div}_\tau(\theta)\mathrm{I} + \nabla\theta^\top\big) \mathrm{Ae}(u_0)\mathrm{n}\big) \cdot \varphi,$$

for all $\varphi \in \mathcal{C}^\infty(\overline{\Omega_0}, \mathbb{R}^d)$. Thus it follows that

$$\int_{\Omega_0} \big(\mathrm{div} \left(\mathrm{Ae}(u_0)\right) \theta^\top + \left(\mathrm{Ae}(u_0)\right) \nabla\theta^\top + \mathrm{A} \left(\nabla u_0 \nabla\theta\right) - \mathrm{Ae} \left(\nabla u_0 \theta\right)\big) : \nabla\varphi$$
$$- \int_{\Omega_0} \mathrm{div}(\theta)\mathrm{Ae}(u_0) : \mathrm{e}\left(\varphi\right) - \int_{\Gamma_0} \nabla\theta^\top \left(\mathrm{Ae}(u_0)\mathrm{n}\right) \cdot \varphi + \int_{\Gamma_0} \left(\theta \cdot \mathrm{n}\right) f \cdot \varphi$$
$$= \int_{\Gamma_0} \left(\theta \cdot \mathrm{n} \left(\partial_{\mathrm{n}} \left(\mathrm{Ae}(u_0)\mathrm{n}\right) - \partial_{\mathrm{n}} \left(\mathrm{Ae}(u_0)\right) \mathrm{n}\right) + \mathrm{Ae}(u_0)\nabla_\tau \left(\theta \cdot \mathrm{n}\right)\right) \cdot \varphi$$
$$- \int_{\Gamma_0} \big(\nabla(\mathrm{Ae}(u_0)\mathrm{n})\theta + \big(\mathrm{div}_\tau(\theta)\mathrm{I} + \nabla\theta^\top\big) \mathrm{Ae}(u_0)\mathrm{n}\big) \cdot \varphi,$$

for all $\varphi \in \mathcal{C}^\infty(\overline{\Omega_0}, \mathbb{R}^d)$, and one deduces by density of $\mathcal{C}^\infty(\overline{\Omega_0}, \mathbb{R}^d)$ in $\mathrm{H}^1(\Omega_0, \mathbb{R}^d)$ that

$$\langle u_0', v - u_0' \rangle_{\mathrm{A},\Omega_0} \geq \int_{\Gamma_{\mathrm{S}_0}} \left(\theta \cdot \mathrm{n} \left(\partial_\mathrm{n}\left(\mathrm{Ae}(u_0)\mathrm{n}\right) - \partial_\mathrm{n}\left(\mathrm{Ae}(u_0)\right)\mathrm{n}\right) + \mathrm{Ae}(u_0)\nabla_\tau\left(\theta\cdot\mathrm{n}\right)\right)\cdot\left(v - u_0'\right)$$
$$- \int_{\Gamma_{\mathrm{S}_0}} \left(\nabla(\mathrm{Ae}(u_0)\mathrm{n})\theta + \sigma_\mathrm{n}(u_0)\left(\mathrm{div}_\tau(\theta)\mathrm{I} + \nabla\theta^\top\right)\mathrm{n}\right)\cdot\left(v - u_0'\right),$$

for all $v \in \mathrm{T}_{\mathrm{N}_{\mathcal{K}_0^1(\Omega_0,\mathbb{R}^d)}}(u_0) \cap \left(\mathbb{R}\left(F_0 - u_0\right)\right)^\perp + \nabla\theta u_0 - \nabla u_0\theta$, which concludes the proof from Subsection 4.2.2. □

## 9.2 Shape gradient of the Signorini energy functional

Thanks to the characterization of the material and shape directional derivatives we can now prove the shape differentiability of the Signorini energy functional (9.2).

**Theorem 9.2.1.** *Consider the framework of Theorem 9.1.2. Then the Signorini energy functional $\mathcal{J}$ admits a shape gradient at $\Omega_0$ in the direction $\theta \in \mathcal{C}^{2,\infty}_\mathrm{D}(\mathbb{R}^d, \mathbb{R}^d)$ given by*

$$\mathcal{J}'(\Omega_0)(\theta) = \int_{\Omega_0} \mathrm{div}\left(\theta\right) \frac{\mathrm{Ae}\left(u_0\right) : \mathrm{e}\left(u_0\right)}{2} - \int_{\Omega_0} \mathrm{div}\left(\mathrm{Ae}\left(u_0\right)\right)\cdot\nabla u_0\theta - \int_{\Omega_0} \mathrm{Ae}\left(u_0\right) : \nabla u_0 \nabla\theta$$
$$- \int_{\Gamma_{\mathrm{S}_0}} \theta\cdot\mathrm{n}\left(f\cdot u_0\right) + \langle \mathrm{Ae}(u_0)\mathrm{n}, \nabla\theta u_0\rangle_{\mathrm{H}^{-1/2}(\Gamma_0,\mathbb{R}^d)\times\mathrm{H}^{1/2}(\Gamma_0,\mathbb{R}^d)}.$$

*Proof.* By following the usual strategy developed in the shape optimization literature (see, e.g., [8, 46]) to compute the shape gradient of $\mathcal{J}$ at $\Omega_0$ in the direction $\theta \in \mathcal{C}^{2,\infty}_\mathrm{D}(\mathbb{R}^d, \mathbb{R}^d)$, one gets

$$\mathcal{J}'(\Omega_0)(\theta) = -\frac{1}{2}\int_{\Omega_0} \mathrm{div}(\theta)\mathrm{Ae}(u_0) : \mathrm{e}(u_0) + \int_{\Omega_0} \mathrm{Ae}(u_0) : \nabla u_0\nabla\theta - \left\langle \overline{u}_0', u_0\right\rangle_{\mathrm{A},\Omega_0}.$$

Moreover, one has

$$\left\langle \overline{u}_0', u_0\right\rangle_{\mathrm{A},\Omega_0} = \left\langle \overline{\overline{u}}_0', u_0\right\rangle_{\mathrm{A},\Omega_0} + \left\langle \nabla\theta u_0, u_0\right\rangle_{\mathrm{A},\Omega_0},$$

and, since $\overline{\overline{u}}_0' \pm u_0 \in \mathrm{T}_{\mathrm{N}_{\mathcal{K}_0^1(\Omega_0,\mathbb{R}^d)}}(u_0) \cap \left(\mathbb{R}\left(F_0 - u_0\right)\right)^\perp$, one deduces from the variational formulation of $\overline{\overline{u}}_0'$ (see Inequality (9.3)) and the divergence formula that

$$\left\langle \overline{u}_0', u_0\right\rangle_{\mathrm{A},\Omega_0} = \int_{\Omega_0} \left(\mathrm{div}\left(\mathrm{Ae}(u_0)\right)\theta^\top + \left(\mathrm{Ae}(u_0)\right)\nabla\theta^\top + \mathrm{A}\left(\nabla u_0\nabla\theta\right) - \mathrm{div}(\theta)\mathrm{Ae}(u_0)\right) : \nabla u_0$$
$$+ \int_{\Gamma_{\mathrm{S}_0}} \theta\cdot\mathrm{n}(f\cdot u_0) - \langle \mathrm{Ae}(u_0)\mathrm{n}, \nabla\theta u_0\rangle_{\mathrm{H}^{-1/2}(\Gamma_0,\mathbb{R}^d)\times\mathrm{H}^{1/2}(\Gamma_0,\mathbb{R}^d)}.$$

Then, using the equality $\mathrm{div}\left(\mathrm{Ae}(u_0)\right)\theta^\top : \nabla u_0 = \mathrm{div}\left(\mathrm{Ae}(u_0)\right)\cdot\nabla u_0\theta$ *a.e.* on $\Omega_0$, one concludes the proof. □

This result can be improved with additional regularity assumption.

**Corollary 9.2.2.** *Consider the framework of Theorem 9.2.1 and assume that* $u_0 \in \mathrm{H}^2(\Omega_0, \mathbb{R}^d)$*. Then the Signorini energy functional* $\mathcal{J}$ *admits a shape gradient at* $\Omega_0$ *in the direction* $\theta \in \mathcal{C}^{2,\infty}_{\mathrm{D}}(\mathbb{R}^d, \mathbb{R}^d)$ *given by*

$$\mathcal{J}'(\Omega_0)(\theta) = \int_{\Gamma_{\mathrm{S}_0}} \left( \theta \cdot \mathrm{n} \left( \frac{\mathrm{Ae}(u_0) : \mathrm{e}(u_0)}{2} - f \cdot u_0 \right) + \mathrm{Ae}(u_0)\mathrm{n} \cdot (\nabla\theta u_0 - \nabla u_0 \theta) \right).$$

*Proof.* Let $\theta \in \mathcal{C}^{2,\infty}_{\mathrm{D}}(\mathbb{R}^d, \mathbb{R}^d)$. Since $u_0 \in \mathrm{H}^2(\Omega_0, \mathbb{R}^d)$, it follows from Theorem 9.2.1 that

$$\begin{aligned}\mathcal{J}'(\Omega_0)(\theta) = &-\frac{1}{2}\int_{\Omega_0} \theta \cdot \nabla(\mathrm{Ae}(u_0) : \mathrm{e}(u_0)) + \int_{\Gamma_0} \theta \cdot \mathrm{n} \frac{\mathrm{Ae}(u_0) : \mathrm{e}(u_0)}{2} + \int_{\Omega_0} \mathrm{Ae}\,(u_0) : \mathrm{e}\,(\nabla u_0 \theta) \\ &- \int_{\Gamma_0} \mathrm{Ae}\,(u_0)\,\mathrm{n} \cdot \nabla u_0 \theta - \int_{\Omega_0} \mathrm{Ae}\,(u_0) : \nabla u_0 \nabla\theta - \int_{\Gamma_{\mathrm{S}_0}} \theta \cdot \mathrm{n}\,(f \cdot u_0) + \int_{\Gamma_{\mathrm{S}_0}} \mathrm{Ae}(u_0)\mathrm{n} \cdot \nabla\theta u_0.\end{aligned}$$

Moreover, since

$$\mathrm{Ae}(u_0) : \mathrm{e}\,(\nabla u_0 \theta) = \mathrm{Ae}(u_0) : \nabla u_0 \nabla\theta + \frac{1}{2}\theta \cdot \nabla(\mathrm{Ae}(u_0) : \mathrm{e}(u_0)) \ \textit{a.e.} \text{ on } \Omega_0,$$

one deduces

$$\mathcal{J}'(\Omega_0)(\theta) = \int_{\Gamma_0} \theta \cdot \mathrm{n} \left( \frac{\mathrm{Ae}(u_0) : \mathrm{e}(u_0)}{2} \right) - \int_{\Gamma_0} \mathrm{Ae}(u_0)\mathrm{n} \cdot \nabla u_0 \theta - \int_{\Gamma_{\mathrm{S}_0}} \theta \cdot \mathrm{n}(f \cdot u_0) + \int_{\Gamma_0} \mathrm{Ae}(u_0)\mathrm{n} \cdot \nabla\theta u_0,$$

which completes the proof since $\theta = 0$ on $\Gamma_{\mathrm{D}}$. □

**Remark 9.2.3.** Consider the framework of Theorem 9.2.1. In the same way as Remark 8.2.4, note that the scalar product $\langle \overline{u}'_0, u_0 \rangle_{\mathrm{A},\Omega_0}$ is linear with respect to the direction $\theta$, while $\overline{u}'_0$ is not. This leads to an expression of the shape gradient $\mathcal{J}'(\Omega_0)(\theta)$ in Theorem 9.2.1 that is linear with respect to the direction $\theta$, thus to the shape differentiability of the Signorini energy functional $\mathcal{J}$ at $\Omega_0$.

# 9.3 Numerical simulations

In this section we numerically solve an example of the shape optimization problem (9.1) in the two-dimensional case $d = 2$, by making use of our theoretical results obtained in Section 9.2. The numerical simulations have been performed using Freefem++ software [44] with P1-finite elements and standard affine mesh, and we use the expression of the shape gradient of $\mathcal{J}$ obtained in Corollary 9.2.2.

## 9.3.1 Numerical methodology

Consider an initial shape $\Omega_0 \in \mathcal{U}_{\mathrm{ref}}$. Note that Corollary 9.2.2 allows to exhibit a descent direction $\theta_0$ of the Signorini energy functional $\mathcal{J}$ at $\Omega_0$ by finding the solution $\theta_0$ to the variational equality

$$\langle \theta_0, \theta \rangle_{\mathrm{A},\Omega_0} = -\mathcal{J}'(\Omega_0)(\theta), \qquad \forall \theta \in \mathrm{H}^1_{\mathrm{D}}(\Omega_0, \mathbb{R}^d),$$

since it satisfies $\mathcal{J}'(\Omega_0)(\theta_0) = -\|\theta_0\|^2_{\mathrm{A},\Omega_0} \leq 0$.

In order to numerically solve the shape optimization problem (9.1) on a given example, we have to deal with the volume constraint $|\Omega| = |\Omega_{\mathrm{ref}}| > 0$. To this aim, as Chapter 8, the Uzawa algorithm is used and one refers to Subsection 8.3.1 for methodological details.

Let us mention that the Signorini problem is numerically solved using the Nitsche method (see, e.g., [21, 23, 64]). In a nutshell, the solution $u_0 \in \mathrm{H}^1_{\mathrm{D}}(\Omega_0, \mathbb{R}^d)$ is approximated by $u_0^h \in \mathbb{V}^h$ which is the solution to the Nitsche formulation

$$\int_{\Omega_0} \mathrm{Ae}(u_0^h) : \mathrm{e}(v^h) - \gamma \int_{\Gamma_{\mathrm{S}_0}} \sigma_{\mathrm{n}}(u_0^h)\sigma_{\mathrm{n}}(v^h) + \frac{1}{\gamma}\int_{\Gamma_{\mathrm{S}_0}} \left[ {u_0}_{\mathrm{n}}^h - \gamma\sigma_{\mathrm{n}}(u_0^h) \right]_+ \left[ v_{\mathrm{n}}^h - \gamma\sigma_{\mathrm{n}}(v^h) \right] = \int_{\Omega_0} f \cdot v^h, \qquad \forall v^h \in \mathbb{V}^h,$$

where $\mathbb{V}^h$ is the classical P1-finite elements space whose elements are null on $\Gamma_{\mathrm{D}}$ (see [23] for numerical analysis details). We also precise that, for all $i \in \mathbb{N}^*$, the difference between the Signorini energy functional $\mathcal{J}$ at the iteration $20 \times i$ and at the iteration $20 \times (i-1)$ is computed. The smallness of this difference is used as a stopping criterion for the algorithm. Note that we have also used the iterative switching algorithm (as we did in Section 8.3) instead of the Nitsche method to solve numerically the solution to the Signorini problem. Both show similar results, however a comparative study of these methods to determine their advantages and drawbacks would be interesting, but has not been performed in this thesis.

### 9.3.2 Example and numerical results

In this subsection, let $d = 2$ and $f \in \mathrm{H}^1(\mathbb{R}^2, \mathbb{R}^2)$ given by

$$\begin{array}{rccl} f: & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & (x,y) & \longmapsto & f(x,y) := \begin{pmatrix} \frac{1}{2}\exp(x^2)\eta(x,y) & 0 \end{pmatrix}, \end{array}$$

where $\eta \in \mathcal{C}^\infty_0(\mathbb{R}^2, \mathbb{R})$ is a cut-off function chosen appropriately so that $f$ satisfies the assumptions of the present chapter. The reference shape $\Omega_{\mathrm{ref}}$ is the unit disk of $\mathbb{R}^2$, and the fixed part $\Gamma_{\mathrm{D}}$ is given by

$$\Gamma_{\mathrm{D}} = \left\{ (\cos\alpha, \sin\alpha) \in \Gamma_{\mathrm{ref}} \mid \alpha \in \left[\frac{\pi}{6}, \frac{5\pi}{6}\right] \cup \left[\frac{7\pi}{6}, \frac{11\pi}{6}\right] \right\},$$

(see Figure 9.1). The volume constraint is $|\Omega_{\mathrm{ref}}| = \pi$ and the initial shape is $\Omega_0 := \Omega_{\mathrm{ref}}$. We assume that all shapes in $\mathcal{U}_{\mathrm{ref}}$ are isotropic, which means that their mechanical properties are identical in all directions (see Remark 1.0.1). In that case, for all $\Omega \in \mathcal{U}_{\mathrm{ref}}$, the Cauchy stress tensor is given, for all $v \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$, by

$$\sigma(v) = 2\mu\mathrm{e}(v) + \lambda\mathrm{tr}\,(\mathrm{e}(v))\,\mathrm{I},$$

where $\mathrm{tr}\,(\mathrm{e}(v))$ is the trace of the matrix $\mathrm{e}(v)$, and $\mu \geq 0, \lambda \geq 0$ are Lamé parameters (see, e.g., [73]). In what follows, we consider $\mu = 0.3846$ and $\lambda = 0.5769$ that correspond to a Young's modulus equal to 1 and of Poisson's ratio equal to 0.3 which is a typical value for a large variety of materials, and one presents the numerical results obtained for this two-dimensional example using the methodology described in Subsection 9.3.1.

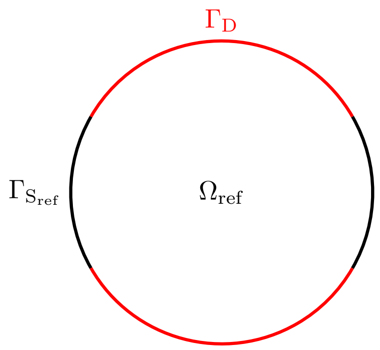


Figure 9.1 – Unit disk $\Omega_{ref}$ and its boundary $\Gamma_{ref} = \Gamma_D \cup \Gamma_{S_{ref}}$.

In Figure 9.2 is represented the initial shape (left) and the shape which solves Problem (9.1) (right). On top are the vector values of the solution $u$ to the Signorini problem ($SP_\Omega$). Note that the black boundary shows where $\sigma_n(u) = 0$, while the yellow boundary shows where $u_n = 0$. At the bottom is shown the values of the integrand of $\mathcal{J}$. It seems that the area where the integrand of $\mathcal{J}$ is the lowest (in orange) has been shifted to the left by "pushing" the left boundary (which corresponds to the part where there is no compressive stress), while in return, the right boundary (which corresponds to the contact part) has been pulled.

Figure 9.3 shows the values of $\mathcal{J}$ (left) and the volume of the shape (right) with respect to the iteration. We observe that $\mathcal{J}$ is lower at the final shape than at the initial shape, with some oscillations due to the Lagrange multiplier in order to satisfy the volume constraint.

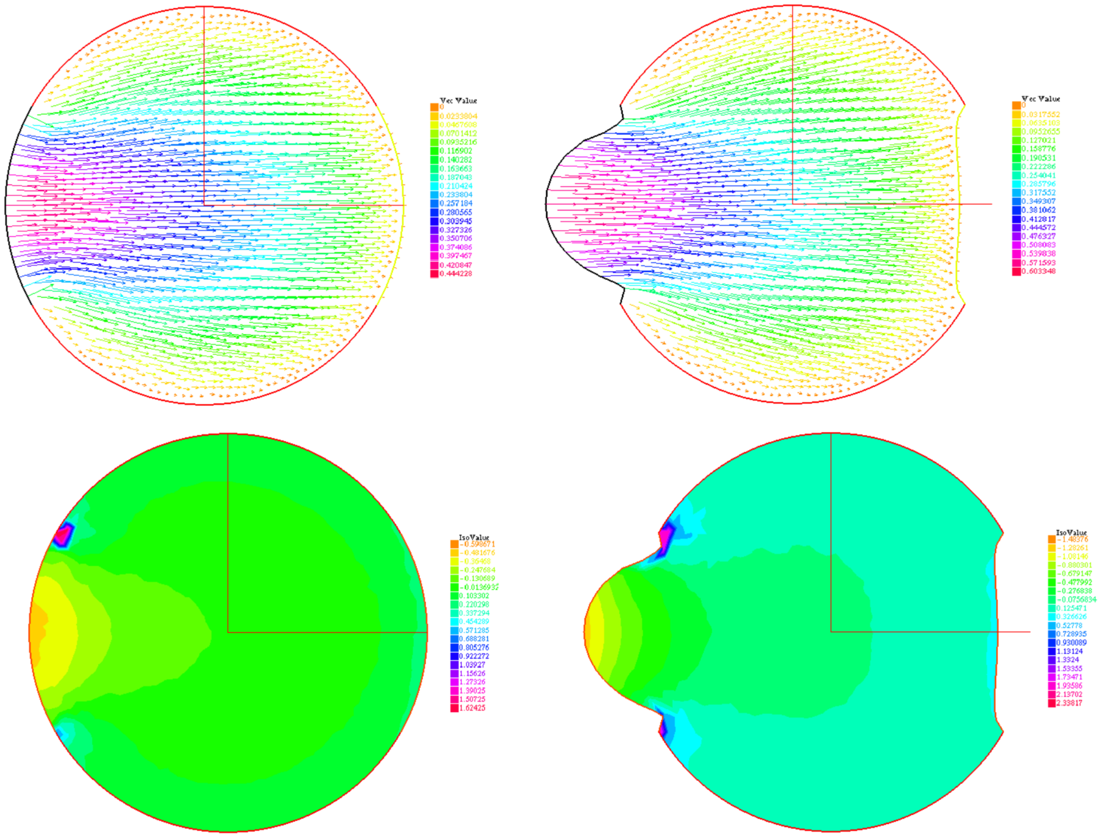


Figure 9.2 – Initial shape (left) and the shape minimizing $\mathcal{J}$ (right), under the volume constraint $|\Omega_{\text{ref}}| = \pi$. On top is shown the vector values of the Signorini solution, while at bottom is shown the values of the integrand of $\mathcal{J}$.

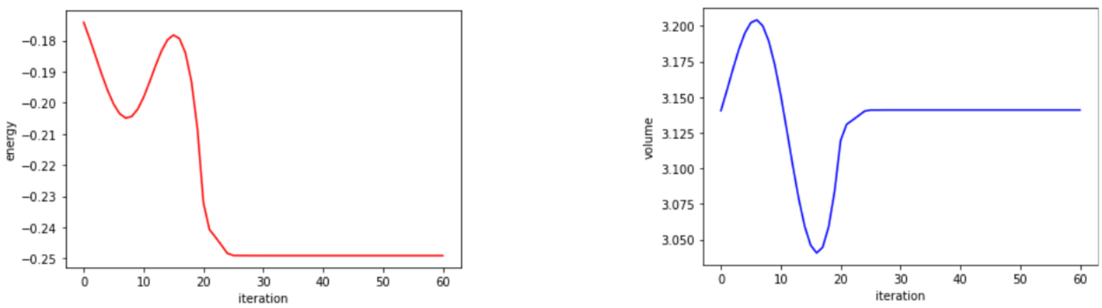


Figure 9.3 – Values of the energy functional (left) and the volume (right) with respect to the iterations.

Chapter 10

# Optimal control for variational inequalities: the Tresca problem in the linear elastic model

Let $d \in \{2,3\}$ and assume that $\Omega$ has a $\mathcal{C}^1$-boundary. We consider the decomposition $\Gamma =: \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{S}}$, where $\Gamma_{\mathrm{D}}$ and $\Gamma_{\mathrm{S}}$ are two measurable (with positive measure) pairwise disjoint subsets of $\Gamma$ such that almost every point of $\Gamma_{\mathrm{S}}$ belongs to $\mathrm{int}_{\Gamma}(\Gamma_{\mathrm{S}})$ (see Remark 4.2.23 for comments on this last assumption). We assume that $\Omega$ is an elastic solid satisfying the linear elastic model (see Chapter 1), and we consider $f \in \mathrm{L}^2(\Omega, \mathbb{R}^d)$, $h \in \mathrm{L}^2(\Gamma_{\mathrm{S}})$, $g_1 \in \mathrm{L}^\infty(\Gamma_{\mathrm{S}})$ such that $g_1 \geq m$ *a.e.* on $\Gamma_{\mathrm{S}}$ for some positive constant $m > 0$ and $g_2 \in \mathrm{L}^\infty(\Gamma_{\mathrm{S}})$ such that $||g_2||_{\mathrm{L}^\infty(\Gamma_{\mathrm{S}})} > 0$. In this chapter we consider the optimal control problem given by

$$\underset{z \in \mathcal{U}}{\mathrm{minimize}} \ \mathcal{J}(z), \tag{10.1}$$

where $\mathcal{J}$ is the cost functional defined by

$$\begin{array}{rccl} \mathcal{J}: & \mathrm{V} & \longrightarrow & \mathbb{R} \\ & z & \longmapsto & \mathcal{J}(z) := \frac{1}{2} \left\| u(\ell(z)) \right\|_{\mathrm{A}}^2 + \frac{\beta}{2} \left\| \ell(z) \right\|_{\mathrm{L}^2(\Gamma_{\mathrm{S}})}^2 , \end{array} \tag{10.2}$$

where $|| \cdot ||_{\mathrm{A}}$ is the norm associated with the scalar product $\langle \cdot, \cdot \rangle_{\mathrm{A}}$ (see (4.1)), V is the open subset of $\mathrm{L}^\infty(\Gamma_{\mathrm{S}})$ defined by

$$\mathrm{V} := \left\{ z \in \mathrm{L}^\infty(\Gamma_{\mathrm{S}}) \mid \exists C(z) > 0, \, \ell(z) > C(z) \ a.e. \text{ on } \Gamma_{\mathrm{S}} \right\},$$

where $\ell$ is the map defined by $z \in \mathrm{L}^\infty(\Gamma_{\mathrm{S}}) \mapsto \ell(z) := g_1 + z g_2 \in \mathrm{L}^\infty(\Gamma_{\mathrm{S}})$, and where $u(\ell(z)) \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ stands for the unique solution to the Tresca friction problem

$$\left\{ \begin{aligned} -\mathrm{div}(\mathrm{Ae}(u)) &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_{\mathrm{D}}, \\ \sigma_{\mathrm{n}}(u) &= h && \text{on } \Gamma_{\mathrm{S}}, \\ \|\sigma_\tau(u)\| \leq \ell(z) \text{ and } u_\tau \cdot \sigma_\tau(u) + \ell(z) \|u_\tau\| &= 0 && \text{on } \Gamma_{\mathrm{S}}, \end{aligned} \right. \tag{$\mathrm{CTP}_{\ell(z)}$}$$

where $\beta > 0$ is a positive constant and where $\mathcal{U}$ is a given nonempty convex subset of V such that $\mathcal{U}$ is a bounded closed subset of $\mathrm{L}^2(\Gamma_{\mathrm{S}})$. Note that the first term in the cost functional $\mathcal{J}$ corresponds to the compliance, while the second term is the energy consumption which is standard in optimal control problems (see, e.g., [55]).

This chapter is organized as follows. In Section 10.1 we prove the existence of a solution to Problem (10.1). In Section 10.2 we prove, under some assumptions, that $\mathcal{J}$ is Gateaux differentiable on V and we characterize its gradient. Finally, in Section 10.3, numerical simulations are performed to solve Problem (10.1) on a two-dimensional example.

## 10.1 Existence

This section is dedicated to the following existence result.

**Proposition 10.1.1.** *There exists $z^* \in \mathcal{U}$ such that $\mathcal{J}(z^*) \leq \mathcal{J}(z)$ for all $z \in \mathcal{U}$.*

*Proof.* In this proof the strong (resp. weak) convergence in Hilbert spaces is denoted by $\rightarrow$ (resp. $\rightharpoonup$) and all limits with respect to the index $i$ will be considered for $i \rightarrow +\infty$. Since $0 \leq \mathcal{J}(z) < +\infty$ for all $z \in \mathcal{U}$, we get that $\inf_{z\in\mathcal{U}} \mathcal{J}(z) \in \mathbb{R}_+$. Considering a minimizing sequence $(z_i)_{i\in\mathbb{N}}$, there exists $N \in \mathbb{N}$ such that $\mathcal{J}(z_i) \leq 1 + \inf\limits_{z\in\mathcal{U}} \mathcal{J}(z)$ for all $i \geq N$, that is

$$\frac{1}{2} \left\| u(\ell(z_i)) \right\|_{\mathrm{A}}^2 + \frac{\beta}{2} \left\| \ell(z_i) \right\|_{\mathrm{L}^2(\Gamma_{\mathrm{S}})}^2 \leq 1 + \inf_{z\in\mathcal{U}} \mathcal{J}(z),$$

for all $i \geq N$. Thus the sequence $(\ell(z_i))_{i\in\mathbb{N}}$ is bounded in $\mathrm{L}^2(\Gamma_{\mathrm{S}})$ and thus, up to a subsequence that we do not relabel, weakly converges to some $g^* \in \mathrm{L}^2(\Gamma_{\mathrm{S}})$. Moreover, since $\mathcal{U}$ is a bounded closed convex subset of $\mathrm{L}^2(\Gamma_{\mathrm{S}})$ (and thus weakly closed in $\mathrm{L}^2(\Gamma_{\mathrm{S}})$), we know that, up to a subsequence that we do not relabel, the sequence $(z_i)_{i\in\mathbb{N}}$ weakly converges to some $z^* \in \mathcal{U}$. Moreover one has

$$\left| \int_{\Gamma_{\mathrm{S}}} \left( \ell(z_i) - g_1 - z^* g_2 \right) v \right| = \int_{\Gamma_{\mathrm{S}}} \left( z_i - z^* \right) g_2 v,$$

for all $v \in \mathrm{L}^2(\Gamma_{\mathrm{S}})$, and, since $g_2 \in \mathrm{L}^\infty(\Gamma_{\mathrm{S}})$, it holds that $g_2 v \in \mathrm{L}^2(\Gamma_{\mathrm{S}})$ and one deduces that $\ell(z_i) \rightharpoonup g_1 + z^* g_2$ in $\mathrm{L}^2(\Gamma_{\mathrm{S}})$ and thus $g^* = g_1 + z^* g_2$. In a similar way, up to a subsequence that we do not relabel, the sequence $(u(\ell(z_i)))_{i\in\mathbb{N}}$ weakly converges in $\mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ to some $u^* \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ and thus $u(\ell(z_i)) \rightarrow u^*$ in $\mathrm{L}^2(\Gamma, \mathbb{R}^d)$ from the compact embedding $\mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d) \hookrightarrow\!\!\!\!\rightarrow \mathrm{L}^2(\Gamma, \mathbb{R}^d)$ (see Proposition 2.1.7). Let us prove that $u(\ell(z_i)) \rightarrow u^*$ in $\mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$. It holds that

$$\left\| u^* - u(\ell(z_i)) \right\|_{\mathrm{A}}^2 = \left\langle u^*, u^* - u(\ell(z_i)) \right\rangle_{\mathrm{A}} - \left\langle u(\ell(z_i)), u^* - u(\ell(z_i)) \right\rangle_{\mathrm{A}},$$

for all $i \in \mathbb{N}$. Using the weak formulation satisfies by $u(\ell(z_i))$, we get that

$$\begin{aligned}\|u^* - u(\ell(z_i))\|_{\mathrm{A}}^2 \leq \langle u^*, u^* - u(\ell(z_i))\rangle_{\mathrm{A}} - \int_\Omega f \cdot (u^* - u(\ell(z_i))) \\ - \int_{\Gamma_{\mathrm{S}}} h\,(u^*_{\mathrm{n}} - u(\ell(z_i))_{\mathrm{n}}) + \int_{\Gamma_{\mathrm{S}}} \ell(z_i)\,(\|u^*_\tau\| - \|u(\ell(z_i))_\tau\|) \\ \leq \langle u^*, u^* - u(\ell(z_i))\rangle_{\mathrm{A}} - \int_\Omega f \cdot (u^* - u(\ell(z_i))) - \int_{\Gamma_{\mathrm{S}}} h\,(u^*_{\mathrm{n}} - u(\ell(z_i))_{\mathrm{n}}) \\ + C\,\|u^* - u(\ell(z_i))\|_{\mathrm{L}^2(\Gamma,\mathbb{R}^d)} \longrightarrow 0,\end{aligned}$$

where $C > 0$ is a constant (depending only on $\Omega$ and on $\max_{i\in\mathbb{N}} ||\ell(z_i)||_{\mathrm{L}^2(\Gamma_{\mathrm{S}})}$). Now let us prove that $u^* = u(g_1 + z^* g_2)$. For $v \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ fixed, it holds that

$$\begin{aligned}\langle u(\ell(z_i)), v - u(\ell(z_i))\rangle_{\mathrm{A}} + \int_{\Gamma_{\mathrm{S}}} \ell(z_i)\,\|v_\tau\| - \int_{\Gamma_{\mathrm{S}}} \ell(z_i)\,\|u(\ell(z_i))_\tau\| \\ \geq \int_\Omega f \cdot (v - u(\ell(z_i))) + \int_{\Gamma_{\mathrm{S}}} h\,(v_{\mathrm{n}} - u(\ell(z_i))_{\mathrm{n}})\,, \end{aligned} \tag{10.3}$$

for all $i \in \mathbb{N}$. Note that:

(i) $|\langle u(\ell(z_i)), v - u(\ell(z_i))\rangle_{\mathrm{A}} - \langle u^*, v - u^*\rangle_{\mathrm{A}}| \leq D\,\|u^* - u(\ell(z_i))\|_{\mathrm{A}} \longrightarrow 0;$

(ii) $\left|\int_\Omega f \cdot (v - u(\ell(z_i))) - \int_\Omega f \cdot (v - u^*)\right| \leq D\,\|f\|_{\mathrm{L}^2(\Omega,\mathbb{R}^d)}\,\|u^* - u(\ell(z_i))\|_{\mathrm{A}} \longrightarrow 0;$

(iii) $\left|\int_{\Gamma_{\mathrm{S}}} h\,(v_{\mathrm{n}} - u(\ell(z_i))_{\mathrm{n}}) - \int_{\Gamma_{\mathrm{S}}} h\,(v_{\mathrm{n}} - u^*_{\mathrm{n}})\right| \leq D\,\|h\|_{\mathrm{L}^2(\Gamma_{\mathrm{S}})}\,\|u^* - u(\ell(z_i))\|_{\mathrm{L}^2(\Gamma,\mathbb{R}^d)} \longrightarrow 0;$

(iv) $\left|\int_{\Gamma_{\mathrm{S}}} \ell(z_i)\,(\|v_\tau\| - \|u(\ell(z_i))_\tau\|) - \int_{\Gamma_{\mathrm{S}}} g^*\,(\|v_\tau\| - \|u^*_\tau\|)\right| \leq$
$\left|\int_{\Gamma_{\mathrm{S}}} (\ell(z_i) - g^*)\,\|v_\tau\|\right| + \left|\int_{\Gamma_{\mathrm{S}}} (\ell(z_i) - g^*)\,\|u^*_\tau\|\right| + D\,\|u^* - u(\ell(z_i))\|_{\mathrm{L}^2(\Gamma,\mathbb{R}^d)} \longrightarrow 0;$

where $D \geq 0$ is a constant (depending only on $\Omega$, A and $v$). Therefore it follows in (10.3) when $i \to +\infty$ that,

$$\langle u^*, v - u^*\rangle_{\mathrm{A}} + \int_{\Gamma_{\mathrm{S}}} g^*\,\|v_\tau\| - \int_{\Gamma_{\mathrm{S}}} g^*\,\|u^*_\tau\| \geq \int_\Omega f \cdot (v - u^*) + \int_{\Gamma_{\mathrm{S}}} h\,(v_{\mathrm{n}} - u^*_{\mathrm{n}})\,.$$

Since this inequality is true for all $v \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ and $g^* = g_1 + z^* g_2$, one deduces that $u^* = u(g_1 + z^* g_2)$, and then

$$\begin{aligned}\mathcal{J}(z^*) = \frac{1}{2}\,\|u(g_1 + z^* g_2)\|_{\mathrm{A}}^2 + \frac{\beta}{2}\,\|g_1 + z^* g_2\|^2_{\mathrm{L}^2(\Gamma_{\mathrm{S}})} \leq \\ \liminf_{i\to+\infty} \left(\frac{1}{2}\,\|u(\ell(z_i))\|_{\mathrm{A}}^2 + \frac{\beta}{2}\,\|\ell(z_i)\|^2_{\mathrm{L}^2(\Gamma_{\mathrm{S}})}\right) \leq \liminf_{i\to+\infty} \mathcal{J}(z_i) = \inf_{z\in\mathcal{U}} \mathcal{J}(z),\end{aligned}$$

which concludes the proof. □

**Remark 10.1.2.** Since the solution to the Tresca friction problem is not linear with respect to the friction term, note that $\mathcal{J}$ is not a strictly convex functional (and thus the uniqueness of the solution to Problem (10.1) is not guaranteed).

## 10.2 Gateaux differentiability of the cost functional

Consider the auxiliary functional

$$\begin{array}{rccl} \mathrm{J}: & \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)\times \mathrm{L}^\infty(\Gamma_{\mathrm{S}}) & \longrightarrow & \mathbb{R} \\ & (v,g) & \longmapsto & \mathrm{J}(v,g) := \frac{1}{2}\left\|v\right\|^2_{\mathrm{A}} + \frac{\beta}{2}\left\|g\right\|^2_{\mathrm{L}^2(\Gamma_{\mathrm{S}})}. \end{array}$$

One can easily prove that J is Fréchet differentiable on $\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)\times \mathrm{L}^\infty(\Gamma_{\mathrm{S}})$ and its Fréchet differential at some $(v,g)\in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)\times \mathrm{L}^\infty(\Gamma_{\mathrm{S}})$, denoted by $\mathrm{dJ}(v,g)$, is given by

$$\mathrm{dJ}(v,g)(\tilde{v},\tilde{g}) = \langle v,\tilde{v}\rangle_{\mathrm{A}} + \beta\left\langle g,\tilde{g}\right\rangle_{\mathrm{L}^2(\Gamma_{\mathrm{S}})},$$

for all $(\tilde{v},\tilde{g})\in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)\times \mathrm{L}^\infty(\Gamma_{\mathrm{S}})$. Now let us introduce the map

$$\begin{array}{rccl} \mathcal{F}: & \mathrm{V} & \longrightarrow & \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)\times \mathrm{L}^\infty(\Gamma_{\mathrm{S}}) \\ & z & \longmapsto & \mathcal{F}(z) := \left(u(\ell(z)),\ell(z)\right), \end{array}$$

where $u(\ell(z))\in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ is the unique solution to the Tresca friction problem $(\mathrm{CTP}_{\ell(z)})$. Hence the cost functional $\mathcal{J}$ is given by the composition $\mathcal{J} = \mathrm{J}\circ\mathcal{F}$.

**Theorem 10.2.1.** *Let $z_0\in\mathrm{V}$ be fixed and let us denote by $u_0 := u(\ell(z_0))$. Assume that:*

(i) *the map $s\in\Gamma_{\mathrm{S}^{u_0,\ell(z_0)}_{\mathrm{R}}}\mapsto \frac{\ell(z_0)(s)}{\|u_{0\tau}(s)\|}\in\mathbb{R}^*_+$ belongs to $\mathrm{L}^4(\Gamma_{\mathrm{S}^{u_0,\ell(z_0)}_{\mathrm{R}}})$ (see below for the set $\Gamma_{\mathrm{S}^{u_0,\ell(z_0)}_{\mathrm{R}}}$);*

(ii) *the parameterized Tresca friction functional $\Phi$ defined in (6.8) is twice epi-differentiable at $u_0$ for $F-u_0\in\partial\Phi(0,\cdot)(u_0)$, with*

$$\mathrm{D}^2_e\Phi(u_0)|F-u_0)(v) = \int_{\Gamma_{\mathrm{S}}}\mathrm{D}^2_e H(s)(u_0(s)|\sigma_\tau(F-u_0)(s))(v(s))\mathrm{d}s, \qquad \forall v\in\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d),$$

*where, for almost all $s\in\Gamma_{\mathrm{S}}$, the map $H(s)$ is defined in Proposition 6.2.2, and $F\in\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ is the unique solution to the Dirichlet-Neumann problem*

$$\begin{cases} -\mathrm{div}(\mathrm{Ae}(F)) = f & \text{in } \Omega, \\ F = 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \mathrm{Ae}(F)\mathrm{n} = h\mathrm{n} & \text{on } \Gamma_{\mathrm{S}}. \end{cases} \tag{10.4}$$

*Then the cost functional $\mathcal{J}$ is Gateaux differentiable at $z_0$ and its differential $\mathrm{d}_G\mathcal{J}(z_0)$ is given by*

$$\mathrm{d}_G\mathcal{J}(z_0)(z) = \int_{\Gamma_{\mathrm{S}^{u_0,\ell(z_0)}_{\mathrm{R}}}} zg_2\left(\beta\left(g_1+z_0g_2\right)-\|u_{0_\tau}\|\right) + \int_{\Gamma_{\mathrm{S}^{u_0,\ell(z_0)}_{\mathrm{D}}}\cup\Gamma_{\mathrm{S}^{u_0,\ell(z_0)}_{\mathrm{S}}}} \beta zg_2\left(g_1+z_0g_2\right),$$

*for all* $z \in \mathrm{L}^\infty(\Gamma_\mathrm{S})$*, where* $\Gamma_\mathrm{S}$ *is decomposed (up to a null set) as* $\Gamma_{\mathrm{S}_\mathrm{R}^{u_0,\ell(z_0)}} \cup \Gamma_{\mathrm{S}_\mathrm{D}^{u_0,\ell(z_0)}} \cup \Gamma_{\mathrm{S}_\mathrm{S}^{u_0,\ell(z_0)}}$ *with*

$$\begin{aligned}
\Gamma_{\mathrm{S}_\mathrm{R}^{u_0,\ell(z_0)}} &:= \left\{ s \in \Gamma_\mathrm{S} \mid u_{0_\tau}(s) \neq 0 \right\}, \\
\Gamma_{\mathrm{S}_\mathrm{D}^{u_0,\ell(z_0)}} &:= \left\{ s \in \Gamma_\mathrm{S} \mid u_{0_\tau}(s) = 0 \ \textit{and} \ \tfrac{\sigma_\tau(u_0)(s)}{\ell(z_0)(s)} \in \mathrm{B}(0,1) \cap (\mathbb{R}\mathrm{n}(s))^\perp \right\}, \\
\Gamma_{\mathrm{S}_\mathrm{S}^{u_0,\ell(z_0)}} &:= \left\{ s \in \Gamma_\mathrm{S} \mid u_{0_\tau}(s) = 0 \ \textit{and} \ \tfrac{\sigma_\tau(u_0)(s)}{\ell(z_0)(s)} \in \partial\mathrm{B}(0,1) \cap (\mathbb{R}\mathrm{n}(s))^\perp \right\}.
\end{aligned}$$

*Proof.* Let $z \in \mathrm{L}^\infty(\Gamma_\mathrm{S})$ and $t > 0$ be sufficiently small such that $z_t := z_0 + tz \in \mathrm{V}$. We denote by $u_t := u(\ell(z_t)) \in \mathrm{H}^1_\mathrm{D}(\Omega, \mathbb{R}^d)$. From Subsection 4.2.4, $u_t \in \mathrm{H}^1_\mathrm{D}(\Omega, \mathbb{R}^d)$ is given by $u_t = \mathrm{prox}_{\Phi(t,\cdot)}(F)$, where $\Phi$ is the parameterized Tresca friction functional defined in (6.8) and $F$ is the unique solution to the Dirichlet-Neumann problem (10.4). From Hypotheses (i), (ii) and since the map $t \in \mathbb{R}_+ \mapsto \ell(z_t) \in \mathrm{L}^\infty(\Gamma_\mathrm{S})$ is differentiable at $t = 0$, with its derivative given by $\ell'(z_0) := zg_2$, one can apply Theorem 6.2.7 to deduce that the map $t \in \mathbb{R}_+ \mapsto u_t \in \mathrm{H}^1_\mathrm{D}(\Omega, \mathbb{R}^d)$ is differentiable at $t = 0$ and its derivative, denoted by $u'_0 \in \mathcal{K}_{u_0, \frac{\sigma_\tau(F_0 - u_0)}{\ell(z_0)}} \subset \mathrm{H}^1_\mathrm{D}(\Omega, \mathbb{R}^d)$, is the unique solution to the variational inequality (which is the weak formulation of a tangential Signorini problem) given by

$$\begin{aligned}
\langle u'_0, v - u'_0 \rangle_\mathrm{A} \geq & \int_{\Gamma_{\mathrm{S}_\mathrm{S}^{u_0,\ell(z_0)}}} \ell'(z_0) \frac{\sigma_\tau(u_0)}{\ell(z_0)} \cdot (v_\tau - u'_{0\tau}) \\
&+ \int_{\Gamma_{\mathrm{S}_\mathrm{R}^{u_0,\ell(z_0)}}} \left( -\ell'(z_0) \frac{u_{0_\tau}}{\|u_{0_\tau}\|} - \frac{\ell(z_0)}{\|u_{0_\tau}\|} \left( u'_{0\tau} - \left( u'_{0\tau} \cdot \frac{u_{0_\tau}}{\|u_{0_\tau}\|} \right) \frac{u_{0_\tau}}{\|u_{0_\tau}\|} \right) \right) \cdot (v_\tau - u'_{0\tau}),
\end{aligned}$$

for all $v \in \mathcal{K}_{u_0, \frac{\sigma_\tau(F_0 - u_0)}{\ell(z_0)}}$, where

$$\mathcal{K}_{u_0, \frac{\sigma_\tau(F_0 - u_0)}{\ell(z_0)}} := \left\{ v \in \mathrm{H}^1_\mathrm{D}(\Omega, \mathbb{R}^d) \mid v_\tau = 0 \ \textit{a.e.} \text{ on } \Gamma_{\mathrm{S}_\mathrm{D}^{u_0,\ell(z_0)}} \text{ and } v_\tau \in \mathbb{R}_- \frac{\sigma_\tau(u_0)}{\ell(z_0)} \ \textit{a.e.} \text{ on } \Gamma_{\mathrm{S}_\mathrm{S}^{u_0,\ell(z_0)}} \right\}.$$

Since $\mathcal{J} = \mathrm{J} \circ \mathcal{F}$, with J Fréchet differentiable on $\mathrm{H}^1_\mathrm{D}(\Omega, \mathbb{R}^d) \times \mathrm{V}$, and

$$\frac{\|\mathcal{F}(z_0 + tz) - \mathcal{F}(z_0) - t\,(u'_0, \ell'(z_0))\|_{\mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d) \times \mathrm{L}^\infty(\Gamma_\mathrm{S})}}{t} = \frac{\|u_t - u_0 - tu'_0\|_\mathrm{A}}{t} \longrightarrow 0,$$

when $t \to 0^+$, we deduce that $\mathcal{J}$ has a right derivative at $z_0$ in the direction $z$ given by

$$\mathcal{J}'(z_0)(z) = \langle u'_0, u_0 \rangle_\mathrm{A} + \beta \left\langle \ell(z_0), \ell'(z_0) \right\rangle_{\mathrm{L}^2(\Gamma_\mathrm{S})}.$$

Furthermore, since $u'_0 \pm u_0 \in \mathcal{K}_{u_0, \frac{\sigma_\tau(F - u_0)}{\ell(z_0)}}$, one deduces that

$$\langle u'_0, u_0 \rangle_\mathrm{A} = \int_{\Gamma_{\mathrm{S}_\mathrm{R}^{u_0,\ell(z_0)}}} \left( -\ell'(z_0) \frac{u_{0_\tau}}{\|u_{0_\tau}\|} - \frac{\ell(z_0)}{\|u_{0_\tau}\|} \left( u'_{0\tau} - \left( u'_{0\tau} \cdot \frac{u_{0_\tau}}{\|u_{0_\tau}\|} \right) \frac{u_{0_\tau}}{\|u_{0_\tau}\|} \right) \right) \cdot u_{0_\tau}.$$

Since

$$\int_{\Gamma_{\mathrm{S}_\mathrm{R}^{u_0,\ell(z_0)}}} \frac{\ell(z_0)}{\|u_{0_\tau}\|} \left( u'_{0\tau} - \left( u'_{0\tau} \cdot \frac{u_{0_\tau}}{\|u_{0_\tau}\|} \right) \frac{u_{0_\tau}}{\|u_{0_\tau}\|} \right) \cdot u_{0_\tau} = 0,$$

we get that

$$\langle u_0', u_0 \rangle_{\mathrm{A}} = -\int_{\Gamma_{\mathrm{S}_{\mathrm{R}}^{u_0,\ell(z_0)}}} \ell'(z_0) \, \|u_{0_\tau}\| \, ,$$

and we can rewrite the right derivative of $\mathcal{J}$ at $z_0$ in the direction $z$ as

$$\begin{aligned} \mathcal{J}'(z_0)(z) = -\int_{\Gamma_{\mathrm{S}_{\mathrm{R}}^{u_0,\ell(z_0)}}} \ell'(z_0) \, \|u_{0_\tau}\| + \int_{\Gamma_{\mathrm{S}}} \beta \ell'(z_0)\ell(z_0) \\ = \int_{\Gamma_{\mathrm{S}_{\mathrm{R}}^{u_0,\ell(z_0)}}} \ell'(z_0) \left(\beta \ell(z_0) - \|u_{0_\tau}\|\right) + \int_{\Gamma_{\mathrm{S}_{\mathrm{D}}^{u_0,\ell(z_0)}} \cup \Gamma_{\mathrm{S}_{\mathrm{S}}^{u_0,\ell(z_0)}}} \beta \ell'(z_0)\ell(z_0), \end{aligned}$$

and thus

$$\mathcal{J}'(z_0)(z) = \int_{\Gamma_{\mathrm{S}_{\mathrm{R}}^{u_0,\ell(z_0)}}} z g_2 \left(\beta \left(g_1 + z_0 g_2\right) - \|u_{0_\tau}\|\right) + \int_{\Gamma_{\mathrm{S}_{\mathrm{D}}^{u_0,\ell(z_0)}} \cup \Gamma_{\mathrm{S}_{\mathrm{S}}^{u_0,\ell(z_0)}}} \beta z g_2 \left(g_1 + z_0 g_2\right).$$

Note that $\mathcal{J}'(z_0)$ is linear and continuous on $\mathrm{L}^\infty(\Gamma_{\mathrm{S}})$. Thus $\mathcal{J}$ is Gateaux differentiable at $z_0$ with its Gateaux differential given by $\mathrm{d}_G\mathcal{J}(z_0) := \mathcal{J}'(z_0)$. The proof is complete. □

**Remark 10.2.2.** In the proof of Theorem 10.2.1, note that the derivative $u_0'$ depends on the pair $(\ell(z_0), \ell'(z_0)) = (g_1 + z_0 g_2, z g_2)$ and thus on the term $z \in \mathrm{L}^\infty(\Gamma_{\mathrm{S}})$. Therefore let us denote by $u_0' := u_0'(z)$. Note that $u_0'(z)$ is not linear with respect to $z$. However one can observe that the scalar product $\langle u_0'(z), u_0 \rangle_{\mathrm{A}}$, that appears in the proof of Theorem 10.2.1, is linear with respect to $z$. Therefore it leads to an expression of $\mathcal{J}'(z_0)$ that is linear with respect to $z$, and thus to the Gateaux differentiability of $\mathcal{J}$ at $z_0$.

## 10.3 Numerical simulations

In this section we assume that $||g_2||_{\mathrm{L}^\infty(\Gamma_{\mathrm{S}})} < m$, where $m > 0$ is the constant introduced at the beginning of Chapter 10 and we take the admissible set $\mathcal{U}$ given by

$$\mathcal{U} := \left\{ z \in \mathrm{L}^2(\Gamma_{\mathrm{S}}) \mid -1 \leq z \leq 1 \ a.e. \text{ on } \Gamma_{\mathrm{S}} \right\},$$

which is a nonempty convex subset of V and is a bounded closed subset of $\mathrm{L}^2(\Gamma_{\mathrm{S}})$. In this section our aim is to numerically solve an example of Problem (10.1) in the two-dimensional case $d = 2$, by making use of our theoretical result obtained in Theorem 10.2.1.

### 10.3.1 Numerical methodology

Starting with an initial control $z_0 \in \mathcal{U}$, we compute $z_d \in \mathrm{L}^\infty(\Gamma_{\mathrm{S}})$ given by

$$z_d := \begin{cases} -g_2 \left(\beta \left(g_1 + z_0 g_2\right) - \|u_{0\tau}\|\right) & \text{on } \Gamma_{\mathrm{S}_{\mathrm{R}}^{u_0,\ell(z_0)}}, \\ -\beta g_2 \left(g_1 + z_0 g_2\right) & \text{on } \Gamma_{\mathrm{S}_{\mathrm{D}}^{u_0,\ell(z_0)}} \cup \Gamma_{\mathrm{S}_{\mathrm{S}}^{u_0,\ell(z_0)}}, \end{cases}$$

which is, from Theorem 10.2.1, a descent direction of the functional $\mathcal{J}$ at $z_0$ since it satisfies

$$\mathrm{d}_G\mathcal{J}(z_0)(z_d) = -\|g_2\left(\beta\left(g_1+z_0g_2\right)-\|u_{0\tau}\|\right)\|^2_{\mathrm{L}^2(\Gamma_{\mathrm{S}_\mathrm{R}^{u_0,\ell(z_0)}})} \\ -\|\beta g_2\left(g_1+z_0g_2\right)\|^2_{\mathrm{L}^2(\Gamma_{\mathrm{S}_\mathrm{D}^{u_0,\ell(z_0)}}\cup\Gamma_{\mathrm{S}_\mathrm{S}^{u_0,\ell(z_0)}})}\leq 0.$$

Then the control is updated as $z_1 = \mathrm{proj}_{\mathcal{U}}\left(z_0+\eta z_d\right)$, where $\eta > 0$ is a fixed parameter and $\mathrm{proj}_{\mathcal{U}}$ is the classical projection operator onto $\mathcal{U}$ considered in $\mathrm{L}^2(\Gamma_\mathrm{S})$. Then the algorithm restarts with $z_1$, and so on.

Let us mention that the numerical simulations have been performed using Freefem++ software [44] with P1-finite elements and standard affine mesh. The Tresca friction problem is numerically solved using an adaptation of iterative switching algorithms (this adaptation is close to the one described in [3, Appendix C] which concerns a scalar Tresca friction problem). We also precise that, for all $i \in \mathbb{N}^*$, the difference between the cost functional $\mathcal{J}$ at the iteration $20\times i$ and at the iteration $20\times(i-1)$ is computed. The smallness of this difference is used as a stopping criterion for the algorithm.

### 10.3.2 Example and numerical results

In this subsection take $d = 2$ and let $\Omega$ be the unit disk of $\mathbb{R}^2$ with its boundary $\Gamma := \partial\Omega$ decomposed as $\Gamma = \Gamma_\mathrm{D}\cup\Gamma_\mathrm{S}$ (see Figure 10.1), where

$$\Gamma_\mathrm{D} := \left\{(\cos\theta,\sin\theta)\in\Gamma \mid 0\leq\theta\leq\tfrac{\pi}{2}\right\},$$
$$\Gamma_\mathrm{S} := \left\{(\cos\theta,\sin\theta)\in\Gamma \mid \tfrac{\pi}{2}<\theta<2\pi\right\}.$$

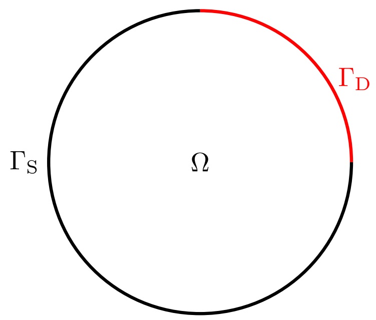


Figure 10.1 – Unit disk $\Omega$ and its boundary $\Gamma = \Gamma_\mathrm{D}\cup\Gamma_\mathrm{S}$.

We assume that $\Omega$ is *isotropic*, in the sense that the Cauchy stress tensor is given by

$$\sigma(v) = 2\mu\mathrm{e}(v)+\lambda\mathrm{tr}\left(\mathrm{e}(v)\right)\mathrm{I},$$

for all $v\in\mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d)$, where $\mathrm{tr}\left(\mathrm{e}(v)\right)$ is the trace of the matrix $\mathrm{e}(v)$ and where $\mu\geq 0$ and $\lambda\geq 0$ are Lamé parameters (see, e.g., [73]). In what follows we take $\mu = 0.3846$ and $\lambda = 0.5769$. This corresponds to a Young's modulus equal to 1 and to a Poisson's ratio equal to 0.3, which is a typical value for a large variety of materials. Let us consider the arbitrary functions $h := 0$ *a.e.* on $\Gamma_\mathrm{S}$, $g_1 := 2$

*a.e.* on $\Gamma_S$, $g_2 \in L^2(\Gamma_S)$ be the function defined by

$$\begin{array}{rccl} g_2 : & \Gamma_S & \longrightarrow & \mathbb{R} \\ & (x,y) & \longmapsto & g_2(x,y) := x^2 - y^2, \end{array}$$

and $f \in L^2(\Omega, \mathbb{R}^2)$ be the function defined by

$$\begin{array}{rccl} f : & \Omega & \longrightarrow & \mathbb{R}^2 \\ & (x,y) & \longmapsto & f(x,y) := \begin{pmatrix} \frac{5-x^2-y^2+xy}{4} & \frac{5-x^2-y^2+xy}{4} \end{pmatrix}. \end{array}$$

With $m := 2$, one has $g_1 \geq m$ *a.e.* on $\Gamma_S$ and $0 < ||g_2||_{L^\infty(\Gamma_S)} < m$, thus the assumptions from the beginning of Chapter 10 and from Section 10.3 are satisfied. We consider the initial control $z_0 \in \mathcal{U}$ given by

$$\begin{array}{rccl} z_0 : & \Gamma_S & \longrightarrow & \mathbb{R} \\ & (x,y) & \longmapsto & z_0(x,y) := \cos(x^2 - y^2). \end{array}$$

We present now the numerical results obtained for the above two-dimensional example using the numerical methodology described in Subsection 10.3.1. Figure 10.2 depicts the control which solves Problem (10.1). It is a *bang–bang* optimal control, that takes exclusively the two values $-1$ and $1$ on the boundary $\Gamma_S$. Figure 10.3 shows the evolution of the value of $\mathcal{J}$ with respect to the iteration. We observe an usual decreasing of the cost functional $\mathcal{J}$ with respect to the iterations.

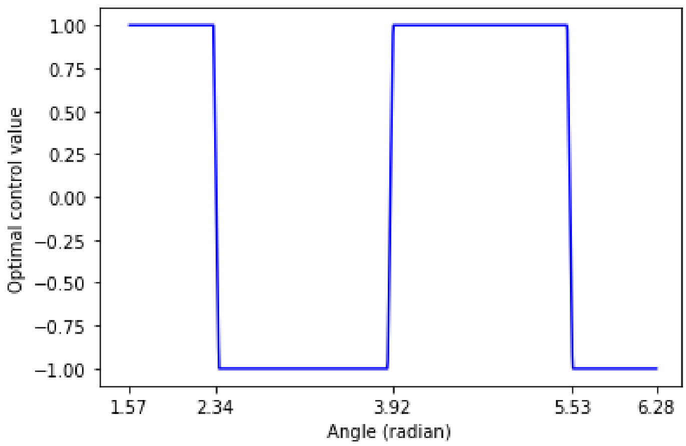


Figure 10.2 – Values of the optimal control on the boundary $\Gamma_S := \left\{(\cos\theta, \sin\theta) \in \Gamma \mid \frac{\pi}{2} < \theta < 2\pi\right\}$.

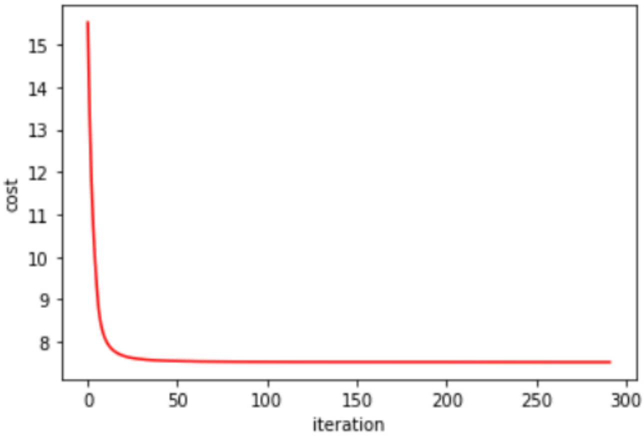


Figure 10.3 – Values of the cost functional $\mathcal{J}$ with respect to the iterations.

# Conclusions and perspectives

This PhD thesis investigated the sensitivity analysis of two mechanical contact problems: the Signorini problem and the Tresca friction problem, both in the scalar model and in the linear elastic model. Using a new methodology based on the theory of variational inequalities and on tools from convex and variational analyses such as the notion of proximal operator and the notion of twice epi-differentiability, we proved that, under appropriate assumptions, the derivative of a Signorini/Tresca friction problem with respect to the data can be characterized as solution to a boundary value problem involving (tangential) Signorini unilateral conditions. Then we apply those results to study shape optimization problems and optimal control problems in which we were able to characterize the (shape) gradient of an appropriate cost functional. An important point and novelty of this manuscript is that we did not use any penalization or regularization procedure to deal with such optimization problems.

In Part III we presented some applications but our method is applicable to other shape optimization problems and optimal control problems. In particular, note that the work in Subsection 6.2.2 could allow us to consider also a shape optimization problem involving the Tresca friction law in the linear elastic model and in the two-dimensional case. Nevertheless, in the three-dimensional case, since the norm in the parameterized Tresca friction functional is a $t$-dependent map (see Remark 6.2.13), we have not been able to study further a shape optimization problem in this framework. Nevertheless, maybe it would be possible to continue the investigation by computing the twice epi-differentiability of the $t$-dependent norm, which is a highly nontrivial work. We summarize in the following tables the shape otimization problems (resp. the optimal control problems) that can be solved with our methodology for the corresponding energy functional (resp. the cost functional defined by (10.2)).

| Shape optimization problems | Scalar model | Elasticity model $d = 2$ | Elasticity model $d = 3$ |
|---|---|---|---|
| Signorini unilateral conditions | ✓ | ✓ | ✓ |
| Tresca friction law | ✓ | ✓ | ? |

| Optimal control problems | Scalar model | Elasticity model $d = 2$ | Elasticity model $d = 3$ |
|---|---|---|---|
| Signorini unilateral conditions | ✓ | ✓ | ✓ |
| Tresca friction law | ✓ | ✓ | ✓ |

In this manuscript we have minimized a self-adjoint functional. Other cost functionals, such as the least-square functional, can pose challenges to correctly define an adjoint problem due to nonlinearities in shape gradients. Note that these difficulties do not appear in the literature when using penalization or regularization procedures (see, e.g., [47]). We do not address this challenge yet, and we believe that it is an interesting area for future research.

In this work we did not have considered the Coulomb friction law. Indeed, the main difficulty of this model comes from the friction threshold which depends on the normal stress, thus leads to a nonvariational problem. Nevertheless, the Tresca model can be seen as a simplified Coulomb friction law with a given friction threshold (see Section 1.2), thus can be considered as a first step towards the treatment of the more complicated mathematical formulation of the Coulomb friction law (for details, see, e.g., [32, 74]).

The inversion of the symbols ME-lim and $\int_\Gamma$ on the $\mathrm{H}^1(\Omega)$-space (see Remark 5.2.5) remains an open question in the literature and a challenge for a future research project. We refer to the work [65] in which the author studied the inversion of the ME-lim symbol and the $\int_\Omega$ symbol over the $\mathrm{L}^2(\Omega)$-space.

Concerning the numerical point, we have used different algorithms to perform our numerical simulations: the iterative switching algorithms and Nitsche methods. The iterative switching algorithms are easily implementable and seem efficient, but the convergence proof of these algorithms is not established yet, while Nitsche methods have been studied rigorously in [21, 22, 23]. These two methods were used to compare them, at least briefly. However, this comparison has not been pursued in depth and some investigations need to be carried out in order to study their advantages and drawbacks, which is an interesting prospect for further work.

To conclude note that, in the shape optimization problems considered all along this manuscript, we do not prove theoretically the existence of an optimal shape. It would be an interesting subject for future investigations. Let us mention some related results that can be found in the literature, e.g, [42, 43, 62, 63, 66].

Appendix A

# Sufficient conditions for the twice epi-differentiability of the parameterized Tresca friction functional

In this first appendix our aim is to prove, in some particular cases which correspond to practical situations, that the parameterized Tresca friction functional is twice epi-differentiable. More precisely, we focus here on the parameterized Tresca friction functional $\Phi$ defined in (5.15), thus we preserve the notations introduced in Subsection 5.2.2 and we assume that, for almost all $s \in \Gamma$, the map $t \in \mathbb{R}_+ \mapsto g_t(s) \in \mathbb{R}_+$ is differentiable at $t = 0$, with its derivative denoted by $g_0'(s)$ and also $g_0' \in \mathrm{L}^2(\Gamma)$. We now want to prove that $\Phi$ is twice epi-differentiable at $u_0$ for $F_0 - u_0 \in \partial\Phi(0,\cdot)(u_0)$, with its second-order epi-derivative given by

$$\mathrm{D}_e^2\Phi(u_0|F_0 - u_0)(w) = \int_\Gamma \mathrm{D}_e^2 G(s)(u_0(s)|\partial_\mathrm{n}(F_0 - u_0)(s))(w(s))\mathrm{d}s,$$

for all $w \in \mathrm{H}^1(\Omega)$. From the characterization of Mosco epi-convergence (see Proposition 3.2.4), it is sufficient to prove that, for all $w \in \mathrm{H}^1(\Omega)$, the two conditions:

(i) for all $(w_t)_{t>0} \subset \mathrm{H}^1(\Omega)$ such that $(w_t)_{t>0} \rightharpoonup w$ in $\mathrm{H}^1(\Omega)$, then

$$\liminf \Delta_t^2\Phi(u_0|F_0 - u_0)(w_t) \geq \iota_{\mathcal{K}_{u_0, \frac{\partial_\mathrm{n}(F_0-u_0)}{g_0}}}(w) + \int_\Gamma g_0'(s)\frac{\partial_\mathrm{n}(F_0 - u_0)(s)}{g_0(s)}w(s)\mathrm{d}s;$$

(ii) there exists $(w_t)_{t>0} \subset \mathrm{H}^1(\Omega)$ such that $(w_t)_{t>0} \to w$ in $\mathrm{H}^1(\Omega)$ and

$$\limsup \Delta_t^2\Phi(u_0|F_0 - u_0)(w_t) \leq \iota_{\mathcal{K}_{u_0, \frac{\partial_\mathrm{n}(F_0-u_0)}{g_0}}}(w) + \int_\Gamma g_0'(s)\frac{\partial_\mathrm{n}(F_0 - u_0)(s)}{g_0(s)}w(s)\mathrm{d}s;$$

are satisfied.

The condition (i) is always satisfied. Indeed, from Proposition 5.2.8, this condition can be rewritten as

$$\liminf \int_{\Gamma} \Delta_t^2 G(s)(u(s)|\partial_{\mathrm{n}}(F_0-u_0)(s))(w_t(s))\mathrm{d}s \geq \int_{\Gamma} \mathrm{D}_e^2 G(s)(u_0(s)|\partial_{\mathrm{n}}(F_0-u_0)(s))(w(s))\mathrm{d}s,$$

which is true thanks to the dense and compact embedding $\mathrm{H}^1(\Omega) \hookrightarrow\!\!\!\!\rightarrow \mathrm{L}^2(\Gamma)$, to the twice epi-differentiability of the function $G(s)$ for almost all $s \in \Gamma$ (see Proposition 5.2.10) and to the classical Fatou's lemma (see, e.g., [16, Lemma 4.1 p.90]).

The condition (ii) is obviously satisfied if $w \notin \mathcal{K}_{u_0, \frac{\partial_{\mathrm{n}}(F_0-u_0)}{g_0}}$. Thus one has only to prove the following assertion:

(ii′) for all $w \in \mathcal{K}_{u_0, \frac{\partial_{\mathrm{n}}(F_0-u_0)}{g_0}}$, there exists $(w_t)_{t>0} \subset \mathrm{H}^1(\Omega)$ such that $(w_t)_{t>0} \to w$ in $\mathrm{H}^1(\Omega)$ and

$$\limsup \Delta_t^2 \Phi(u_0|F_0-u_0)(w_t) \leq \int_{\Gamma} g_0'(s) \frac{\partial_{\mathrm{n}}(F_0-u_0)(s)}{g_0(s)} w(s)\mathrm{d}s.$$

Unfortunately we are not able to prove this assertion in a general setting yet, that is without any additional assumptions on $u_0$ and on $\Gamma$, and in any dimension $d \geq 1$. Nevertheless, in this appendix, we prove this assertion in some particular cases which correspond to practical situations, providing sufficient conditions. In particular, in the next sections, we consider the additional assumption

(A) the map $t \in \mathbb{R}_+ \mapsto g_t \in \mathrm{L}^2(\Gamma)$ is differentiable at $t = 0$.

## A.1 First example of sufficient condition: $u = 0$ almost everywhere on $\Gamma$

In this first example, we assume that $u_0 = 0$ almost everywhere on $\Gamma$, therefore $\Gamma_{\mathrm{N}}^{u_0,g_0}$ has a null measure. Let $w \in \mathcal{K}_{u_0, \frac{\partial_{\mathrm{n}}(F_0-u_0)}{g_0}}$. Then, taking the sequence $w_t = w$ for all $t > 0$, one gets

$$\begin{aligned}\Delta_t^2 \Phi(u_0|F_0-u_0)(w) &= \int_{\Gamma_{\mathrm{S}+}^{u_0,g_0} \cup \Gamma_{\mathrm{S}-}^{u_0,g_0}} \frac{g_t(s)|u_0(s)+tw(s)| - g_t(s)|u_0(s)| + t\partial_{\mathrm{n}}(F_0-u_0)(s)w(s)}{t^2}\mathrm{d}s \\ &= \int_{\Gamma_{\mathrm{S}+}^{u_0,g_0}} \frac{g_t(s)-g_0(s)}{t} w(s)\mathrm{d}s - \int_{\Gamma_{\mathrm{S}-}^{u_0,g_0}} \frac{g_t(s)-g_0(s)}{t} w(s)\mathrm{d}s \\ &\longrightarrow \int_{\Gamma} g_0'(s) \frac{\partial_{\mathrm{n}}(F_0-u_0)(s)}{g_0(s)} w(s)\mathrm{d}s,\end{aligned}$$

when $t \to 0^+$ from Assumption (A). Therefore Condition (ii′) is satisfied.

## A.2 Second example of sufficient condition: truncature

In this second example, we introduce two disjoint subsets of $\Gamma$ given by

$$\Gamma_{\mathrm{N}+}^{u_0,g_0} := \{s \in \Gamma \mid u_0(s) > 0\} \qquad \text{and} \qquad \Gamma_{\mathrm{N}-}^{u_0,g_0} := \{s \in \Gamma \mid u_0(s) < 0\}.$$

Hence it follows that $\Gamma^{u_0,g_0}_{\mathrm{N}} = \Gamma^{u_0,g_0}_{\mathrm{N}+} \cup \Gamma^{u_0,g_0}_{\mathrm{N}-}$, $\partial_{\mathrm{n}} u_0 = -g_0$ *a.e.* on $\Gamma^{u_0,g_0}_{\mathrm{N}+}$ and that $\partial_{\mathrm{n}} u_0 = g_0$ *a.e.* on $\Gamma^{u_0,g_0}_{\mathrm{N}-}$. Now let us assume that there exists $C > 0$ such that $|u_0| \geq C$ on $\Gamma^{u_0,g_0}_{\mathrm{N}+} \cup \Gamma^{u_0,g_0}_{\mathrm{N}-}$. Let us consider $w \in \mathcal{K}_{u_0,\frac{\partial_{\mathrm{n}}(F_0-u_0)}{g_0}}$ and the truncature $w_t \in \mathrm{H}^1(\Omega)$ of $w$ defined by

$$w_t(x) := \begin{cases} \frac{1}{\sqrt{t}} & \text{if } w(x) \geq \frac{1}{\sqrt{t}}, \\ w(x) & \text{if } |w(x)| \leq \frac{1}{\sqrt{t}}, \\ -\frac{1}{\sqrt{t}} & \text{if } w(x) \leq -\frac{1}{\sqrt{t}}, \end{cases}$$

for almost all $x \in \Omega$ and for all $t > 0$. One deduces from Marcus-Mizel theorem (see [56]) that $w_t \to w$ in $\mathrm{H}^1(\Omega)$ when $t \to 0^+$. Moreover, for all $t \leq C^2$, one gets

$$\Delta^2_t \Phi(u_0|F_0 - u_0)(w_t) = \int_{\Gamma^{u_0,g_0}_{\mathrm{S}+} \cup \Gamma^{u_0,g_0}_{\mathrm{N}+}} \frac{g_t(s) - g_0(s)}{t} w_t(s) \mathrm{d}s - \int_{\Gamma^{u_0,g_0}_{\mathrm{S}-} \cup \Gamma^{u_0,g_0}_{\mathrm{N}-}} \frac{g_t(s) - g_0(s)}{t} w_t(s) \mathrm{d}s \longrightarrow \int_{\Gamma} g_0'(s) \frac{\partial_{\mathrm{n}}(F_0 - u_0)(s)}{g_0(s)} w(s) \mathrm{d}s,$$

when $t \to 0^+$ from Assumption (A), therefore Condition (ii′) is satisfied.

## A.3 Third example of sufficient condition: truncature and dilatation

In this third example, we take $d = 2$ and we assume that $u_0$ and $\partial_{\mathrm{n}} u_0$ are continuous on $\Gamma$, and that $\Gamma$ is diffeomorphic to the circle $\mathrm{S}^1 := \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. From this last assumption, for simplicity, we assume in the sequel that $\Gamma = \mathrm{S}^1$. Let us assume that $\Gamma = \Gamma^{u_0,g_0}_{\mathrm{S}+} \cup \Gamma^{u_0,g_0}_{\mathrm{N}+}$ (in this particular case, the hypothesis on the continuity of $\partial_{\mathrm{n}} u_0$ is useless, see Remark A.3.1) where $\Gamma^{u_0,g_0}_{\mathrm{N}+}$ has already been defined in the previous example, and with the following parameterizations

$$\Gamma^{u_0,g_0}_{\mathrm{N}+} = \{(\cos\theta, \sin\theta) \in \Gamma \mid \theta \in ]\gamma_1, \gamma_2[\},$$

$$\Gamma^{u_0,g_0}_{\mathrm{S}+} = \{(\cos\theta, \sin\theta) \in \Gamma \mid \theta \in [\gamma_2, \gamma_1 + 2\pi]\},$$

such that $-\pi < \gamma_1 < \gamma_2 < \pi$ (see Figure A.1). From the continuity of $u_0$, there exists $c > 0$ such that $u_0 \geq c$ on the set $\{(\cos\theta, \sin\theta) \in \Gamma, \theta \in [\chi_1, \chi_2]\} \subset \Gamma^{u_0,g_0}_{\mathrm{N}+}$, with $\gamma_1 < \chi_1 < \chi_2 < \gamma_2$. Let us consider $\omega_1 \in ]-\pi, \gamma_1[$, $\omega_2 \in ]\gamma_2, \pi[$, and also $\alpha_t$, $\beta_t$ defined, for $t > 0$ such that $\sqrt{t} \leq c$, by

$$\alpha_t := \inf \Big\{\alpha \in [\gamma_1, \chi_1] \mid \forall\theta \in [\alpha, \chi_1], u_0(\cos\theta, \sin\theta) \geq \sqrt{t}\Big\},$$

$$\beta_t := \inf \Big\{\beta \in [\chi_2, \gamma_2] \mid \forall\theta \in [\chi_2, \beta], u_0(\cos\theta, \sin\theta) \geq \sqrt{t}\Big\}.$$

From the continuity of $u_0$, ones deduces that $\alpha_t \to \gamma_1$ and $\beta_t \to \gamma_2$ when $t \to 0^+$.

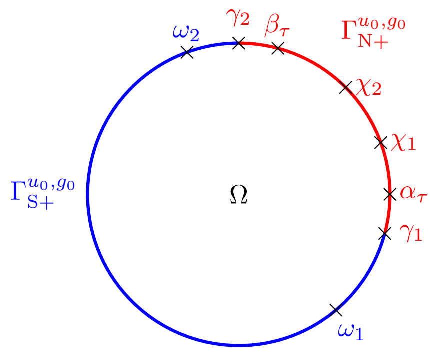


Figure A.1 – Illustration of the boundary $\Gamma$

Let $w \in \mathcal{K}_{u_0, \frac{\partial_{\mathrm{n}}(F_0 - u_0)}{g_0}}$, and let $y_t \in \mathrm{H}^1(\Omega)$ be the truncature of $w$ given by

$$y_t(x) := \begin{cases} \frac{1}{\sqrt{t}} & \text{if } w(x) \geq \frac{1}{\sqrt{t}}, \\ w(x) & \text{if } |w(x)| \leq \frac{1}{\sqrt{t}}, \\ -\frac{1}{\sqrt{t}} & \text{if } w(x) \leq -\frac{1}{\sqrt{t}}, \end{cases}$$

for almost all $x \in \Omega$ and for all $t > 0$. As in the previous section, one gets $y_t \to w$ in $\mathrm{H}^1(\Omega)$, and thus $y_{t|\Gamma} \to w_{|\Gamma}$ in $\mathrm{H}^{1/2}(\Gamma)$ when $t \to 0^+$. Let us consider, for $t > 0$ sufficiently small, the dilatation $z_t := y_{t|\Gamma} \circ d_t$ of $y_{t|\Gamma}$, with $d_t$ given by

$$\begin{array}{rccl} d_t: & \Gamma & \longrightarrow & \Gamma \\ & (x_1,x_2) & \longmapsto & \begin{cases} (x_1,x_2) & \text{if } (x_1,x_2) \in \Gamma_{\mathrm{S+}}^{u_0,g_0} \backslash \{(\cos\theta, \sin\theta), \theta \in [\omega_1,\omega_2]\}, \\ d^{\omega_1,\alpha_t}(x_1,x_2) & \text{if } 2\arctan\left(\frac{x_2}{x_1+1}\right) \in [\omega_1, \alpha_t], \\ d^{\alpha_t,\chi_1}(x_1,x_2) & \text{if } 2\arctan\left(\frac{x_2}{x_1+1}\right) \in [\alpha_t, \chi_1], \\ (x_1,x_2) & \text{if } 2\arctan\left(\frac{x_2}{x_1+1}\right) \in [\chi_1, \chi_2], \\ d^{\chi_2,\beta_t}(x_1,x_2) & \text{if } 2\arctan\left(\frac{x_2}{x_1+1}\right) \in [\chi_2, \beta_t], \\ d^{\beta_t,\omega_2}(x_1,x_2) & \text{if } 2\arctan\left(\frac{x_2}{x_1+1}\right) \in [\beta_t, \omega_2], \end{cases} \end{array}$$

where

$$d^{\omega_1,\alpha_t}(x_1,x_2) = (\cos\theta^{\omega_1,\alpha_t}, \sin\theta^{\omega_1,\alpha_t}), \text{ with } \theta^{\omega_1,\alpha_t} = \frac{(\gamma_1-\omega_1)\,2\arctan\left(\frac{x_2}{x_1+1}\right) + w_1(\alpha_t-\gamma_1)}{\alpha_t-\omega_1},$$

$$d^{\alpha_t,\chi_1}(x_1,x_2) = (\cos\theta^{\alpha_t,\chi_1}, \sin\theta^{\alpha_t,\chi_1}), \text{ with } \theta^{\alpha_t,\chi_1} = \frac{(\chi_1-\gamma_1)\,2\arctan\left(\frac{x_2}{x_1+1}\right) + \chi_1(\gamma_1-\alpha_t)}{\chi_1-\alpha_t},$$

$$d^{\chi_2,\beta_t}(x_1,x_2) = \left(\cos\theta^{\chi_2,\beta_t}, \sin\theta^{\chi_2,\beta_t}\right), \text{ with } \theta^{\chi_2,\beta_t} = \frac{(\gamma_2-\chi_2)\,2\arctan\left(\frac{x_2}{x_1+1}\right) + \chi_2(\beta_t-\gamma_2)}{\beta_t-\chi_2},$$

$$d^{\beta_t,\omega_2}(x_1,x_2)=\left(\cos\theta^{\beta_t,\omega_2},\sin\theta^{\beta_t,\omega_2}\right),\ \text{with}\ \theta^{\beta_t,\omega_2}=\frac{(\omega_2-\gamma_2)\,2\arctan\left(\frac{x_2}{x_1+1}\right)+\omega_2\,(\gamma_2-\beta_t)}{\omega_2-\beta_t}.$$

Note that, since $-\pi<\omega_1<\omega_2<\pi$, then $d_t$ is a well-defined bijective Lipschitz continuous map, and its inverse is also a bijective Lipschitz continuous map. Thus it follows that $z_t\in\mathrm{H}^{1/2}(\Gamma)$ and also $z_t\to w_{|\Gamma}$ in $\mathrm{H}^{1/2}(\Gamma)$ when $t\to 0^+$. Then, for $t>0$ sufficiently small, we denote by $w_t\in\mathrm{H}^1(\Omega)$ a lift of $z_t\in\mathrm{H}^{1/2}(\Gamma)$, such that $w_t\to w$ in $\mathrm{H}^1(\Omega)$ when $t\to 0^+$. Therefore, by denoting

$$m_t(s)=\frac{g_t(s)|u_0(s)+tw_t(s)|-g_t(s)|u_0(s)|-t\partial_{\mathrm{n}}(F_0-u_0)(s)w_t(s)}{t^2},$$

for $t>0$ sufficiently small and for almost all $s\in\Gamma$, it follows that

$$\begin{aligned}\Delta_t^2\Phi(u_0|F_0-u_0)(w_t)=&\int_{\{(\cos\theta,\sin\theta),\ \theta\in[-\pi,\omega_1]\}}m_t(s)\mathrm{d}s\\&+\int_{\{(\cos\theta,\sin\theta),\ \theta\in[\omega_1,\alpha_t]\}}m_t(s)\mathrm{d}s+\int_{\{(\cos\theta,\sin\theta),\ \theta\in[\alpha_t,\chi_1]\}}m_t(s)\mathrm{d}s\\&+\int_{\{(\cos\theta,\sin\theta),\ \theta\in[\chi_1,\chi_2]\}}m_t(s)\mathrm{d}s+\int_{\{(\cos\theta,\sin\theta),\ \theta\in[\chi_2,\beta_t]\}}m_t(s)\mathrm{d}s\\&+\int_{\{(\cos\theta,\sin\theta),\ \theta\in[\beta_t,\omega_2]\}}m_t(s)\mathrm{d}s+\int_{\{(\cos\theta,\sin\theta),\ \theta\in[\omega_2,\pi]\}}m_t(s)\mathrm{d}s.\end{aligned}$$

Then, from the definition of $d_t$ and Assumption (A), one deduces that

$$\Delta_t^2\Phi(u_0|F_0-u_0)(w_t)\longrightarrow\int_\Gamma g_0'(s)\frac{\partial_{\mathrm{n}}(F_0-u_0)(s)}{g_0(s)}w(s)\mathrm{d}s,$$

when $t\to 0^+$, and thus Condition (ii$'$) is satisfied.

**Remark A.3.1.** In the case where $\Gamma=\Gamma_{\mathrm{S}+}^{u_0,g_0}\cup\Gamma_{\mathrm{N}+}^{u_0,g_0}$, the hypothesis $\partial_{\mathrm{n}}u_0$ continuous on $\Gamma$ is useless. Nevertheless, in the general case $\Gamma=\Gamma_{\mathrm{N}+}^{u_0,g_0}\cup\Gamma_{\mathrm{N}-}^{u_0,g_0}\cup\Gamma_{\mathrm{D}}^{u_0,g_0}\cup\Gamma_{\mathrm{S}-}^{u_0,g_0}\cup\Gamma_{\mathrm{S}+}^{u_0,g_0}$, with the hypotheses $u_0$ and $\partial_{\mathrm{n}}u_0$ continuous on $\Gamma$, one can get with appropriate assumptions on the angles (as we did for $\omega_1$ and $\omega_2$), the twice epi-differentiability of the parameterized Tresca friction functional: a part of $\Gamma_{\mathrm{S}-}^{u_0,g_0}$ (resp. $\Gamma_{\mathrm{S}+}^{u_0,g_0}$, resp. $\Gamma_{\mathrm{N}-}^{u_0,g_0}$) is never side to side with a part of $\Gamma_{\mathrm{N}+}^{u_0,g_0}$ (resp. $\Gamma_{\mathrm{N}-}^{u_0,g_0}$, resp. $\Gamma_{\mathrm{N}+}^{u_0,g_0}$), and thus, using an appropriate dilatation, one can obtain the same result.

Appendix B

# Iterative switching algorithms

In this second appendix we describe the iterative switching algorithm for the Signorini problem proposed in [4]. Then, being inspired by this procedure, we propose hereafter an adapted iterative switching algorithm in order to solve numerically the Tresca friction problem. We focus here on the scalar Signorini problem given in Subsection 5.1.1 and the scalar Tresca friction problem given in Section 5.2.2.

## B.1 Iterative switching algorithm for the scalar Signorini problem

Consider the scalar Signorini problem

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega, \\ u \leq 0,\ \partial_{\mathrm{n}} u \leq g \text{ and } u\,(\partial_{\mathrm{n}} u - g) = 0 & \text{on } \Gamma. \end{cases}$$

The algorithm is based on the following iterations.

Step 1: We start by solving the scalar Dirichlet problem

$$\begin{cases} -\Delta u_0 + u_0 = f & \text{in } \Omega, \\ u_0 = 0 & \text{on } \Gamma, \end{cases}$$

then we verify if the Signorini unilateral conditions are satisfied, that is $\partial_{\mathrm{n}} u_0 \leq g$ on $\Gamma$. Otherwise we go to the second step.

Step 2: Let $\Gamma_{\mathrm{N}}^1 \subset \Gamma$ such that $\partial_{\mathrm{n}} u_0 > g$, and $\Gamma_{\mathrm{D}}^1 := \Gamma \backslash \overline{\Gamma_{\mathrm{N}}^1}$. Then we solve the scalar Dirichlet-Neumann problem given by

$$\begin{cases} -\Delta u_1 + u_1 = f & \text{in } \Omega, \\ u_1 = 0 & \text{on } \Gamma_{\mathrm{D}}^1, \\ \partial_{\mathrm{n}} u_1 = g & \text{on } \Gamma_{\mathrm{N}}^1. \end{cases}$$

If $\partial_{\mathrm{n}} u_1 \leq g$ on $\Gamma_{\mathrm{D}}^1$ and $u_1 \leq 0$ on $\Gamma_{\mathrm{N}}^1$, then $u_1$ is the solution to the scalar Signorini problem. Otherwise we go to the third step.

Step 3: Let $\tilde{\Gamma}_{\mathrm{N}}^2 \subset \Gamma_{\mathrm{D}}^1$ such that $\partial_{\mathrm{n}} u_1 > g$, and $\tilde{\Gamma}_{\mathrm{D}}^2 := \Gamma_{\mathrm{D}}^1 \backslash \overline{\tilde{\Gamma}_{\mathrm{N}}^2}$. Moreover, let $\tilde{\tilde{\Gamma}}_{\mathrm{D}}^2 \subset \Gamma_{\mathrm{N}}^1$ such that $u_1 > 0$, and $\tilde{\tilde{\Gamma}}^2_{\mathrm{N}} := \Gamma^1_{\mathrm{N}} \backslash \tilde{\tilde{\Gamma}}^2_{\mathrm{D}}$. Then let us defined $\Gamma^2_{\mathrm{D}} := \tilde{\Gamma}^2_{\mathrm{D}} \cup \tilde{\tilde{\Gamma}}^2_{\mathrm{D}}$ and $\Gamma^2_{\mathrm{N}} := \tilde{\Gamma}^2_{\mathrm{N}} \cup \tilde{\tilde{\Gamma}}^2_{\mathrm{N}}$. Then solve

$$\begin{cases} -\Delta u_2 + u_2 &= f \quad \text{in } \Omega, \\ u_2 &= 0 \quad \text{on } \Gamma^2_{\mathrm{D}}, \\ \partial_{\mathrm{n}} u_2 &= g \quad \text{on } \Gamma^2_{\mathrm{N}}. \end{cases}$$

If $\partial_{\mathrm{n}} u_2 \leq g$ on $\Gamma^2_{\mathrm{D}}$, $u_2 \leq 0$ on $\Gamma^2_{\mathrm{N}}$, then $u_2$ is the solution to the scalar Signorini problem. Otherwise repeat this step.

## B.2 Iterative switching algorithm for the scalar Tresca friction problem

In this section we adapt the above strategy to the scalar Tresca friction problem given by

$$\begin{cases} -\Delta u + u &= f \quad \text{in } \Omega, \\ |\partial_{\mathrm{n}} u| \leq g \text{ and } u\partial_{\mathrm{n}} u &= -g|u| \quad \text{on } \Gamma. \end{cases}$$

Step 1: We start by solving the scalar Dirichlet problem given by

$$\begin{cases} -\Delta u_0 + u_0 &= f \quad \text{in } \Omega, \\ u_0 &= 0 \quad \text{on } \Gamma. \end{cases}$$

If $|\partial_{\mathrm{n}} u_0| \leq g$ then $u_0$ is the solution to the scalar Tresca friction problem. Otherwise we go to the second step.

Step 2: Let $\Gamma^1_{\mathrm{T-,N}} \subset \Gamma$ such that $\partial_{\mathrm{n}} u_0 < -g$, and let $\Gamma^1_{\mathrm{T+,N}} \subset \Gamma$ such that $\partial_{\mathrm{n}} u_0 > g$. We define $\Gamma^1_{\mathrm{D}} = \Gamma \backslash \overline{\Gamma^1_{\mathrm{T-,N}} \cup \Gamma^1_{\mathrm{T+,N}}}$, and we solve the scalar Dirichlet-Neumann problem given by

$$\begin{cases} -\Delta u_1 + u_1 &= f \quad \text{in } \Omega, \\ u_1 &= 0 \quad \text{on } \Gamma^1_{\mathrm{D}} \subset \Gamma, \\ \partial_{\mathrm{n}} u_1 &= -g \quad \text{on } \Gamma^1_{\mathrm{T-,N}}, \\ \partial_{\mathrm{n}} u_1 &= g \quad \text{on } \Gamma^1_{\mathrm{T+,N}}. \end{cases}$$

If $|\partial_{\mathrm{n}} u_1| \leq g$ on $\Gamma^1_{\mathrm{D}}$ and $u_1 \partial_{\mathrm{n}} u_1 = -g|u_1|$ on $\Gamma^1_{\mathrm{T-,N}} \cup \Gamma^1_{\mathrm{T+,N}}$, then $u_1$ is the solution to the scalar Tresca friction problem. Otherwise we go to the third step.

Step 3: Let $\tilde{\Gamma}^2_{\mathrm{T-,N}} \subset \Gamma^1_{\mathrm{D}}$ such that $\partial_{\mathrm{n}} u_1 < -g$, and $\tilde{\Gamma}^2_{\mathrm{T+,N}} \subset \Gamma^1_{\mathrm{D}}$ such that $\partial_{\mathrm{n}} u_1 > g$, and we define $\tilde{\Gamma}^2_{\mathrm{D}} = \Gamma^1_{\mathrm{D}} \backslash \overline{\tilde{\Gamma}^2_{\mathrm{T-,N}} \cup \tilde{\Gamma}^2_{\mathrm{T+,N}}}$.
Let $\tilde{\tilde{\Gamma}}^2_{\mathrm{D}} \subset \Gamma^1_{\mathrm{T-,N}}$ such that $u_1 \partial_{\mathrm{n}} u_1 \neq -g|u_1|$, that is $u_1 < 0$. We define $\tilde{\tilde{\Gamma}}^2_{\mathrm{T-,N}} := \Gamma^1_{\mathrm{T-,N}} \backslash \overline{\tilde{\tilde{\Gamma}}^2_{\mathrm{D}}}$.
Let $\tilde{\tilde{\tilde{\Gamma}}}^2_{\mathrm{D}} \subset \Gamma^1_{\mathrm{T+,N}}$ such that $u_1 \partial_{\mathrm{n}} u_1 \neq -g|u_1|$, that is $u_1 > 0$. We define $\tilde{\tilde{\Gamma}}^2_{\mathrm{T+,N}} := \Gamma^1_{\mathrm{T+,N}} \backslash \overline{\tilde{\tilde{\tilde{\Gamma}}}^2_{\mathrm{D}}}$.

Then we define $\Gamma^2_{\mathrm{D}} := \tilde{\Gamma}^2_{\mathrm{D}} \cup \tilde{\tilde{\Gamma}}^2_{\mathrm{D}} \cup \tilde{\tilde{\tilde{\Gamma}}}^2_{\mathrm{D}}$, $\Gamma^2_{\mathrm{T-,N}} := \tilde{\Gamma}^2_{\mathrm{T-,N}} \cup \tilde{\tilde{\Gamma}}^2_{\mathrm{T-,N}}$, $\Gamma^2_{\mathrm{T+,N}} := \tilde{\Gamma}^2_{\mathrm{T+,N}} \cup \tilde{\tilde{\Gamma}}^2_{\mathrm{T+,N}}$, and we solve

$$\left\{ \begin{array}{rcll} -\Delta u_2 + u_2 & = & f & \text{in } \Omega, \\ u_2 & = & 0 & \text{on } \Gamma^2_{\mathrm{D}}, \\ \partial_{\mathrm{n}} u_2 & = & -g & \text{on } \Gamma^2_{\mathrm{T-,N}}, \\ \partial_{\mathrm{n}} u_2 & = & g & \text{on } \Gamma^2_{\mathrm{T+,N}}. \end{array} \right.$$

If $|\partial_{\mathrm{n}} u_2| \leq g$ on $\Gamma^2_{\mathrm{D}}$ and $u_2 \partial_{\mathrm{n}} u_2 = -g|u_2|$ on $\Gamma^2_{\mathrm{T-,N}} \cup \Gamma^2_{\mathrm{T+,N}}$, then $u_2$ is solution to the scalar Tresca friction problem. Otherwise repeat this step.

# Bibliography


[1] R. Adams and J. Fournier. *Sobolev Spaces*, volume 140 of *Pure and Applied Mathematics*. Elsevier, 2003.

[2] S. Adly and L. Bourdin. Sensitivity analysis of variational inequalities via twice epi-differentiability and proto-differentiability of the proximity operator. *SIAM Journal on Optimization*, 28(2):1699–1725, 2018.

[3] S. Adly, L. Bourdin, and F. Caubet. Sensitivity analysis of a Tresca-type problem leads to Signorini's conditions. *ESAIM Control Optim. Calc. Var.*, 28:Paper No. 29, 29, 2022.

[4] J. M. Aitchison and M. W. Poole. A numerical algorithm for the solution of Signorini problems. *J. Comput. Appl. Math.*, 94(1):55–67, 1998.

[5] C. D. Aliprantis and O. Burkinshaw. *Principles of Real Analysis.* 3rd Edition, Academic Press, New York and London, 1998.

[6] G. Allaire. Shape Optimization by the Homogenization Method. Applied Mathematical Sciences, Vol 146. *Applied Mechanics Reviews*, 56(2):B26–B27, 03 2003.

[7] G. Allaire. *Analyse numerique et optimisation.* Mathématiques et Applications. Éditions de l'École Polytechnique., 2007.

[8] G. Allaire. *Conception optimale de structures.* Mathématiques et Applications. Springer-Verlag Berlin Heidelberg, 2007.

[9] G. Allaire, F. Jouve, and A. Maury. Shape optimisation with the level set method for contact problems in linearised elasticity. *The SMAI journal of computational mathematics*, 3:249–292, 2017.

[10] S. S. Antman. The influence of elasticity on analysis: modern developments. *Bull. Amer. Math. Soc. (N.S.)*, 9(3):267–291, 1983.

[11] H. Attouch. *Variational convergence for functions and operators.* Applicable Mathematics Series. Pitman (Advanced Publishing Program), Boston, MA, 1984.

[12] F. Ben Belgacem and Y. Renard. Hybrid finite element methods for the Signorini problem. *Mathematics of Computation*, 72:1117–1145, 2003.

[13] M. Biegert. On traces of Sobolev functions on the boundary of extension domains. *Proc. Amer. Math. Soc.*, 137(12):4169–4176, 2009.

[14] H. Brezis. Problèmes unilatéraux. *J. Math. Pures Appl*, 51:1–168, 1972.

[15] H. Brezis. *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, volume 5 of *North-Holland Mathematics Studies, No. 5. Notas de Matemática (50).* North-Holland Publishing Co., Amsterdam, 1973.

[16] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations.* Universitext. Springer, New York, 2011.

[17] P. Chadwick. *Continuum Mechanics: Concise Theory and Problems.* Dover Publications Inc, 2003.

[18] B. Chaudet-Dumas and J. Deteix. Shape derivatives for the penalty formulation of elastic contact problems with tresca friction. *SIAM Journal on Control and Optimization*, 58(6):3237–3261, 2020.

[19] B. Chaudet-Dumas and J. Deteix. Shape derivatives for an augmented lagrangian formulation of elastic contact problems. *ESAIM: Control, Optimisation and Calculus of Variations*, 27(14):23, 2021.

[20] Z. Chen. *Finite Element Methods And Their Applications.* Scientific Computation. Springer Berlin, Heidelberg, 2005.

[21] F. Chouly. An adaptation of nitsches method to the tresca friction problem. *Journal of Mathematical Analysis and Applications*, 411(1):329–339, 2014.

[22] F. Chouly, M. Fabre, P. Hild, R. Mlika, J. Pousin, and Y. Renard. An overview of recent results on Nitsche's method for contact problems. In B. S., B. E., L. M., and O. M., editors, *UCL Workshop 2016*, volume 121 of *Geometrically Unfitted Finite Element Methods and Applications*, pages 93–141, London, United Kingdom, Jan. 2016. UCL (University College London), Springer.

[23] F. Chouly and P. Hild. A Nitsche-based method for unilateral contact problems: numerical analysis. *SIAM Journal on Numerical Analysis*, 51(2):p. 1295–1307, 2013.

[24] C.-A. Coulomb. Théorie des machines simples, en ayant égard au frottement de leur partie et à la raideur des cordages. *Mémoires de mathématique et de physique*, 10:161–332, 1785.

[25] R. Dautray and J.-L. Lions. *Mathematical Analysis and Numerical Methods for Science and Technology: Volume 2: Functional and Variational Methods.* Springer-Verlag Berlin Heidelberg, 2000.

[26] F. Demengel and G. Demengel. *Convexité dans les espaces fonctionnels - Théorie et illustration par les exemples.* Ellipses, 2004.

[27] J. Deny. Théorie de la capacité dans les espaces fonctionnels. *Séminaire Brelot-Choquet-Deny. Théorie du potentiel*, 9, 1964-1965. talk:1.

[28] C. N. Do. Generalized second-order derivatives of convex functions in reflexive banach spaces. *Transactions of the American Mathematical Society*, 334(1):281–301, 1992.

[29] G. Duvaut. Problèmes unilatéraux en élasticité. In *Colloque d'analyse fonctionnelle (Bordeaux, 1971)*, number 31-32 in Mémoires de la Société Mathématique de France, pages 151–158. Société mathématique de France, 1972.

[30] G. Duvaut. *Mécanique des milieux continus.* Éditions Dunod, 2001.

[31] G. Duvaut and J.-L. Lions. *Les inéquations en mécanique et en physique.* Travaux et Recherches Mathématiques, No. 21. Dunod, Paris, 1972.

[32] C. Eck, J. Jarusek, and M. Krbec. *Unilateral contact problems.* Volume 270 of Pure and Applied Mathematics (Boca Raton), Chapman  Hall/CRC, Boca Raton, FL. 2005.

[33] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.

[34] K. Feng and Z.-C. Shi. *Mathematical Theory of Elastic Structures.* Springer, 1996.

[35] G. Fichera. Problemi elastostatici con vincoli unilaterali: Il problema di Signorini con ambigue condizioni al contorno. *Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur. Sez. Ia (8)*, 7:91–140, 1963/64.

[36] P. Fulmański, A. Laurain, J.-F. Scheid, and J. Sokoł owski. A level set method in shape and topology optimization for variational inequalities. *Int. J. Appl. Math. Comput. Sci.*, 17(3):413–430, 2007.

[37] R. Glowinski, J.-L. Lions, and R. Trémolières. *Numerical Analysis of Variational Inequalities*, volume 8 of *Studies in Mathematics and Its Applications.* North-Holland, Amsterdam, 1981.

[38] P. Grisvard. *Elliptic Problems in Nonsmooth Domains.* Society for Industrial and Applied Mathematics, 2011.

[39] J. Hadamard. Mémoire sur le problème d'analyse relatif à l'équilibre des plaques élastiques encastrées. *Bull. Soc. Math.*, 1907.

[40] A. Haraux. How to differentiate the projection on a convex set in Hilbert space. Some applications to variational inequalities. *Journal of the Mathematical Society of Japan*, 29(4):615 – 631, 1977.

[41] J. Haslinger, I. Hlaváček, and J. Nečas. Numerical methods for unilateral problems in solid mechanics. In *Handbook of numerical analysis, Vol. IV*, Handb. Numer. Anal., IV, pages 313–485. North-Holland, Amsterdam, 1996.

[42] J. Haslinger and A. Klarbring. Shape optimization in unilateral contact problems using generalized reciprocal energy as objective functional. *Nonlinear Anal.*, 21(11):815–834, 1993.

[43] J. Haslinger and P. Neittaanmäki. On the existence of optimal shapes in contact problems. *Numer. Funct. Anal. Optim.*, 7(2-3):107–124, 1984/85.

[44] F. Hecht. New development in freefem++. *J. Numer. Math.*, 20(3-4):251–265, 2012.

[45] A. Henrot, I. Mazari, and Y. Privat. Shape optimization of a Dirichlet type energy for semilinear elliptic partial differential equations. working paper or preprint, Apr. 2020.

[46] A. Henrot and M. Pierre. *Shape Variation and Optimization : a Geometrical Analysis.* Tracts in Mathematics Vol. 28. European Mathematical Society, 2018.

[47] M. Hintermüller and A. Laurain. Optimal shape design subject to elliptic variational inequalities. *SIAM Journal on Control and Optimization*, 49(3):1015–1047, 2011.

[48] J.-B. Hiriart-Urruty and C. Lemaréchal. *Fundamentals of convex analysis.* Grundlehren Text Editions. Springer-Verlag, Berlin, 2001.

[49] N. Kikuchi and J. T. Oden. *Contact Problems in Elasticity.* Society for Industrial and Applied Mathematics, 1988.

[50] D. Kinderlehrer. Remarks about Signorini's problem in linear elasticity. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 8(4):605–645, 1981.

[51] F. Kuss. *Méthodes duales pour les problèmes de contact avec frottement.* Thèse, Université de Provence - Aix-Marseille I, July 2008.

[52] J.-L. Lions. Sur les problèmes unilatéraux. In *Séminaire Bourbaki : vol. 1968/69, exposés 347-363*, number 11 in Séminaire Bourbaki. Springer-Verlag, 1971.

[53] D. Luft, V. H. Schulz, and K. Welker. Efficient techniques for shape optimization with variational inequalities using adjoints. *SIAM Journal on Optimization*, 30(3):1922–1953, 2020.

[54] K. Malanowski. Remarks on Differentiability of Metric Projections onto Cones of Nonnegative Functions. *Journal of Convex Analysis*, 10(1):285—294, 2003.

[55] A. Manzoni, A. Quarteroni, and S. Salsa. *Optimal control of partial differential equations—analysis, approximation, and applications*, volume 207 of *Applied Mathematical Sciences.* Springer, Cham, 2021.

[56] M. Marcus and V. J. Mizel. Absolute continuity on tracks and mappings of Sobolev spaces. *Arch. Rational Mech. Anal.*, 45:294–320, 1972.

[57] A. Maury. *Shape optimization for contact and plasticity problems thanks to the level set method.* Thèse, Université Pierre et Marie Curie - Paris VI, 2016.

[58] F. Mignot. Contrôle dans les inéquations variationelles elliptiques. *Journal of Functional Analysis*, 22(2):130–185, 1976.

[59] G. J. Minty. Monotone (nonlinear) operators in Hilbert space. *Duke Mathematical Journal*, 29(3):341–346, 1962.

[60] J. J. Moreau. Fonctions convexes duales et points proximaux dans un espace hilbertien. *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 255:2897–2899, 1962.

[61] J. J. Moreau. Proximité et dualité dans un espace hilbertien. *Bulletin de la Société Mathématique de France*, 93:273–299, 1965.

[62] F. Murat and J. Simon. Etude de problemes d'optimal design. In *Lecture Notes in Computer Science book series*, volume 41, pages 54–62, Berlin, Heidelberg, 1976. Springer Berlin Heidelberg.

[63] F. Murat and J. Simon. Sur le contrôle par un domaine géométrique. In *Internal Report No 76 015.* Laboratoire d'Analyse Numérique de l'Université Paris 6, 1976.

[64] J. Nitsche. Über ein variationsprinzip zur lösung von dirichlet-problemen bei verwendung von teilräumen, die keinen randbedingungen unterworfen sind. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 36:9–15, 1971.

[65] D. Noll. Second order differentiability of integral functionals on sobolev spaces and $L^2$-spaces. *Journal für die reine und angewandte Mathematik*, 436:1–17, 1993.

[66] Y. Privat, R. Robin, and M. Sigalotti. Existence of surfaces optimizing geometric and PDE shape functionals under reach constraint. working paper or preprint, June 2022.

[67] R. T. Rockafellar. On the maximal monotonicity of subdifferential mappings. *Pacific Journal of Mathematics*, 33(1):209 – 216, 1970.

[68] R. T. Rockafellar. Maximal monotone relations and the second derivatives of nonsmooth functions. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 2(3):167–184, 1985.

[69] R. T. Rockafellar. Generalized second derivatives of convex functions and saddle functions. *Trans. Amer. Math. Soc.*, 322(1):51–77, 1990.

[70] R. T. Rockafellar and R. J.-B. Wets. *Variational Analysis*, volume 317 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag Berlin Heidelberg, 1998.

[71] N. Saito. Regularity of solutions to some variational inequalities for the Stokes equations. Number 1181, pages 182–193. 2001. Variational problems and related topics (Japanese) (Kyoto, 2000).

[72] N. Saito and H. Fujita. Regularity of solutions to the Stokes equations under a certain nonlinear boundary condition. In *The Navier-Stokes equations: theory and numerical methods (Varenna, 2000)*, volume 223 of *Lecture Notes in Pure and Appl. Math.*, pages 73–86. Dekker, New York, 2002.

[73] J. Salençon. *Handbook of continuum mechanics (general concepts, thermoelasticity)*. Springer-Verlag, Berlin, 2001.

[74] M. Shillor, M. Sofonea, and J. J. Telega. *Models and analysis of quasistatic: variational methods*. Springer-Verlag, Berlin, 2004.

[75] A. Signorini. Sopra alcune questioni di statica dei sistemi continui. *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze*, Ser. 2, 2(2):231–251, 1933.

[76] A. Signorini. Questioni di elasticità non linearizzata e semilinearizzata. *Rend. Mat. Appl., V. Ser.*, 18:95–139, 1959.

[77] J. Sokołowski. Sensitivity analysis of contact problems with prescribed friction. *Appl. Math. Optim.*, 18(2):99–117, 1988.

[78] J. Sokołowski and J. Zolésio. *Introduction to shape optimization*, volume 16 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1992.

[79] J. Sokołowski and J. Zolésio. Shape sensitivity analysis of contact problem with prescribed friction. *Nonlinear Analysis: Theory, Methods & Applications*, 12(12):1399–1411, 1988.

[80] M. Tucsnak and G. Weiss. *Observation and Control for Operator Semigroups*. Birkhäuser Advanced Texts Basler Lehrbücher. Birkhäuser Basel, 2009.

[81] B. Velichkov. *Existence and regularity results for some shape optimization problems*. Theses, Université de Grenoble, Nov. 2013.

UNIVERSITÉ DE PAU ET DES PAYS DE L'ADOUR

**Titre :** Analyse de sensibilité et optimisation pour des problèmes de contact

**Mot clés :** Opérateur proximal, épi-différentiabilité du second ordre, inéquations variationnelles, optimisation de forme, loi de contact unilatéral de Signorini, loi de friction de Tresca

**Résumé :** L'analyse de sensibilité de problèmes décrits par des inéquations variationnelles est un domaine prometteur pour le traitement de problèmes de contrôle optimal et d'optimisation de forme en mécanique du contact. L'objectif de cette thèse est de mener une telle analyse sans utiliser les méthodes classiques de régularisation et/ou de pénalisation qui perturbent la nature non lisse des modèles physiques d'origine. Nous proposons alors une nouvelle méthodologie, basée sur des outils avancés issus de l'analyse convexe et non lisse, tels que l'opérateur proximal et le concept d'épi-différentiabilité du second ordre. Plus précisément, les modèles considérés sont ceux issus de la mécanique du contact, décrivant le contact entre des solides déformables qui se touchent sur des parties de leurs bords sans s'interpénétrer et qui peuvent éventuellement glisser l'un contre l'autre, provoquant ainsi des phénomènes de frottement. La condition de non-perméabilité peut être décrite par la loi de contact unilatéral de Signorini, tandis que le frottement peut être décrit par la loi de friction de Tresca. Ces deux lois se traduisent par des inégalités et/ou des termes non lisses dans les formulations variationnelles correspondantes. À l'aide de notre approche, nous procédons à une analyse de sensibilité de ces modèles et étudions ainsi des problèmes associés de contrôle optimal et d'optimisation de forme.

**Title:** Sensitivity analysis and optimization for contact problems

**Keywords:** Proximal operator, twice epi-differentiability, variational inequalities, shape optimization, Signorini's unilateral conditions, scalar Tresca friction law

**Abstract:** Sensitivity analysis of problems described by variational inequalities is a promising field for the treatment of optimal control and shape optimization problems in contact mechanics. The objective of this PhD thesis is to carry out such analysis without using classical regularization and/or penalization procedures which perturb the non-smooth nature of the original physical models. Then we propose a new methodology, based on advanced tools from convex and non-smooth analyses, such as the proximal operator and the concept of second order epi-differentiability. More precisely, the models considered are those from contact mechanics, describing the contact between a deformable body with a rigid foundation without penetrating it, and possibly allows sliding modes which causes friction phenomena. The non-permeability condition can be described by the Signorini unilateral conditions, while the friction can be described by the Tresca friction law, both resulting in inequalities and/or non-smooth terms in the corresponding variational formulations. Using our approach, we perform a sensitivity analysis of these models and thus study associated optimal control and shape optimization problems.